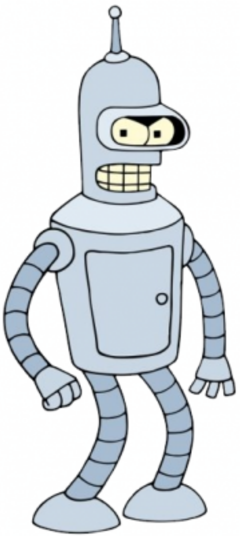

Figure: Bender

Geometry of Bending Surfaces

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The URL for these Beamer Slides: *"Geometry of Bending Surfaces"*

http://www.math.utah.edu/~treiberg/BendingSlides.pdf

- Manfredo doCarmo, *Differential Geometry of Curves and Surfaces,* Prentice Hall, Englewood Cliffs, 1976.
- Erwin Kreyszig, *Differential Geometry,* Dover, New York, 1991; orig. U. Toronto Press, Toronto, 1959.
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4. Outline.

- Surfaces
  - Examples. Definition. Moving Frame. Coframe. Metric.
- Intrinsic Geometry is Preserved by Bending = Local Isometry
  - Length of Curve. Example of Torus. Rigid Motions Preserve Metric.
- Extrinsic Geometry
  - Second Fundamental Form. Mean Curvature. Gaussian Curvature.
  - Examples: Sphere, Graph, Torus. Gauss Theorem.
- Ruled Surfaces
  - Moving Frame along Space Curve
  - Gaussian Curvature of Ruled Surface.
  - Examples: Hyperbolic Paraboloid, Helicoid, Hyperboloid.
  - Tanget Developables are Isometric to the Plane.
- What Can a Flat Surface be Bent Into?
  - Non-umbillic Flat Surfaces are Ruled.
  - Flat Ruled Surfaces are Plane, Cylinder, Cone or Tangent Developable.
- Global Geometry
  - Complete Flat surfaces are Planes or clinders
  - Compact, Closed Constant Gaussian Curvature Surface is Sphere.

5. Examples of Surfaces.

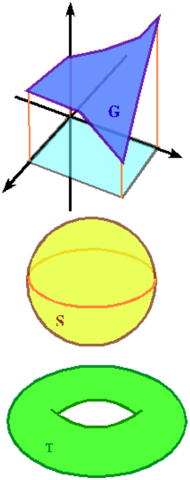


Figure: Surfaces

Some examples of what should be surfaces.

- Graphs of functions
  $\mathbb{G}^2 = \{\big(x, y, f(x, y)\big) \in \mathbf{R}^3 : (x, y) \in U\}$
  where $U \subset \mathbf{R}^2$ is an open set.
- Level sets, *e.g.*,
  $\mathbb{S}^2 = \{(x, y, z) \in \mathbf{R}^3 : x^2 + y^2 + z^2 = 1\}$
  This is the standard unit sphere.
- Parameterized Surfaces, *e.g.*,
  $\mathbb{T}^2 = \Big\{\Big((a+b\cos\psi)\cos\theta, (a+b\cos\psi)\sin\theta, b\sin\psi\Big) : \theta, \psi \in \mathbf{R}\Big\}$
  is the torus with radii $a > b > 0$ constructed as a surface of revolution about the z-axis.

A surface can locally be given by a curvilinear coordinate chart, also called a parameterization. Let $U \subset \mathbf{R}^2$ be open. Let

$$X : U \to \mathbf{R}^3$$

be a smooth function. Then we want $M = X(U)$ to be a piece of a surface. At each point $P \in X(U)$ we can identify tangent vectors to the surface. If $P = X(a)$ some $a \in U$, then

$$X_i(a) = \frac{\partial X}{\partial u^i}(a)$$

for $i = 1, 2$ are vectors in $\mathbf{R}^3$ tangent to the coordinate curves. To avoid singularities at P, we shall assume that all $X_1(P)$and $X_2(P)$ are linearly independent vectors. Then the tangent plane to the surface at P is

$$T_P M = \operatorname{span}\{X_1(P), X_2(P)\}.$$

For the graph $\mathbb{G}^2 = \{(x, y, z) \in \mathbf{R}^3 : \; z = f(x, y) \text{ and } (x, y) \in U \;\}$ one coordinate chart covers the whole surface, $X : U \to \mathbb{G}^2 \cap V = \mathbb{G}^2$, where

$$X(u^1, u^2) = (u^1, u^2, f(u^1, u^2)).$$

where $V = \{(u^1, u^2, u^3) : (u^1, u^2) \in U \text{ and } u^3 \in \mathbf{R}\}$.
The tangent vectors are thus

$$\begin{aligned} X_1(u^1, u^2) &= \left(1, 0, \frac{\partial f}{\partial u^1}(u^1, u^2)\right), \\ X_2(u^1, u^2) &= \left(0, 1, \frac{\partial f}{\partial u^2}(u^1, u^2)\right) \end{aligned} \tag{1}$$

which are linearly independent for every $(u^1, u^2) \in U$.

Definition

A connected subset $M \subset \mathbf{R}^3$ is a *regular surface* if to each $P \in M$, there is an open neighborhood $P \in V \subset \mathbf{R}^2$, and a map

$$X : U \to V \cap M$$

of an open set $U \subset \mathbf{R}^2$ onto $V \cap M$ such that

1. X is differentiable. (In fact, we shall assume X is smooth ($\mathcal{C}^\infty$)
2. X is a homeomorphism (X is continuous and has a continuous inverse)
3. The tangent vectors $X_1(a)$ and $X_2(a)$ are linearly independent for all $a \in U$.

For simplicity, we shall assume our surfaces are also orientable.

The Euclidean structure of $\mathbf{R}^3$, the usual dot product, gives a way to measure lengths and angles of vectors. If $V = (v_1, v_2, v_3)$ then its length

$$|V| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{V \bullet V}$$

If $W = (w_1, w_2, w_3)$ then the angle $\alpha = \angle(V, W)$ is given by

$$\cos\alpha = \frac{V \bullet W}{|V|\,|W|}.$$

If $\gamma : [a, b] \to M \subset \mathbf{R}^3$ is a continuously differentiable curve, its length is

$$\mathrm{L}(\gamma) = \int_a^b |\dot{\gamma}(t)|\,dt.$$

It is convenient to find an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ for each tangent space. One way, but not the only way to find an orthonormal basis is to use the Gram-Schmidt algorithm:

$$\mathbf{e}_1 = \frac{X_1}{|X_1|}, \qquad \mathbf{e}_2 = \frac{X_2 - (X_2 \bullet \mathbf{e}_1)\mathbf{e}_1}{|X_2 - (X_2 \bullet \mathbf{e}_1)\mathbf{e}_1|}. \tag{2}$$

Then $\mathbf{e}_1$ and $\mathbf{e}_2$ vary smoothly point to point and span the tangent space. We can also define a unit normal vector which is perpendicular to the tangent plane by

$$\mathbf{e}_3 = \frac{X_1 \times X_2}{|X_1 \times X_2|} = \mathbf{e}_1 \times \mathbf{e}_2.$$

The resulting moving frame, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is orthonormal

$$\mathbf{e}_i \bullet \mathbf{e}_j = \delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

It is also convenient to introduce the moving coframe of one-forms (linear functionals). For any $V \in T_pM$,

$$\omega^i(V) = \mathbf{e}_i \bullet V.$$

They satisfy duality equations

$$\omega^A(\mathbf{e}_B) = \delta^A{}_B.$$

One forms may be integrated along curves.

If the surface M is given in terms of local parameters $X(u^1, u^2)$ then the one forms are expresses in terms of differentials du^1 and du^2. These are not orthonormal but are dual to coordinate directions

$$du^1(X_1) = 1, \qquad du^1(X_2) = 0, \qquad du^2(X_1) = 0, \qquad du^2(X_2) = 1.$$

Thus if $\omega^i = p\,du^1 + q\,du^2$ we get the coefficients from

$$p = \omega^i(X_1), \qquad q = \omega^i(X_2).$$

Then if the vector field is $V = v^1\mathbf{e}_1 + v^2\mathbf{e}_2$ then $\omega^i(V) = v^i$ and the length squared is

$$|V|^2 = \left(\omega^1(V)\right)^2 + \left(\omega^2(V)\right)^2.$$

For short we write the Riemannian Metric

$$ds^2 = \left(\omega^1\right)^2 + \left(\omega^2\right)^2$$

where s is for arclength. It also called the First Fundamental Form.

The area two form is

$$dA = \omega^1 \wedge \omega^2.$$

Two forms may be integrated along surfaces.

If $\gamma : [a, b] \to M$ is a curve on the surface then we may factor through the coordinate chart. There are continuously differentiable $u(t) = (u^1(t), u^2(t)) \in U$ so that

$$\gamma(t) = X(u^1(t), u^2(t)) \qquad \text{for all } t \in [a, b].$$

We write its velocity vector

$$\begin{aligned}\dot\gamma(t) &= X_1(u^1(t), u^2(t))\,\dot u^1(t) + X_2(u^1(t), u^2(t))\,\dot u^2(t)\\ &= v^1(t)\mathbf{e}_1(\gamma(t)) + v^2(t)\mathbf{e}_1(\gamma(t))\end{aligned}$$

so that the length of the curve

$$\begin{aligned}L(\gamma) &= \int_\gamma ds\\ &= \int_a^b \sqrt{[\omega^1(\dot\gamma(t))]^2 + [\omega^2(\dot\gamma(t))]^2}dt.\\ &= \int_a^b \sqrt{[v^1(t)]^2 + [v^2(t)]^2}dt.\\ &= \int_a^b |\dot\gamma(t)|\, dt\end{aligned}$$

For the graph $\mathbb{G}^2 = \{(x, y, z) \in \mathbf{R}^3 : \; z = f(x, y) \text{ and } (x, y) \in U \;\}$ take the patch $X(u^1, u^2) = (u^1, u^2, f(u^1, u^2))$. Hence

$$X_1 = \begin{pmatrix} 1 \\ 0 \\ f_1 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 0 \\ 1 \\ f_2 \end{pmatrix},$$

Orthonormalizing $\{X_1, X_2\}$ using (2),

$$\mathbf{e}_1 = \frac{1}{\sqrt{1 + f_1^2}} \begin{pmatrix} 1 \\ 0 \\ f_1 \end{pmatrix}, \quad \mathbf{e}_2 = \frac{1}{\sqrt{1 + f_1^2}\sqrt{1 + f_1^2 + f_2^2}} \begin{pmatrix} -f_1 f_2 \\ 1 + f_1^2 \\ f_2 \end{pmatrix}$$

so

$$X_1 = \sqrt{1 + f_1^2}\, \mathbf{e}_1, \quad X_2 = \frac{f_1 f_2}{\sqrt{1 + f_1^2}} \mathbf{e}_1 + \frac{\sqrt{1 + f_1^2 + f_2^2}}{\sqrt{1 + f_1^2}} \mathbf{e}_2.$$

If $\omega^i = a^i du^1 + b^i du^2$ then $\omega^i(X_1) = a^i$ and $\omega^i(X_2) = b^i$. It follows that

$$\omega^1 = \sqrt{1+f_1^2}\, du^1 + \frac{f_1 f_2}{\sqrt{1+f_1^2}}\, du^2, \quad \omega^2 = \frac{\sqrt{1+f_1^2+f_2^2}}{\sqrt{1+f_1^2}}\, du^2.$$

and so the Riemannian metric is

$$\begin{aligned} ds^2 &= (\omega^1)^2 + (\omega^2)^2 \\ &= (1+f_1^2)\,(du^1)^2 + 2f_1 f_2\, du^1\, du^2 + (1+f_2^2)\,(du^2)^2, \end{aligned}$$

as expected. Also

$$dA = \omega^1 \wedge \omega^2 = \sqrt{1+f_1^2+f_2^2}\, du^1 \wedge du^2,$$

also as expected. It gives the usual formula for area

$$\mathrm{A}(X(D)) = \int_D \sqrt{1+f_1^2+f_2^2}\, du_1\, du_2.$$

16. Intrinsic Geometry is Preserved by Bending.

Geometric quantities determined by the metric are called intrinsic. A bending of one surface may into another is a local isometry, a diffeomorphism that preserves lengths of curves, hence all intrinsic quantities. Equivalently, the Riemannian metrics are preserved. Thus if

$$f : (M^2, ds^2) \to (\tilde{M}^2, d\tilde{s}^2)$$

is an isometry, then $f : M^n \to \tilde{M}^n$ is a diffeomorphism and $f^* d\tilde{s}^2 = ds^2$. This means that for at every point $u \in M$, and every tangent vector V, W of M, the corresponding inner products are the same in M and $\tilde{M}$:

$$ds^2(V(u), W(u))_u = d\tilde{s}^2(df_u(V(u)), df_u(V(u)))_{\tilde{u}}$$

where $\tilde{u} = \tilde{u}(u)$ correspond under the map and $df_u : T_u M \to T_{\tilde{u}} \tilde{M}$ is the differential. We have written the first fundamental form $ds^2(V, W) = v^1 w^1 + v^2 w^2$.

WARNING: funtional analysts and geometric group theorists define "isometry" in a slightly different way.

Lets work out the geometry of a torus $\mathbb{T}^2$. Suppose that the torus has radii $0 < b < a$ and is given parametrically as

$$X(\theta, \psi) = \begin{pmatrix} \cos\theta(a + b\cos\psi) \\ \sin\theta(a + b\cos\psi) \\ b\sin\psi \end{pmatrix},$$

$$X_\theta(\theta, \psi) = (a + b\cos\psi) \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix} = (a + b\cos\psi)\mathbf{e}_1,$$

$$X_\psi(\theta, \psi) = b \begin{pmatrix} -\cos\theta\sin\psi \\ -\sin\theta\sin\psi \\ \cos\psi \end{pmatrix} = b\,\mathbf{e}_2,$$

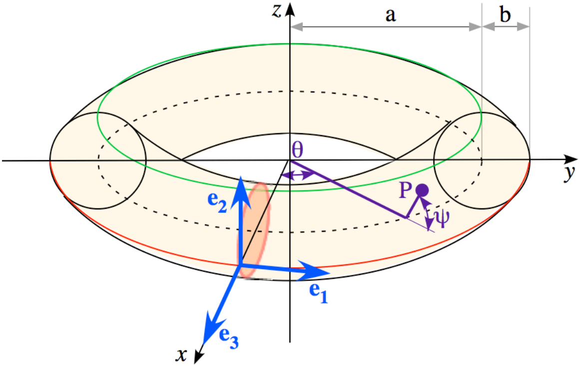
$$\frac{X_\theta \times X_\psi}{|X_\theta \times X_\psi|} = \begin{pmatrix} \cos\theta\cos\psi \\ \sin\theta\cos\psi \\ \sin\psi \end{pmatrix} = \mathbf{e}_3$$

By duality,

$$\omega^1 = (a + b\cos\psi)\,d\theta, \qquad \omega^2 = b\,d\psi$$

Thus the metric of $\mathbb{T}^2$ is

$$ds^2 = (\omega^1)^2 + (\omega^2)^2 = (a + b\cos\psi)^2\,d\theta^2 + b^2\,d\psi^2.$$



Extrinsic Geometry deals with how M sits in its ambient space.

Near $P \in M$, the surface may be parameterized as the graph over its tangent plane, where $f(u_1, u_2)$ is the “height” above the tangent plane

$$X(u_1, u_2) = P + u_1\mathbf{e}_1(P) + u_2\mathbf{e}_2(P) + f(u_1, u_2)\mathbf{e}_3(p). \qquad (3)$$

So $f(0) = 0$ and $Df(0) = 0$. The Hessian of f at 0 gives the shape operator at P. It is also called the Second Fundamental Form.

$$h_{ij}(P) = \frac{\partial^2 f}{\partial u_i\, \partial u_j}(0)$$

The Mean Curvature and Gaussian Curvature at P are

$$H(P) = \frac{1}{2}\,\mathrm{tr}(h_{ij}(P)), \qquad K(P) = \det(h_{ij}(P)).$$

20. Sphere Example.

The sphere about zero of radius $r > 0$ is an example

$$\mathbb{S}^2_r = \{(x, y, z) \in \mathbf{R}^3 : x^2 + y^2 + z^2 = r^2\}.$$

Let $P = (0, 0, -r)$ be the south pole. By a rotation (an isometry of $\mathbb{R}^3$), any point of $\mathbb{S}^2_r$ can be moved to P with the surface coinciding. Thus the computation of $H(P)$ and $K(P)$ will be the same at all points of $\mathbb{S}^2_r$. If $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$ and $\mathbf{e}_3 = (0, 0, 1)$, the height function of (3) near zero is given by

$$f(u_1, u_2) = r - \sqrt{r^2 - u_1^2 - u_2^2}.$$

The Hessian is

$$\frac{\partial^2 f}{\partial u_i\, u_j}(u) = \begin{pmatrix} \frac{r^2 - u_2^2}{\left(r^2 - u_1^2 - u_2^2\right)^{3/2}} & \frac{-u_1 u_2}{\left(r^2 - u_1^2 - u_2^2\right)^{3/2}} \\ \frac{-u_1 u_2 u_2^2}{\left(r^2 - u_1^2 - u_2^2\right)^{3/2}} & \frac{r^2 - u_1^2}{\left(r^2 - u_1^2 - u_2^2\right)^{3/2}} \end{pmatrix}$$

Thus the second fundamental form at P is

$$h_{ij}(P) = f_{ij}(0) = \begin{pmatrix} \frac{1}{r} & 0 \\ 0 & \frac{1}{r} \end{pmatrix} \qquad \text{so } H(P) = \frac{1}{r} \text{ and } K(P) = \frac{1}{r^2}$$

If $X(u_1, u_2) = (u_1, u_2, f(u_1, u_2))$, by correcting for the slope at different points one finds for all $(u_1, u_2) \in U$,

$$H(u_1, u_2) = \frac{\left(1 + f_2^2\right) f_{11} - 2 f_1\, f_2\, f_{12} + \left(1 + f_1^2\right) f_{22}}{2\left(1 + f_1^2 + f_2^2\right)^{3/2}},$$

$$K(u_1, u_2) = \frac{f_{11}\, f_{22} - f_{12}^2}{\left(1 + f_1^2 + f_2^2\right)^2}.$$

Because $\mathbf{e}_A \cdot \mathbf{e}_A = 1$, taking the directional derivative $d\mathbf{e}_A \cdot \mathbf{e}_A = 0$, so that $d\mathbf{e}_A \perp \mathbf{e}_A$ and we may express the rate of rotation of the frame

$$d\mathbf{e}_A = \omega_A{}^B \mathbf{e}_B,$$

where summation over repeated indices is assumed. $\omega_A{}^B$ are called connection forms. Differentiation of $\mathbf{e}_A \cdot \mathbf{e}_B = \delta_{AB}$ implies $\omega_A{}^B$ is skew and satisfies

$$d\omega^A = \omega^B \wedge \omega_B{}^A. \tag{4}$$

Also, differentiating the normal $d\mathbf{e}_3 = \omega_3{}^i \mathbf{e}_i$, where lower case Roman indices run over $i, j, k, \ldots = 1, 2$. Moreover

$$\omega_3{}^i = -h_{ij}\omega^j$$

recovers the second fundamental form.

Indeed, for surfaces of the form (3),

$$X(u_1, u_2) = P + u_1\mathbf{e}_1(P) + u_2\mathbf{e}_2(P) + f(u_1, u_2)\mathbf{e}_3(p),$$

the normal vector equals

$$\mathbf{e}_3 = \frac{1}{\sqrt{1 + f_1^2 + f_2^2}} \begin{pmatrix} -f_1 \\ -f_2 \\ 1 \end{pmatrix}$$

so that at P where $f(0) = f_1(0) = f_2(0) = 0$, we get

$$\begin{aligned} d\mathbf{e}_3(P) &= -\sum_{i,j=1}^{2} \frac{\partial^2 f}{\partial u_i\, \partial u_j}(0)\, du^i\, \mathbf{e}_j(P) \\ &= -\sum_{i,j=1}^{2} h_{ij}(P)\omega^i\, \mathbf{e}_j(P). \end{aligned}$$

$$d\mathbf{e}_3 = d\begin{pmatrix}\cos\theta\cos\psi\\ \sin\theta\cos\psi\\ \sin\psi\end{pmatrix} = \begin{pmatrix}-\sin\theta\cos\psi\\ \cos\theta\cos\psi\\ 0\end{pmatrix} d\theta + \begin{pmatrix}-\cos\theta\sin\psi\\ -\sin\theta\sin\psi\\ \cos\psi\end{pmatrix} d\psi$$

$$= \cos\psi\, d\theta\, \mathbf{e}_1 + \frac{d}{\psi}\mathbf{e}_2 = {\omega_3}^1\mathbf{e}_1 + {\omega_1}^2\mathbf{e}_2$$

It follows that

$$\begin{aligned} {\omega_3}^1 &= -h_{1j}\,\omega^j = \frac{\cos\psi}{a + b\cos\psi}\omega^1 \\ {\omega_3}^2 &= -h_{2j}\,\omega^j = \frac{1}{b}\omega^2. \end{aligned}$$

Thus, the second fundamental form is diagonal too and

$$h_{ij} = \begin{pmatrix} -\frac{\cos\psi}{a+b\cos\psi} & 0\\ 0 & -\frac{1}{b}\end{pmatrix}.$$

Since

$$h_{ij} = \begin{pmatrix} -\frac{\cos\psi}{a+b\cos\psi} & 0 \\ 0 & -\frac{1}{b} \end{pmatrix}.$$

it follows that

$$\begin{aligned} H(P) &= \tfrac{1}{2}\,\mathrm{tr}(h_{ij}(P)) = -\tfrac{1}{2}\left(\tfrac{\cos\psi}{a+b\cos\psi} + \tfrac{1}{b}\right) \\ K(P) &= \det(h_{ij}(P)) = \tfrac{\cos\psi}{b(a+b\cos\psi)}. \end{aligned}$$

So for the outside part of the torus $|\psi| < \frac{\pi}{2}$, the surface is on one side of the tangent plane so $K > 0$.

On the inside of the torus $\frac{\pi}{2} < |\psi| \le \pi$, the surface is a saddle on both sides of the tangent plane where $K < 0$.

On the circles on top and bottom $\psi = \pm\frac{\pi}{2}$, the ψ direction leaves the tangent plane but the θ directions stay in it, therefore there $K = 0$.

One imagines that one can roll up a piece of paper in $\mathbf{R}^3$ without changing lengths of curves in the surface. Thus the plane $\mathbb{P}^2$ and the cylinder $\mathbb{Z}^2$ are locally isometric. Let us check by computing the Riemannian metrics at corresponding points. For any $(u_1, u_2) \in \mathbf{R}^2$ the plane is parameterized by $X(u^1, u^2) = (u^1, u^2, 0)$ so $\mathbf{e}_1 = X_1 = (1, 0, 0)$, $\mathbf{e}_2 = X_2 = (0, 1, 0)$ and so $\omega^1 = du^1$, $\omega^2 = du^2$ and thus

$$ds_{\mathbb{P}}^2 = (du^1)^2 + (du^2)^2.$$

Cylinder $\mathbb{Z}^2 = \{(x, y, z) : y^2 + z^2 = r^2\}$ of radius r is parameterized $Z(v^1, v^2) = (v^1, r\cos v^2, r\sin v^2)$ so $\tilde{\mathbf{e}}_1 = Z_1 = (1, 0, 0)$, $r\tilde{\mathbf{e}}_2 = Z_2 = (0, -r\sin v^2, r\cos v^2)$ and so $\tilde{\omega}^1 = dv^1$, $\tilde{\omega}^2 = r du^2$ and thus

$$ds_{\mathbb{Z}}^2 = (dv^1)^2 + r^2 (dv^2)^2.$$

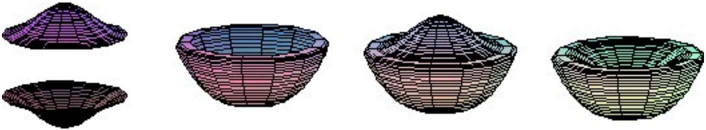
The map $f : X(u_1, u_2) \mapsto Z(u_1, u_2/r)$ is an isometry because it pulls back the same metric and lengths of curves agree: $v^2 = u^2/r$ means $f^*(dv^2) = du^2/r$ so

$$f^*(ds_{\mathbb{Z}}^2) = f^*(dv^1)^2 + r^2 f^*(dv^2)^2 = (du^1)^2 + (du^2)^2 = ds_{\mathbb{P}}^2.$$

By manipulating half of a rubber ball that has been cut through its equator, one sees that the cap can be deformed into a football shape without distorting intrinsic lengths of curves and angles of vectors. The spherical cap is deformable through isometries: it is not rigid. (Rigid means that any isometry has to be a rigid motion of $\mathbf{R}^3$: composed of rotations, translations or reflections.)

It turns out by Herglotz's Theorem, all $\mathcal{C}^3$ closed $K > 0$ surfaces (hence surfaces of convex bodies which are simply connected) are rigid.

It is not true for nonconvex surfaces. The lid may be glued on up side down or right side up to give locally isometric surfaces of revolution (they can be bent into each other), but they are not congruent.



So far, the formula for the Gauss Curvature has been given in terms of the second fundamental form and thus may depend on the extrinsic geometry of the surface. However, Gauss discovered a formula that he deemed excellent:

Theorem (Gauss's Theorema Egregium 1828)

Let $M^2 \subset \mathbf{R}^3$ be a smooth regular surface. Then the Gauss Curvature may be computed intrinsically from the metric and its first and second derivatives.

In other words, the Gauss Curvature coincides at corresponding points of isometric surfaces.

The word has the same Latin root as "egregious" or "gregarious."

The proof depends on a fact about Euclidean Three Space, called the Riemann Curvature Equation. For ANY moving frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, the connction forms satisfy

$$d\omega_A{}^C - \sum_{B=1}^{3} \omega_A{}^B \wedge \omega_B{}^C = 0. \tag{5}$$

It is a tensor: it is independent of choice of frame. It is easily seen to be zero in the standard basis of $\mathbb{E}^3$.

Applied to the $\omega_1{}^2$, we see that the Gauss Curvature may be computed from differentiation of $\omega_1{}^2$. Indeed, by (5),

$$\begin{aligned} d\omega_1{}^2 = \omega_1{}^3 \wedge \omega_3{}^2 &= -\sum_{i,j=1}^{2} h_{1i} h_{2j} \omega^i \wedge \omega^j \\ &= -(h_{11}h_{22} - h_{12}h_{21})\omega^1 \wedge \omega^2 = -K\omega^1 \wedge \omega^2. \end{aligned}$$

Now $\omega_1{}^2$ and therefore $d\omega_1{}^2$ and K can be computed intrinsically. $\square$

It turns out that the Gaussian Curvature can be measured intrinsically in another way.

The ball of radius r around a point in a surface $P \in M^2$ is the set

$$B_r(P) = \{x \in M^2 : \mathrm{dist}(x, P) < r\}.$$

If the area of the ball for small r is expanded in series, the Gaussian curvature at P appears as a coefficient in the expansion. It can be viewed as a correction to the Euclidean area growth, which is quadratic on the nose.

$$\mathrm{Area}(B_r(P)) = \pi r^2 + \frac{\pi}{12} K(P) r^4 + \cdots$$

31. Illustrate Gauss's Theorem for the Torus.

Recall the coframe of $\mathbb{T}^2$ is

$$\omega^1 = (a + b\cos\psi)d\theta, \qquad \omega^2 = bd\psi$$

We solve (4) for ${\omega_1}^2 = -{\omega_2}^1 = p\,d\theta + q\,d\psi$ such that

$$bd\psi \wedge (-p\,d\theta - q\,d\psi) = \omega^2 \wedge {\omega_2}^1 = d\omega^1 = -b\sin\psi\,d\psi \wedge d\theta,$$
$$(a + b\cos\psi)d\theta \wedge (p\,d\theta + q\,d\psi) = \omega^1 \wedge {\omega_1}^2 = d\omega^2 = 0.$$

The second says $q = 0$ and the first $p = \sin\psi$ so ${\omega_1}^2 = \sin\psi\,d\theta$. Differentiating,

$$-Kb(a + b\cos\psi)\,d\theta \wedge d\psi = -K\,\omega^1 \wedge \omega^2 = d{\omega_1}^2 = \cos\psi\,d\psi \wedge d\theta.$$

Therefore, we recover the Gauss Curvature we found before extrinsically

$$K = \frac{\cos\psi}{b(a + b\cos\psi)}.$$

32. Bending.

We imagine starting with a flexible and inextensible two dimensional surface $S \subset \mathbb{E}^3$ and ask if it is possible to bend it into another surface $M \subset \mathbb{E}^3$?

Let's call $f : S \to M$, the mapping of corresponding points a bending if it preserves lengths along the surface. So f is a local isometry: it preserves the metric, hence all intrinsic geometry, such as angles, areas, geodesic curvatures of curves and the Gauss curvature of the surface. However it may have different second fundamental forms at corresponding points.

Let $f : S \to M$ be a bending. We assume it is smooth: if $X : U \to S$ is a local parameterization for S, then $\tilde{X} = f \circ X : U \to M$ is a local parameterization of M. It is a local isometry: it preserves lengths of vectors:

$$g_{ij} = \frac{\partial X}{\partial u^i} \bullet \frac{\partial X}{\partial u^j} = \frac{\partial \tilde{X}}{\partial u^i} \bullet \frac{\partial \tilde{X}}{\partial u^i} = \tilde{g}_{ij}.$$

There are trivial bendings, the rigid motions, consisting of rotations, reflections and translations of space. As rigid motions preserve lengths and corresponding moving frames, both the metrics and the second fundamental forms are preserved.

Theorem (Invariance under Rigid Motions.)

Let $S \in \mathbb{E}^3$ be a surface and $X : U \to \mathbb{E}^3$ be a local coordinate chart for S. Let $B : \mathbb{E}^3 \to \mathbb{E}^3$ be a rigid motion, i.e., $B(x) = \mathcal{R}x + v$ where $\mathcal{R} \in \mathbf{O}(3)$ is an orthogonal matrix and $v \in \mathbb{E}^3$ a translation vector. Let $\tilde{X} = B \circ X$. Then after an appropriate choice of normal vectors, the first and second fundamental forms coincide at corresponding points

$$ds^2 = d\tilde{s}^2, \qquad h_{ij} = \tilde{h}_{ij}.$$

Conversely, if there are two connected surfaces whose parameterizations $X, \tilde{X} : U \to \mathbb{E}^3$ satisty $ds^2 = d\tilde{s}^2$ and $h_{ij} = \tilde{h}_{ij}$, then $\tilde{X} = B \circ X$ for some rigid motion B.

We consider what surfaces are bendings of pieces of the plane. it turns out they must be planes, cylinders, cones and tangent developables.

Imagine sweeping out a surface by moving a line. Such a surface is called a ruled surface. It may be given by specifying a unit speed regular space curve $\alpha(u^2)$ and a unit direction vector $V(u^2)$. Locally it is parameterized by

$$X(u^1, u^2) = \alpha(u^2) + u^1 V(u^2).$$

Examples are solutions of the the equations in $\mathbb{E}^3$ giving hyperbolic paraboloids

$$z = xy$$

or hyperboloids of one sheet

$$x^2 + y^2 - z^2 = 1.$$

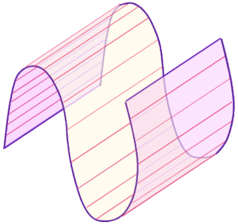


Figure: Generalized Cylinder $X(u^1, u^2) = \alpha(u^2) + u^1 V$.

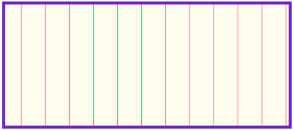


Figure: Unrolled Generalized Cylinder in Plane $Y(v^1, v^2) = (v^1, v^2, 0)$.

When direction $V = V(u^2)$ is constant, then the ruling lines are parallel. If also $\dot{\alpha}$ is not parallel to V then the surface $\mathcal{C}$ is a Generalized Cylinder.

Let $\alpha(u^2)$ be a unit speed curve in the plane perpendicular to V. Then $X_1 = V$, $X_2 = \alpha'(u^2)$ and the metric is

$$ds^2_{\mathcal{C}} = (du^1)^2 + (du^2)^2.$$

If $\mathcal{P}$ is surface in the plane, it has metric

$$ds^2_{\mathcal{P}} = (dv^1)^2 + (dv^2)^2.$$

The map $F : X(u^1, u^2) \mapsto Y(u^1, u^2)$ is an isometry: if $\gamma(t) = X(u^1(t), u^2(t))$ is a curve in $\mathcal{C}$ then $F \circ \gamma$ is a curve in $\mathcal{P}$ with the same length because

$$|\gamma'(t)|_{\mathcal{C}} = |(F \circ \gamma)'|_{\mathcal{P}}.$$

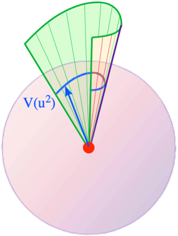


Figure: Generalized Cone $X(u^1, u^2) = P + u^1 V(v^2)$.

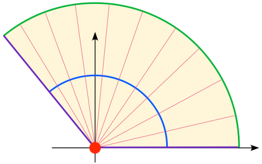


Figure: Unrolled Gen. Cone $Y(v^1, v^2) = (v^1 \cos v^2, v^1 \sin v^2, 0)$

If $V(u^2)$ is a curve on the unit sphere and and $\alpha(u^2) = P$ is constant, all the ruling lines pass through $P \in \mathbb{E}^3$, then the surface $\mathcal{V}$ is a Generalized Cone. If α' is a unit vector, then its metric is

$$ds^2_{\mathcal{V}} = (du^1)^2 + (u^1)^2(du^2)^2.$$

If $\mathcal{Q}$ is surface Y in the plane, it has metric (in polar coordinates)

$$ds^2_{\mathcal{Q}} = (dv^1)^2 + (v^1)^2(dv^2)^2.$$

The map $F : X(u^1, u^2) \mapsto Y(u^1, u^2)$ is an isometry: if $\gamma(t) = X(u^1(t), u^2(t))$ is a curve in $\mathcal{V}$ then $F \circ \gamma$ is a curve in $\mathcal{Q}$ with the same length because

$$|\gamma'(t)|_{\mathcal{V}} = |(F \circ \gamma)'|_{\mathcal{Q}}.$$

Let $\alpha(u^2) : (a, b) \to \mathbb{R}^2$ be a smooth space curve. Assume that $T'(u^2) = \alpha''(u^2) \neq 0$. Since $T \bullet T = 1$ we have $T \bullet T' = 0$ so that T' is orthogonal to T. If

$$N = \frac{T'}{|T'|}, \qquad \kappa = |T'|, \qquad B = T \times N$$

then $\{T, N, B\}$ the orthonormal frame along the curve. Similarly by differentiating $N \bullet N = 1$, $T \bullet N = 0$, $T \bullet B = 0$, $N \bullet B = 0$ and $B \bullet B = 1$, we find the Frenet Equations

$$T' = \kappa N, \quad N' = -\kappa N + \tau B, \qquad B' = -\tau N.$$

for some functions $\kappa(u^2) > 0$ and $\tau(u^2)$ called the curvature and torsion of a space curve.

If the function $\tau \equiv 0$ then B is constant and the curve is in the plane $B^\perp$. If $\tau \neq 0$ then the curve leaves its osculating plane and is called twisted.

38. Metric and Curvature of a Ruled Surface.

If $\alpha(u^2)$ is parametrized by arclength and $V(u^2)$ unit, the ruled surface is

$$X(u^1,u^2) = \alpha(u^2) + u^1 V(u^2).$$

$T = \alpha'$ and $V \bullet V' = 0$. By solving $d\omega_1^2 = -K\,\omega^1 \wedge \omega^2$,

$$X_1 = V; \qquad X_2 = T + u^1 V'$$

$$\omega^1 = du^1 + T \bullet V\, du^2, \qquad \omega^2 = |T + u^1 V' - (T \bullet V)V| du^2$$

$$ds^2 = (du^1)^2 + 2V \bullet T\, du^1 du^2 + |T + u^1 V'|^2 (du^2)^2.$$

$$d\omega^1 = 0, \qquad d\omega^2 = \frac{T \bullet V' + u^1|V'|^2}{|T + u^1 V' - (T \bullet V)V|} du^1 \wedge du^2,$$

$$\omega_1^2 = \frac{T \bullet V' + u^1|V'|^2}{|T + u^1 V' - (T \bullet V)V|} du^2,$$

$$K = \frac{(T \bullet V)^2|V'|^2 - |V'|^2 + (T \bullet V')^2}{|T + u^1 V' - (T \bullet V)V|^4} = -\frac{(T \bullet V \times V')^2}{|T + u^1 V' - (T \bullet V)V|^4}$$

If X is a generalized cylinder, V is constant, $V' = 0$ and $K = 0$.
If X is a tangent developable, $V = T$, $V' = T' = \kappa N$ so $K = 0$.

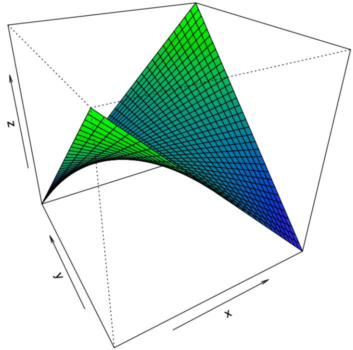


Figure: Hyperbolic Paraboloid.

The surface $z = xy$ is obviously a ruled surface.

Set $u^2 = x$ and $u^1 = y\sqrt{1+x^2}$,

$$X(u^1, u^2) = \begin{pmatrix} u^2 \\ 0 \\ 0 \end{pmatrix} + \frac{u^1}{\sqrt{1+(u^2)^2}} \begin{pmatrix} 0 \\ 1 \\ u^2 \end{pmatrix}.$$

$T = (1, 0, 0)$, $V = [1 + (u^2)^2]^{-1/2}(0, 1, u^2)$, $V' = [1 + (u^2)^2]^{-3/2}(0, -u^2, 1)$ we get $T \bullet V = 0$, $T \bullet V \times V' = [1 + (u^2)^2]^{-1}$ so

$$K = -\frac{[1 + (u^2)^2]^2}{\{[1 + (u^2)^2]^2 + (u^1)^2\}^2} = -\frac{1}{\{1 + x^2 + y^2\}^2}.$$

Or use the formula for a graph.

40. Helicoid is Example of Ruled Surface..

If $\alpha(u^2) = (0, 0, u^2)$ the vertical line and $V(u^2) = (\cos u^2, \sin u^2, 0)$ rotates about a circle in the x-y plane, the resulting surface is a helicoid. Plugging into the formula or solving from scratch,

$$X_1 = \begin{pmatrix} \cos u^2 \\ \sin u^2 \\ 0 \end{pmatrix} = \mathbf{e}_1, \quad X_2 = \begin{pmatrix} -u^1 \sin u^2 \\ u^1 \cos u^2 \\ 1 \end{pmatrix} = \sqrt{1 + (u^1)^2}\, \mathbf{e}_2.$$

so $\omega^1 = du^1$, $\omega^2 = \sqrt{1 + (u^1)^2}\, du^2$. Thus

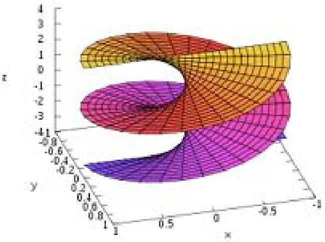


Figure: Helicoid

$$\omega_1^2 = \frac{u^1}{\sqrt{1+(u^1)^2}}\, du^2, \; d\omega_1^2 = \frac{1}{\{1+(u^1)^2\}^{3/2}}\, du^1\, du^2$$

and $K = -\frac{1}{[1+(u^1)^2]^2}$.

For different u^2 the rulings are skew lines in space, whose closest points occur on the centerline $\alpha(u^2)$. This line is called the line of stricture.

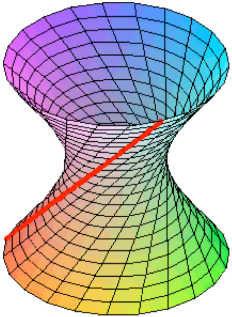


Figure: Hyperboloid

If $\alpha(u^2) = (\cos u^2, \sin u^2, 0)$ is the horizontal circle and

$V(u^2) = \frac{1}{\sqrt{2}}(-\sin u^2, \cos u^2, 1)$ the 45° vector in the $\alpha^\perp$ plane, the resulting surface is a hyperboloid of one sheet. Then

$$\begin{aligned} x^2 + y^2 &= \left(\cos u^2 - \frac{u^1}{\sqrt{2}} \sin u^2\right)^2 + \\ &\quad \left(\sin u^2 + \frac{u^1}{\sqrt{2}} \cos u^2\right)^2 \\ &= 1 + \frac{(u^1)^2}{2} \\ &= 1 + z^2. \end{aligned}$$

Then

$$T = \alpha' = \begin{pmatrix} -\sin u^2 \\ \cos u^2 \\ 0 \end{pmatrix}, V = \frac{1}{\sqrt{2}} \begin{pmatrix} -\sin u^2 \\ \cos u^2 \\ 1 \end{pmatrix}, V' = \frac{1}{\sqrt{2}} \begin{pmatrix} -\cos u^2 \\ -\sin u^2 \\ 0 \end{pmatrix}$$

so

$$T \bullet V = \frac{1}{\sqrt{2}},\ T \bullet V' = 0,\ |V'|^2 = \frac{1}{2},\ T \bullet V \times V' = -\frac{1}{2}$$

so by the curvature formula

$$K = -\frac{|T \bullet V \times V'|^2}{|T + u^1 V' - (T \bullet V)V|^4} = -\frac{1}{[1 + (u^1)^2]^2}.$$

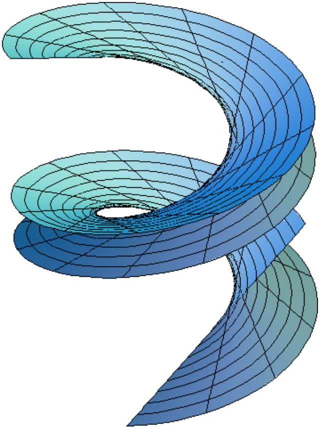


Figure: Tangent Developable

Let's consider a special ruled surface M, the tangent surface. This time, let's assume that $V(u^2) = \alpha'(u^2)$. Since $T(u^2) = \alpha'(u^2)$ is a unit vector in the tangent direction, the surface is swept out by lines tangent to a space curve. In this case, we must assume that $T'(u^2) = \alpha''(u^2) \neq 0$.

The surface is locally given by

$$X(u^1, u^2) = \alpha(u^2) + u^1 T(u^2).$$

Then the tangent vectors are

$$X_1 = T(u^2), \qquad X_2 = T(u^2)+u^1\kappa(u^2)\, N(u^2)$$

so that $\mathbf{e}_1 = N(u^2)$ and $\mathbf{e}_2 = T(u^2)$ is moving frame along M for all $u^1 > 0$. Note that the tangent vectors are independent iff $\kappa(u^2) > 0$.

44. Tangent Developable Surface.

The fact that the $\mathbf{e}_3$ is constant along the generator for all $u^1 > 0$ distinguishes the developable surfaces among the ruled surfaces. Another description is that this surface is the envelope of a family of planes in $\mathbb{E}^3$. Here the family is given for $u^2 \in (a, b)$ by the osculating planes of α:

$$\{Z \in \mathbb{E}^3 : (Z - \alpha(u^2)) \bullet B(u^2) = 0\}.$$

For $X(u^1, u^2) = \alpha(u^2) + u^1 T(u^2)$, we have

$$X_1 = T(u^2), \qquad X_2 = T(u^2) + u^1 \kappa(u^2)\, N(u^2).$$

Since $\mathbf{e}_1 = N(u^2)$ and $\mathbf{e}_2 = T(u^2)$, the dual frames are

$$\omega^1 = u^1 \kappa\, du^2, \qquad \omega^2 = du^1 + du^2;$$

and the metric is

$$ds^2 = (\omega^1)^2 + (\omega^2)^2 = (du^1)^2 + 2du^1 du^2 + (1 + (u^2)^2 \kappa^2)(du^2)^2.$$

The metric is

$$ds^2 = (\omega^1)^2 + (\omega^2)^2 = (du^1)^2 + 2du^1 du^2 + (1 + (u^2)^2\kappa^2)(du^2)^2.$$

Note that the metric DOES NOT DEPEND on τ. Thus if $\gamma(u^2)$ is the unit speed PLANE CURVE with the same curvature $\kappa(u^2)$ (and torsion zero) then

$$Y(u^1, u^2) = \gamma(u^2) + u^1\,\gamma'(u^2)$$

is a parameterization of a piece of that plane and has the SAME METRIC as $X(u^1, u^2)$ at corresponding points. This is called developing the surface into the plane. Because $Y(u^1, u^2)$ is planar, its Gauss Curvature is dead zero. Since it's isometric, the curvature of $X(u^1, u^2)$ is dead zero too. Thus a piece of the plane can be bent into a tangent developable surface.

Theorem

Let $P \in M \in \mathbb{E}^3$ be a point in a flat surface. Suppose that M is not umbillic at P. Then M is a ruled surface in some neighborhood of P.

The hypotheses tell us that $h_{ij}(P) \neq 0$, so it is nonzero in a coordinate neighborhood $U \subset M$. By rotating the orgthonormal frame at points of U we may arrange that at all points of U,

$$h_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & h_{22} \end{pmatrix} \tag{6}$$

where $h_{22} \neq 0$ in U. For any $P_0 \in U$, let $\gamma(u^1) \in U$ be a curve such that $\gamma(0) = P_0$ and

$$\gamma'(u^1) = \mathbf{e}_1(\gamma(u^1))$$

for all u^1. We show that $\gamma(s^1)$ is a straight line in $\mathbb{E}^3$, since P_0 is any, U is foliated by generating lines. It suffices to show that

$$\gamma''(0) = D_{\mathbf{e}_1}\mathbf{e}_1 = 0.$$

But

$$D_{\mathbf{e}_1}\mathbf{e}_1 = {\omega_1}^2(\mathbf{e}_1)\mathbf{e}_2 + {\omega_1}^3(\mathbf{e}_1)\mathbf{e}_3. \tag{7}$$

Now ${\omega_1}^3 = h_{11}\omega^1 + h_{12}\omega^2 = 0$ by (6) so the second term vanishes. Also $h_{11} = h_{12} = 0$ in U so

$$0 = dh_{11} = h_{111}\omega^1 + h_{112}\omega^2 + 2h_{12}{\omega_1}^2,$$
$$0 = dh_{12} = h_{121}\omega^1 + h_{122}\omega^2 + h_{22}{\omega_1}^2 + h_{12}{\omega_2}^1.$$

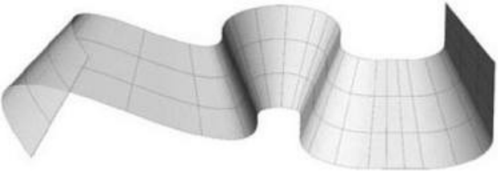
The first tells us $h_{111} = h_{112} = 0$. By the Codazzi equations, $h_{112} = h_{121} = 0$. So the second says

$$\begin{aligned} 0 = \mathbf{e}_1 h_{12} &= h_{121}\omega^1(\mathbf{e}_1) + h_{122}\omega^2(\mathbf{e}_1) + h_{22}{\omega_1}^2(\mathbf{e}_1) + h_{12}{\omega_2}^1(\mathbf{e}_1) \\ &= 0 + 0 + h_{22}{\omega_1}^2(\mathbf{e}_1) + 0. \end{aligned}$$

But $h_{22} \neq 0$ so

$${\omega_1}^2(\mathbf{e}_1) = 0.$$

and the first term of (7) vanishes also. □



Theorem

Let M be a flat ruled surface. Then M consists of pieces of planes, generalized cylinders, generalized cones and tangent developables.

Locally, M is given by a unit speed space curve $\alpha(u^2)$ and a unit tangent vector field $V(u^2)$ with

$$X(u^1, u^2) = \alpha(u^2) + u^1 V(u^2).$$

From the computation for general ruled surfaces, curvature vanishes if and only if at all points,

$$T \bullet V \times V' = 0, \tag{8}$$

in other words T, V and V' are linearly dependent.

In the first case, assume that $V' = 0$ for in a neighborhood $U \subset M$. Then V is constant and U is a generalized cylinder.

Now assume $V' \neq 0$ in a neighborhood. Since $V \bullet V' = 0$, the vectors V and V' are independent. By (8),

$$T(u^2) = f(u^2)V(u^2) + g(u^2)V'(u^2) \tag{9}$$

for some smooth functions f and g.
In case $f = g'$ in a neighborhood U,

$$T = \alpha' = g'V + gV' = (gV)'$$

Thus

$$\alpha - gV = P$$

is constant, so that

$$X(u^1, u^2) = \alpha(u^2) + u^1 V(u^2) = P + (g(u^2) + u^1)V(u^2).$$

Hence u is a generalized cone.

Now assume $V' \neq 0$ and $f \neq g'$ in a neighborhood of M. Define a new curve and coordinate

$$\tilde{\alpha}(u^2) = \alpha(u^2) - g(u^2)V(u^2), \qquad \tilde{u}^1 = \frac{u^1 + g(u^2)}{f(u^2) - g'(u^2)}.$$

Hence, by (9)

$$\tilde{\alpha}' = T - g'V - gV' = (f - g')V$$

thus

$$\begin{aligned} X(u^1, u^2) &= \alpha + u^1 V = \tilde{\alpha} + gV + u^1 V = \tilde{\alpha} + \tilde{u}^1 (f - g')V \\ &= \tilde{\alpha}(u^2) + \tilde{u}^1 \tilde{\alpha}'(u^2). \end{aligned}$$

Hence the surface is a tangent developable in U.

It may be that none of the conditions considered are satisfied. In different open pieces of M, the surface may be any of a cylinder, cone or tangent developable, which meet along segments of generators. $\square$

51. Unrolling A Sheet of Paper with Circle Cut Out.

Figure: Circular Hole

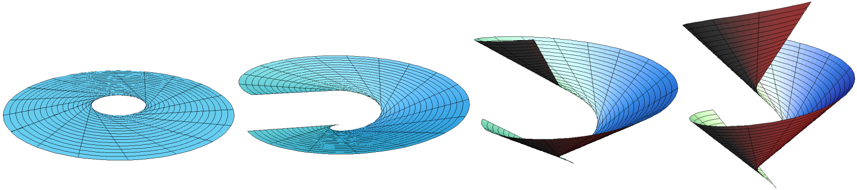
The curvature K of the hole cannot be decreased in the paper. If the paper is bent so that the circle becomes helical,

$$\alpha(t) = (a\cos t, a\sin t, bt),$$

its curvature as a space curve must satisfy

$$K \le \frac{a}{a^2+b^2} \text{ or } \left(a - \frac{1}{2K}\right)^2 + b^2 \le \frac{1}{4K^2}.$$

That is why the farthest the circle can be unrolled is so the circle is the edge of regression , *e.g.*, at pitch $b = \frac{1}{2K}$ with radius $a = \frac{1}{2K}$.



If we ask to bend the entire plane, then we must rule out surfaces whose generating lines cross. This is an example of global differential geometry. It was proved amazingly recently. "Complete" means that any intrinsic straight line in S can be continued forever without running into a boundary of S.

Theorem (P. Hartman & L. Nirenberg, 1959)

Let S be a complete, connected surface with zero Gaussian curvature. Then S is a cylinder or plane.

There are theorems about the rigidity of other surfaces. In contrast to zero curvature, the following theorem was proved much earlier.

Theorem (H. Liebmann 1899)

Let S be a compact connected, regular surface with constant Gaussian curvature K. Then S is the round sphere.

We outline Hilbert's Proof which appeared in the second ever volume of *Proceedings of the American Mathematical Society*, 1901.

The first thing that is proved is that if S is a compact surface then its Gaussian curvature must be positive

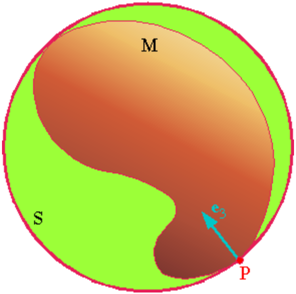


Figure: M and smallest surrounding sphere S

As $M \subset \mathbb{E}^3$ is compact, there is a smallest sphere S that contains M. Lets say the radius is $1/k$. Let P be a point in common and let $\mathbf{e}_3$ be the inward normal. As M is inside the sphere we can compare the Hessian of M with the Hessian of S in the $\mathbf{e}_3$ direction at P

$$h_{ij} \geq k\delta_{ik}.$$

There $K = \det(h_{ij}) \geq k^2 > 0.$

Next we show that the surface is umbillic: the eigenvalues k_1 and k_2 are equal to $\sqrt{K}$ so $h_{ij} = \sqrt{K}\,\delta_{ij}$. If not, then we may assume that $k_1(x) \le k_2(x) = K/k_1(x)$ are not constant, and a some point $P \in M$, $k_1(P) = \min_{x\in M} k_1(x) < \sqrt{K}$. At the same point $k_2(x)$ is a global maximum. Thus $k_1 - k_2 < 0$ at P. On the other hand, if we take a moving frame near P so that $\mathbf{e}_1$ and $\mathbf{e}_2$ are eigendirections of h_{ij} for k_1 and k_2 respectively, then

$$h_{ij} = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$$

in a neighborhood of P. By a computation of second (covariant) derivatives of h_{ij},

$$h_{1122} - h_{2211} = h_{11}h_{22}(h_{11} - h_{22}) < 0. \tag{10}$$

On the other hand, at a minimum $(h_{11})_{22} \ge 0$ and at a maximum $(h_{22})_{11} \le 0$ so that at P, the left side of (10) is nonnegative, which is a contradiction.

Finally, we show that a connected surface whose second fundamental form is $h_{ij} = k\delta_{ij}$, where $k = \sqrt{K}$ is constant, is a piece of the sphere.

If we consider the vector function

$$Y = X + \frac{1}{k}\mathbf{e}_3$$

its derivatives are

$$\begin{aligned} D_{\mathbf{e}_i} Y &= D_{\mathbf{e}_i} X - \frac{1}{k} D_{\mathbf{e}_i}\mathbf{e}_3 = \mathbf{e}_i + \frac{1}{k}\sum_{p=1}^{2} {\omega_3}^p(\mathbf{e}_i)\mathbf{e}_p \\ &= \mathbf{e}_i - \frac{1}{k}\sum_{p,q=1}^{2} h_{pq}\omega^q(\mathbf{e}_i)\mathbf{e}_p\mathbf{e}_i - \frac{1}{k}\sum_{p,q=1}^{2} k\delta_{pq}\,{\delta^q}_i\mathbf{e}_p \\ &= \mathbf{e}_i - \mathbf{e}_i = 0. \end{aligned}$$

Thus Y is a fixed point and X is a fixed distance from Y:

$$|X - Y| = \frac{1}{k}. \quad \square$$


UNBENDABLE

Thanks!