An ODE for the Boundary Layer

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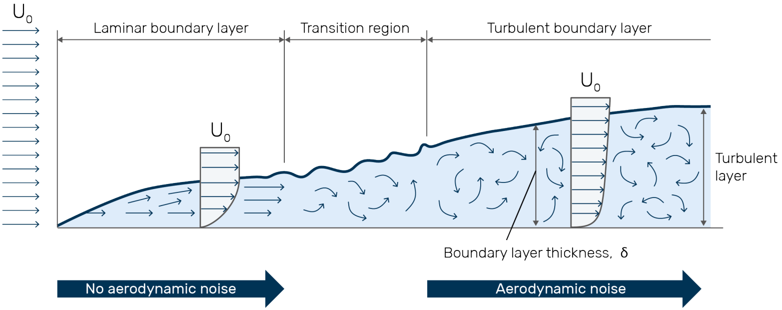
Moving a hull through water or an wing through air requires overcoming a drag force. This force results from viscosity, which is the ratio how much a fluid shears for a given force. The fluid sticks to the fluid boundary and is dragged along by the motion in a small boundary layer. Outside the layer, the fluid motion is more or less the wing velocity U_0.



Flow in the boundary layer near the wing tip is smooth or laminar and transitions to turbulent farther down the wing. Boundary layer thickness and velocity profiles as a function of distance to the wing are shown.

A model of a laminar boundary layer is presented here. Starting from the incompressible Navier-Stokes Equations which describe the motion of a viscous fluid in the plane, we neglect small terms to derive the boundary layer equations of Prandtl. By simplifying the geometry following Blasius, the boundary layer equation reduces to an boundary value problem for ordinary differential equations. Following Serrin, we solve the ODE for the velocity in the boundary layer [see Wilson].

We may regard our study of the boundary layer as an example of doing applied mathematics. Using expertise in fluid mechanics, the appropriate physical model is simplified by omitting negligible quantities derived by asymptotic analysis and experiments. By further simplifying the geometry, we obtain the equations of Blasius, which are tractable by calculus methods. Finally, last but not least, the calculation is informed by mathematical theorems about the short and long time existence, uniqueness and regularity of solutions of these ODE's. These theorems are proved in Math 5410.

The equations of fluid motion are studied in the fluid region of physical two and three dimensional space. Restricting to $d = 2$, fluid motion is described by the fluid velocity vector function $\big(u(x,y,t), v(x,y,t)\big)$ at time t and $p(x,y,t)$ is the pressure. If the flow is laminar and not turbulent, which is the case if motion is not too fast, the Navier Stokes Equations are

$$\begin{aligned} u_t + u\,u_x + v\,u_y &= -\tfrac{1}{\rho} p_x + \nu(u_{xx} + u_{yy}) \\ v_t + u\,v_x + v\,v_y &= -\tfrac{1}{\rho} p_y + \nu(v_{xx} + v_{yy}) \end{aligned}$$

where ρ is the density, p is the pressure and ν os the kinematic viscosity.

A fluid is incompressible if a blob of fluid does not change its area under the flow, thus it satisfies the continuity condition

$$u_x + v_y = 0. \tag{1}$$

If we denote by $\Phi(x, y, t)$ the position at time t of a fluid particle that starts at (x, y), the rate of change of area of a blob $G \subset \mathbf{R}^2$ at time t, $\Phi(G, t) \subset \mathbf{R}^2$, is given by the total flux through the boundary $\partial\Phi(G, t)$. Using the Divergence Theorem

$$\begin{aligned} \tfrac{d}{dt}\,\mathrm{A}(\phi(G,t)) &= \tfrac{d}{dt}\int_{\Phi(G,t)} dx\,dy = \int_{\partial\Phi(G,t)} un_x + vn_y\,ds \\ &= \int_{\Phi(G,t)} u_x + v_y\,dx\,dy \end{aligned}$$

where (n_x, n_y) is the unit outward normal vector at $(x, y) \in \partial\Phi(G, t)$ and ds is arclength. Assuming the flow does not change the area, $\frac{d}{dt}\,\mathrm{A} = 0$ for every blob G, the velocity must satisfy (1).

In addition to incompressibility, we assume that the motion is stationary, which means that (u, v) does not depend on t, and that the density is constant throughout, $\rho = \rho_0$

The resulting incompressible Navier-Stokes equations reduce to

$$\begin{aligned} u_x + v_y &= 0 \\ u\,u_x + v\,u_y &= -\tfrac{1}{\rho_0} p_x + \nu(u_{xx} + u_{yy}) \\ u\,v_x + v\,v_y &= -\tfrac{1}{\rho_0} p_y + \nu(v_{xx} + v_{yy}) \end{aligned} \tag{2}$$

Let us consider a viscous incompressible fluid which flows parallel to the x-axis with constant speed U_∞. To simplify the geometry, imagine that the wing is a flat plate immersed in the fluid in such a way that that the coordinate are relative to a wing which we model by the half axis $\{(x,y) \in \mathbf{R}^2 : x \geq 0, y = 0\}$ and we consider fluid in the positive halfplane $y > 0$. The fluid has zero velocity (no slip boundary condition) at the wing

$$u(x,0) = v(x,0) = 0 \qquad \text{for } x > 0.$$

The wake is not very pronounced at large distances from the plate. This gives the boundary condition, expressed as

$$\lim_{y\to\infty} u(x,y) = U_\infty, \qquad \lim_{y\to\infty} v(x,y) = 0. \tag{3}$$

for each fixed $x > 0$.

Here is a simplified deduction of Prandtl's Boundary Layer Equations. Since v is relatively small in the boundary layer, the last equation of (2) says p is practically independent of y. Thus we may neglect p_x since this vanishes for large values of y where the stream is unaffected by the plate. Also, since u and u_x vanish at the interface, it follows that u_{xx} is negligible compared to u_{yy} in the boundary layer. Thus we may neglect the red terms of (2) resulting in the The resulting equation is the Prandtl Boundary Layer Equations.

$$
\begin{aligned}
u_x + v_y &= 0 \\
u\, u_x + v\, u_y &= -\frac{1}{\rho_0} p_x + \nu(u_{xx} + u_{yy})
\end{aligned}
$$

Figure: Ludwig Prandtl (1875–1953)

Prandtl received his Dr. Phil. at Munich Technical University in 1899 under the guidance of August Föppl. He began teaching in Hannover. In 1904 he delivered a ground breaking paper describing the boundary layer and its importance for drag and streamlining, explaining the concept of stall for the first time.

For this, he was promoted to Director of Technical Physics in Göttingen. Under his guidance, it became a powerhouse of aerodynamics, leading the world until the end of World War II. Its spin off became the Max Planck Institute for Dynamics and Self-Organization.

Start from the Prandtl Boundary Layer Equations

$$
\begin{gathered}
u_x + v_y = 0 \\
u\, u_x + v\, u_y = \nu\, u_{yy} \\
u = v = 0 \quad \text{for } y = 0, \qquad u \to U_\infty \quad \text{as } y \to \infty.
\end{gathered} \tag{4}
$$

Assuming that the velocity is given by a stream function, we automatically satisfy the incompressibility condition, namely, suppose there is a function $w(x, y)$ such that

$$
\begin{aligned}
u &= w_y \\
v &= -w_x .
\end{aligned}
$$

This vector field is perpendicular to the gradient $\nabla w = (w_x, w_y)$ thus the level sets of the stream function are flow lines of the fluid. Substituting, we find that

$$w_y\, w_{xy} - w_x\, w_{yy} = \nu w_{yyy}, \tag{5}$$

$$w_x = w_y = 0 \text{ at } y = 0 \text{ and } x > 0, \tag{6}$$

$$w_y \to U_\infty \text{ as } y \to \infty \text{ for all } x. \tag{7}$$

(5) admits a similarity solution. Assume we the variables scale by

$$
\begin{aligned}
x &= \lambda^a \tilde{x}, \\
y &= \lambda^b \tilde{x}, \\
w &= \lambda^c \tilde{w}.
\end{aligned}
$$

Substituting into (5) yields

$$\lambda^{2c-a-2b} \tilde{w}_y \, w_{\tilde{x}\tilde{y}} - \lambda^{2c-a-2b} \tilde{w}_{\tilde{x}} \, w_{\tilde{y}\tilde{y}} = \nu \lambda^{c-3b} w_{\tilde{y}\tilde{y}\tilde{y}}$$

so

$$\tilde{w}_y \, w_{\tilde{x}\tilde{y}} - \tilde{w}_{\tilde{x}} \, w_{\tilde{y}\tilde{y}} = \nu \lambda^{-c+a-b} w_{\tilde{y}\tilde{y}\tilde{y}}$$

implying two conditions are

$$
\begin{gathered}
\tilde{w}_y \, w_{\tilde{x}\tilde{y}} - \tilde{w}_{\tilde{x}} \, w_{\tilde{y}\tilde{y}} = \nu \lambda^{-c+a-b} w_{\tilde{y}\tilde{y}\tilde{y}} \\
\lambda^{c-b} \tilde{w}_{\tilde{y}} = 0 \text{ at } \tilde{y} = 0 \text{ and } \tilde{x} > 0.
\end{gathered}
$$

To be invariant, for any choice of λ the function $\tilde{w}$ must satisfy the Prandtl equations. This implies we should take

$$\begin{aligned} -c + a - b &= 0 \\ c - b &= 0. \end{aligned}$$

Hence $b = \frac{a}{2}$ and $c = \frac{a}{2}$. Taking $a = 1$, the functions

$$w(x, y; \lambda) = \lambda^{1/2} \tilde{w} \left(\lambda^{-1} x, \lambda^{-1/2} y \right)$$

are all solutions of (5) for every λ is w is a solution.

This suggests that we seek solutions in a class of functions that has the same invariance. Thus for example, we make the Ansatz

$$w(x,y) = Bx^{1/2}f(t), \qquad \text{where } t = \frac{Ay}{\sqrt{x}} \tag{8}$$

where A and B are positive constants. For $x > 0$ we have $t \to 0$ if and only if $y \to 0$. Substituting,

$$w_x = \frac{B}{2}x^{-1/2}f - \frac{AB}{2}x^{-1}yf', \qquad w_y = ABf'$$

Thus for $x > 0$, the boundary conditions (6) and (7) are

$$f(t) = f'(t) \to 0 \text{ as } t \to 0, \tag{9}$$

$$f'(t) \to \frac{U_\infty}{AB} \text{ as } t \to \infty \text{ for all } x. \tag{10}$$

Substituting into (8) into (5),

$$-\frac{1}{2}Bf\,f'' = \nu A f'''.$$

Supposing that

$$\frac{1}{2}B = \nu A \qquad \text{and} \qquad \frac{U_\infty}{AB} = 1$$

so

$$A = \left(\frac{U_\infty}{2\nu}\right)^{1/2}, \qquad B = (2\nu U_\infty)^{1/2},$$

we have

$$w(x,y) = (2\nu U_\infty x)^{1/2} f(t), \qquad \text{where } t = \left(\frac{U_\infty}{2\nu x}\right)^{1/2} y. \tag{11}$$

(11) solves (5), (6), (7) provided f solves the Blasius Equations:

$$\begin{aligned} f'''(t) + f(t)f''(t) &= 0 \qquad \text{for } t > 0, \\ f(0) = f'(0) &= 0, \\ f'(t) \to 1 \qquad &\text{as } t \to \infty. \end{aligned} \tag{12}$$

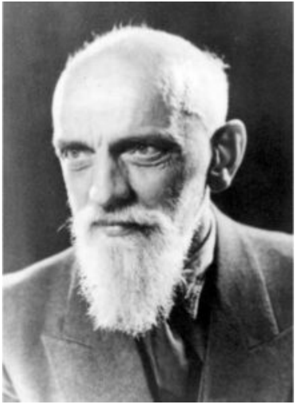


Figure: Heinrich Blasius (1883–1970)

Born in Berlin, Blasius, the sixth student of Prandtl, completed his Dr. Phil. at Gottingen in 1907. Blasius provided a mathematical basis for boundary-layer drag but also showed as early as 1911 that the resistance to flow through smooth pipes could be expressed in terms of the Reynolds number for both laminar and turbulent flow. After six years in science he changed to Ingenieurschule Hamburg (today: University of Applied Sciences Hamburg) and became a Professor.

The Blasius problem is a boundary value problem since the values of the solution are specified at two points, at the origin and at infinity. We shall modify a solution of an initial value problem. That is, we consider

$$\begin{gathered} z'''(t) + z(t)z''(t) = 0 \qquad \text{for } t > 0, \\ z(0) = z'(0) = 0, z''(0) = \gamma \end{gathered}$$

and show that when $\gamma = 1$ the solution almost satisfies (12), but with a different value at the boundary. By changing scale we obtain the desired solution of (12).

We claim that the solution $z(t;\gamma)$ exists for all time $t > 0$ and all $\gamma > 0$.

We shall follow Serrin's method as described by Wilson. The existence of a solution follows the big theorem proved in Math 5410.

Theorem (Existence and Uniqueness Theorem)

Let $F(t, y_1 \ldots, y_n)$ be a continuously differentiable function defined on $\mathcal{D} = \mathbf{R} \times \mathbf{R}^n$ and $(t_0, x_0, \ldots, x_{n-1}) \in \mathcal{D}$. Then there is a unique maximal continuously differentiable solution $y(t) = \psi(t)$ of the initial value problem

$$\frac{d^n y}{dt^n} = F\left(t, y, \frac{dy}{dt}, \frac{d^2 y}{dt^2}, \ldots, \frac{d^{n-1} y}{dt^{n-1}}\right),$$

$$y = x_0, \frac{dy}{dt} = x_1, \ldots, \frac{d^{n-1} y}{dt^{n-1}} = x_{n-1}, \qquad \text{when } t = t_0.$$

The interval of existence is an open interval $\tau_- < t < \tau_+$. If $\tau_+ < \infty$, then $(\psi, \psi', \psi'', \ldots, \psi^{(n-1)})$ becomes unbounded as $t \to \tau_+$.

According to the existence theorem, for any $\gamma > 0$, the Blasius IVP

$$\begin{gathered} y'''(t) = -y(t)y''(t), \\ y(0) = y'(0) = 0, y''(0) = \gamma \end{gathered} \tag{13}$$

has a unique thrice continuously differentiable solution $\psi(t;\gamma)$ defined on the interval $0 \leq t < \tau_+(\gamma)$.

Suppose that it is known that $\tau_+ = \infty$ and that $\lim_{t\to+\infty} \psi'(t,1) = L^2$ exists, where $\lambda > 0$. Then the Blasius Equation (12) may be solved by making the change of variable

$$y(s) = \frac{1}{L}\psi\left(\frac{s}{L}, 1\right).$$

To show $\tau_+ = \infty$, we first show that $\psi'' > 0$ for $0 < t < \tau_+$. If this is not the case, since we are given that $\psi''(0) = 1$, there is a first time $0 < t_0 < \tau_+$ where $\psi''(t_0; \gamma) = 0$. Let

$$A = \psi(t_0; \gamma), \qquad B = \psi'(y_0; \gamma), \qquad \phi(t) = A + B(t - t_0).$$

Then both $\psi(t, \gamma)$ and $\phi(t)$ satisfy the initial value problem

$$y'''(t) = -y(t)y''(t),$$
$$y(t_0) = A,\ y'(y_0) = B, y''(t_0) = 0$$

on $[0, t_0]$. Consequently, by the uniqueness of solutions of the IVP, $\phi(t) = \psi(t)$ for $0 \le t < \tau_+$. This is a contradiction because $\psi''(0) = \gamma > 0$ but $\phi''(0) = 0$.

We show now $\tau_+ = \infty$. From $\psi'' > 0$ on $[0, \tau_+)$, it follows by integration that ψ' and ψ are positive and increasing on $[0, \tau_+)$. Multiplying (13) by the positive integrating factor $\exp(\int \psi(s)\, ds)$,

$$\left(e^{\int_0^t \psi(s;\gamma)\, ds}\psi''(t;\gamma)\right)' = e^{\int_0^t \psi(s;\gamma)\, ds}\left(\psi'''(t;\gamma) + \psi(t;\gamma)\psi''(t;\gamma)\right) = 0$$

implies the product is constant $e^{\int_0^t \psi(s;\gamma)\, ds}\psi''(t;\gamma) = \psi''(0;\gamma) = \gamma$ so

$$\psi''(t) = \gamma e^{-\int_0^t \psi(s)\, ds}. \tag{14}$$

Thus ψ'' is decreasing and

$$0 < \psi'' < \gamma \qquad \text{on } 0 < t < \tau_+ \tag{15}$$

Integrating this inequality twice yields

$$0 < \psi' < \gamma t, \qquad 0 < \psi < \tfrac{\gamma}{2} t^2, \qquad \text{on } 0 < t < \tau_+.$$

This says the solution stays bounded and does not blow up on a finite interval so the existence time has to be infinite $\tau_+ = \infty$ by the existence theorem.

Next we show that $\lim_{t\to+\infty} \psi'(t)$ exists. We already know that $\psi'(t)$ is monotone increasing. Thus we only need to show that it is bounded above. For $t \geq 1$,

Let $c = \psi(1) > 0$. Using $\psi > 0$ and is increasing for $t > 0$, we have $\psi(t) \geq c$ for $t \geq 1$. By (14)

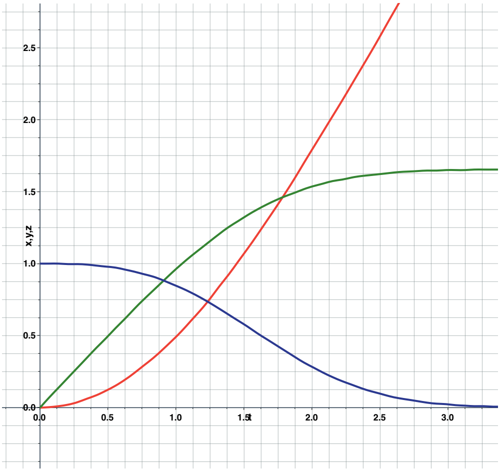
$$\psi''(t) = \gamma e^{-\int_0^1 \psi(s)\,ds - \int_1^t \psi(s)\,ds} \leq \gamma e^{-\int_1^t c\,ds} = e^{-c(t-1)}.$$

Integrating, for $t \geq 1$,

$$\psi'(t) \leq \psi'(1) + \gamma \int_1^\infty e^{-c(s-1)}\,ds = \psi'(1) + \frac{\gamma}{c}$$

which is bounded, finishing the argument that $\lim_{t\to+\infty} \psi'(t)$ exists. This completes the solution of the Blasius Problem. $\square$

23. Numerical Solution of Blasius Equation (13).



At fixed x, t is proportional to y , red$= f(t)$, blue$=f''(t)$, green$= f'(t)$ which is proportional to fluid velocity in the boundary layer.

Here is another two point boundary value problem for the Blasius Equation, with the condition at infinity replaced by a condition at a finite time $T = 1$. For $0 \le t \le T$,

$$\begin{aligned} f'''(t) + f(t)f''(t) &= 0 \\ f(0) = f'(0) &= 0 \\ f'(1) &= 1. \end{aligned} \tag{16}$$

As before, we consider the corresponding initial value problem with $\gamma > 0$

$$z'''(t) + z(t)z''(t) = 0$$
$$z(0) = z'(0) = 0, \quad z''(0) = \gamma$$

We use a shooting method: we continuously vary γ starting from zero and show that the solution at the endpoint $\psi'(1;\gamma)$ varies continuously from small to arbitrarily large values. Thus, for the correct choice of $\gamma = \gamma_0$, the solution is the trajectory that satisfies $\psi'(1,\gamma_0) = 1$.

We have already established that the solution exists on $0 \le t \le 1$ for every $\gamma > 0$. It also depends continuously on initial conditions.

Theorem (Continuous Dependence Theorem)

Let $F(t, y_1 \dots, y_n)$ be a continuously differentiable function defined on $\mathcal{D} = \mathbf{R} \times \mathbf{R}^n$ and $(t_0, x_0, \dots, x_{n-1}) \in \mathcal{D}$. The unique maximally defined solution $y(t) = \psi(t; t_0, x_0, \dots, x_{n-1})$ of the initial value problem

$$\frac{d^n y}{dt^n} = F\left(t, y, \frac{dy}{dt}, \frac{d^2 y}{dt^2}, \dots, \frac{d^{n-1} y}{dt^{n-1}}\right),$$

$$y = x_0, \frac{dy}{dt} = x_1, \dots, \frac{d^{n-1} y}{dt^{n-1}} = x_{n-1}, \qquad \text{when } t = t_0.$$

is a continuously differentiable function in the set $\{(t, t_0, x_0, \dots, x_{n-1}) \in \mathbf{R} \times \mathbf{R} \times \mathbf{R}^n : \tau_-(t_0, x_0, \dots, x_{n-1}) < t < \tau_+(t_0, x_0, \dots, x_{n-1})\}$.

Thus the solution of the Blasius problem $\psi(t, \gamma)$ depends continuously on $(t, \gamma) \in [0, 1] \times (0, \infty)$.

We now show that $\psi(1,\gamma) < 1$ for γ sufficiently small and $\psi(1,\gamma) > 1$ for γ sufficiently large. By the Intermediate Value Theorem, there is an in between γ_0 for which $\psi(1,\gamma_0) = 1$. Consequently $z(t) = \psi(t;\gamma_0)$ solves the Boundary Value Problem (16).

Recall (15), for all $t \in [0,1]$ and $\gamma > 0$ the solution ψ'' is decreasing and

$$0 < \psi'' < \gamma, \qquad 0 < \psi' < \gamma t, \qquad 0 < \psi < \tfrac{\gamma}{2} t^2.$$

If $\gamma < 1$ then $\psi'(1,\gamma) < 1$. Suppose on the other hand that $\gamma > 1$ is so large that

$$6\left(1 - e^{-\gamma/6}\right) > 1.$$

Since $\psi'' > 0$, rewriting the Blasius Equation,

$$\frac{d}{dt}\ln(\psi'') = \frac{\psi'''}{\psi''} = -\psi > -\frac{\gamma}{2}t^2,$$

and integrating, we find using $\psi''(0,\gamma) = \gamma$,

$$\ln\left(\psi''(t,\gamma)\right) - \ln(\gamma) = -\int_0^t \psi(s;\gamma)\,ds > -\frac{\gamma}{6}t^3.$$

Exponentiating,

$$\frac{1}{\gamma}\psi''(t,\gamma) = e^{-\int_0^t \psi(s;\gamma)\,ds} > e^{-\gamma s^3/6}.$$

Finally integrating, and using $s^3 \le s$ for $0 \le s \le 1$,

$$\psi'(t,\gamma) > \gamma \int_0^1 e^{-\gamma s^3/6}\,ds \ge \gamma \int_0^1 e^{-\gamma s/6}\,ds = 6\left(1 - e^{-\gamma/6}\right) > 1$$

by our assumption on γ. The existence of the solution now follows from the Intermediate Value Theorem as described.

28. References

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