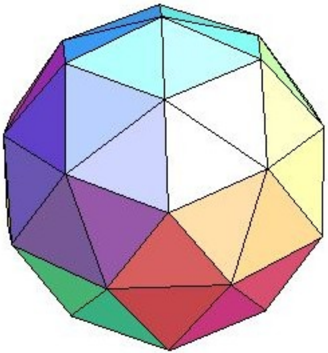

Figure 1: A Rigid Polyhedron.

Bending Polyhedra

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- Herman Gluck, Almost all closed surfaces are rigid, *Lecture Notes in Mathematics* 438, Springer, 1974.

4. Outline.

- Polyhedral Surfaces
- Rigidity
  - Rigid Motion. Congruence. Isometry.
- Examples of Flexible Polyhedra.
  - By Bricard, Connelly, Steffen.
- Infinitesimal Rigidity.
  - Examples of Infinitesimally Flexible Polyhedra.
    - Pyramid, Gluck's Octahedron
  - Algebraic Formulation
- ω-Rigidity
  - Equivalent to Infinitesimal Rigidity
- Infinitesimal Rigidity Implies Rigidity.
- Infinitesimal Rigidity Theorem.
  - Cauchy's Lemma on Spherical Graphs.
  - Alexandrov's Lemma about Convex Vertices.
- New Sharper Rigidity Theorems

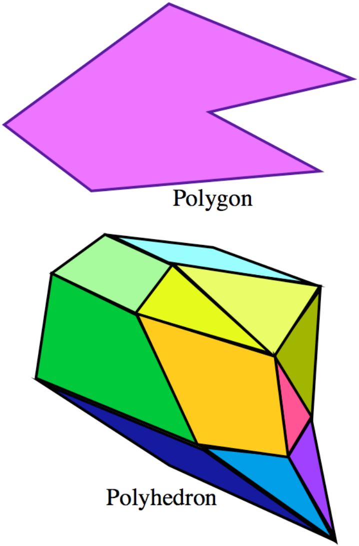


Figure 2: Polyhedron.

A polygon is a connected open plane set whose boundary consists finitely many different line segments or rays glued end to end. A polyhedron is a piecewise flat surface in three space consisting of finitely many planar polygons glued pairwise along their sides. A polyhedron is assumed to be closed: each side of every polygon is glued to the side of another polygon.

We shall abuse notation and call polyhedron together with its interior a "polyhedron."

6. Restrict Attention to Spherical Polyhedra.

Assume our polyhedron P is a triangulated sphere with V vertices. It is determined as the convex hull of V points in $\mathbf{R}^3$. Certain pairs of points determine lines which meets P in an edge and certain triples determine planes that meet P in a face. P is determined by the coordinates of all vertices strung together

$$(p_1, \ldots, p_V) \in \mathbf{R}^{3V}.$$

Since three side edges determine a triangular face, we may also view the polyhedron as being given as the union of these outer edges .

More generally, a framework is an arbitrary set of vertices in $\mathbf{R}^3$ and a list pairs of vertices connected by edges.

Rigidity Conjecture [Euler, 1766] All polyhedra are rigid.

It turns out that most polyhedra are rigid, but there are flexible examples as we shall show.

We now make precise two notions: “rigidity” and “infinitesimal rigidity.”

Two polyhedra $(p_1, \ldots, p_V)$ and $(q_1, \ldots, q_V)$ are congruent if there is a rigid motion $h : \mathbf{R}^3 \to \mathbf{R}^3$ such that $hP = Q$. Equivalently

$$|p_i - p_j| = |q_i - q_j| \qquad \text{for ALL pairs } (i,j),\ 1 \le i,j \le V,$$

where $|\cdot|$ is the Euclidean norm of $\mathbf{R}^3$.

Rigid Motions in $\mathbf{R}^3$ are given by rotations, reflections, and translations. Given a fixed rotation matrix $\mathcal{R}$, reflection $\pm$ about the origin and translation vector $\mathcal{T}$, then the rigid motion is given by

$$h(p) = \pm\mathcal{R}p + \mathcal{T}.$$

The rotations matrix is special orthogonal: it satisfies $\mathcal{R}^T\mathcal{R} = I$ and $\det(\mathcal{R}) = 1$. It moves the standard basis vectors to new orthonormal vectors that form its columns.

Lets move our polyhedron through a family of rigid motions. If $\mathcal{R}(\tau)$ and $\mathcal{T}(\tau)$ are a smoothly varying matrices and vectors with $\mathcal{R}(0) = I$ and $\mathcal{T}(0) = 0$, it turns out that the velocity of a point undergoing the corresponding family of rigid motion is given by

$$\frac{d}{d\tau}\bigg|_{\tau=0} \Big(\pm\mathcal{R}(\tau)p + \mathcal{T}(\tau) \Big) = \pm r \times p + \delta$$

where r is the instantaneous rotation (angular velocity) vector and δ is the instantaneous translational velocity vector.

$\mathcal{R}'(0)$ is skew symmetric so may be written as cross product:

$$\mathcal{R}'(0)p = r \times p.$$

In contrast, P and Q are isometric if each face of P is congruent to the corresponding face of Q. Assuming each face is triangular, this is equivalent to

$$|p_i - p_j| = |q_i - q_j| \qquad \text{for all pairs } (i,j) \in \mathcal{E},$$

where the set of of pairs of vertices corresponding to edges is given by

$$\mathcal{E} = \{(i,j) : 1 \le i,j \le V \text{ and an edge of } K \text{ connects } v_i \text{ to } v_j\}$$

Congruent $\implies$ Isometric.

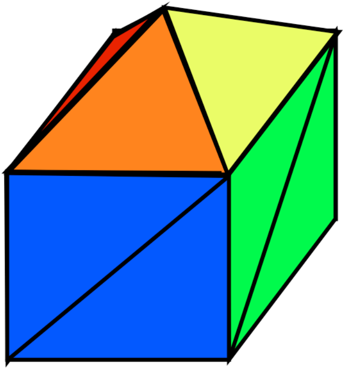


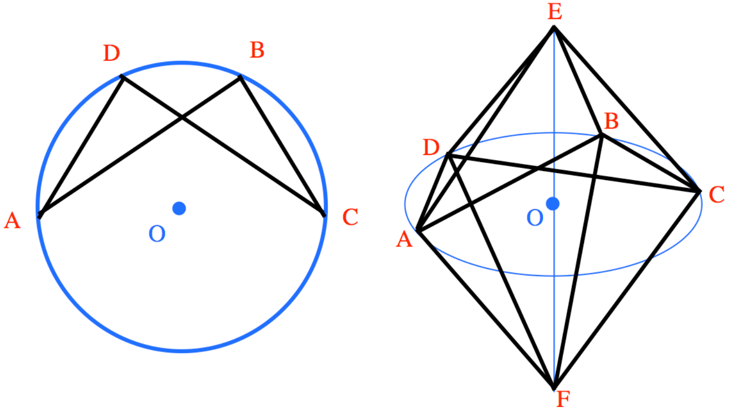
Figure 3: Houses P and Q.

P is a house with roof on top and Q is reconstructed from the same faces except the roof is installed hanging down.

P and Q are isometric but not congruent.

P is Rigid if any continuous deformation of P that keeps the distances fixed between any pairs of points in each surface triangle (*i.e., the edge lengths fixed) keeps the distances fixed between any pairs of points on the surface (i.e., all pairs, not just the edges). In other words, any continuous family of isometries must be a family of congruences.*

A deformation is called isometeric or a flex if it preserves lengths of edges. The houses are not an example of a flex because there is no continuous family of isometries deforming one house to the other.



Vertices of skew rectangle $ABCD$ pass through circle with center O. Its radius changes as rectangle flexes in the plane. Choose points E and E above and below O. The lengths AE, BE, CE, DE are equal. As $ABCD$ flexes these segments meet at point above O so four upper triangles flex. Similarly four lower triangles flex. Found by Bricard in 1897, this octahedron is not embedded since triangles ABE and CDE intersect.

Figure 4: R. Connelly

The fact that embedded polyhedra can flex was settled by R. Connelly who constucted a flexing polyhedron in 1978. Connelly's original example has a triangulation with 62 triangles.

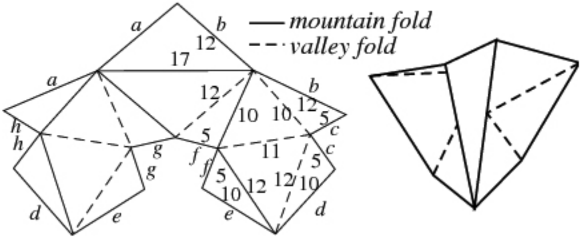


Figure 5: Pieces of an Embedded Flexible Polyhedron

This gluing diagram gives Klaus Steffen's 1978 simplification. One takes two modified copies of the Bricard octahedron with a pair triangles removed *ahdeg* and *bcdef*. The resulting "crinkled" disks flex. These disks are glued to a third disk (two 12 by 17 triangles.).

15. Second Notion: Infinitesimal Rigidity.

A stronger notion of rigidity the liniarized version, infinitesimal rigidity, is closer to what an engineer would consider to mean rigidity of a framework.

A family $(z_i, \ldots, z_V)$ of vectors in $\mathbf{R}^3$ is called an infinitesimal deformation of P if

$$(p_i - p_j) \bullet (z_i - z_j) = 0 \qquad \text{for } i < j \text{ and } (i,j) \in \mathcal{E}.$$

We always have the infinitesimal deformations that come as velocities of rigid motion. If we fix instantaneous rotation and translation vectors and let $\delta_i = r \times p_i + t$ be the velocity of vertices under rigid motion then

$$(p_i - p_j) \bullet (\delta_i - \delta_j) = (p_i - p_j) \bullet [r \times (p_i - p_j)] = 0$$

for all i, j. These infinitesimal congruences are the trivial deformations.

One imagines that the vertices deform along a differentiable path $p(\tau)$ where $\tau \in (-\epsilon, \epsilon)$ in such a way that edge lengths are constant to first order at $\tau = 0$. It follows that for $(i, j) \in \mathcal{E}$,

$$\begin{aligned} 0 &= \frac{d}{d\tau}\bigg|_{\tau=0} \big(p_i(\tau) - p_j(\tau)\big) \bullet \big(p_i(\tau) - p_j(\tau)\big) \\ &= 2\big(p_i(0) - p_j(0)\big) \bullet \big(p_i'(0) - p_j'(0)\big). \end{aligned}$$

So in this situation the velocities $p_i'(0)$ are an infinitesimal deformation.

If P moves as a rigid body then $p(t) = \mathcal{R}(t)p(0) + \mathcal{T}(t)$ then $p'(0) = r \times p(0) + \delta$ is an infinitesimal congruence. If the only infinitesimal deformations of P are infinitesimal congruences, then we say P is infinitesimally rigid.

Otherwise P admits an infinitesimal deformation that is not an infinitesimal congruence, which is called an infinitesimal flex.

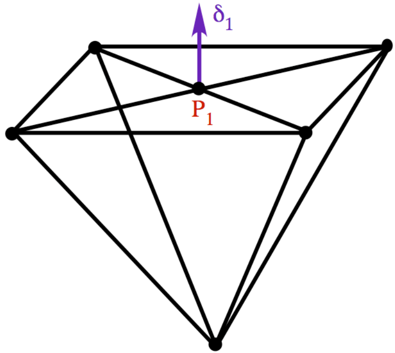


Figure 6: Infinitesimal Flex of Pyramid

Let P be a square pyramid whose base is subdivided into four triangles centered at p_1. Let δ_1 be perpendicular to the base and $\delta_j = 0$ for $j \neq 1$. Then δ is an infinitesimal flex of P.

However, P is rigid.

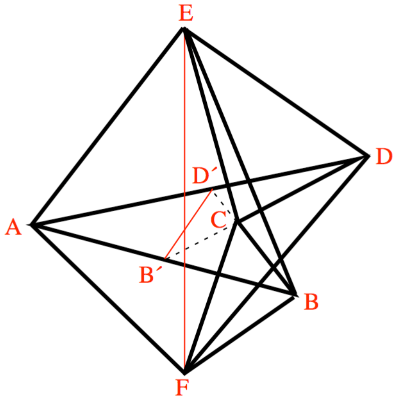


Figure 7: Gluck's Infinitesimally flexible Octohedron

The non-convex quadrilateral $ABCD$ flexes in the horizontal plane. Let B' be the intersection of the lines AB and CD. Let D' be the intersection of the lines AD and BC. Let E be a vertex above a point on the segment $B'D'$. F is a vertex below this line. $ABCDEF$ is octhedron P gotten by suspending the quadrilateral from E and F. It turns out that P is infinitesimally flexible.

It also turns out that P is rigid.

From here on in, we assume that P is a polyhedral surface homeomorphic to $\mathbb{S}^2$. If V is the number of vertices, E the number of edges and F the number of faces in the triangulation of P, then Euler's Formula gives

$$V - E + F = 2.$$

Assuming each face is triangular, this means there are two faces for each edge and three edges for each face so $3F = 2E$. Hence

$$E = 3V - 6.$$

Define the rigidity map $L : \mathbf{R}^{3V} \to \mathbf{R}^{3V-6}$ for $i < j$ and $(i, j) \in \mathcal{E}$ by

$$L(z) = \big(\cdots, (p_i - p_j) \bullet (z_i - z_j), \cdots\big)$$

$Lz = 0$ is a system of $3V - 6$ equations in $3V$ unknowns. It has the 6 dimensional infinitesimal congruences in its kernel.

Theorem 1.

If P does not degenerate to a line in three space, then P is infinitesimally rigid if and only the kernel of $L : \mathbf{R}^{3V} \to \mathbf{R}^{3V-6}$ consts of the infinitesimal congruences, which is equivalent to $\dim(\ker(L)) = 6$.

Let ω_{ij} be real numbers such that for each vertex v_i,

$$\omega_{ij} = \omega_{ji} \tag{1}$$

$$0 = \sum_{j:(i,j)\in\mathcal{E}} \omega_{ij}(p_i - p_j) \qquad \text{for all } 1 \le i \le V. \tag{2}$$

We call this set of numbers ω a ω-bending of P. If P admits a non-zero ω-bending, then we say P is ω-flexible.

Suppose that $p(t)$ is a differentiable deformation of $p(0) = P$ and that $\alpha_{ij}(t)$ be the dihedral angle between the two triangles along edge (i,j) measured from the inside. Then the angular velocity of a face with respect to its neighbor across edge $p_i p_j$ at $t = 0$ is

$$w(i,j) = \alpha'_{ij}(0)\frac{p_i(0) - p_j(0)}{|p_i(0) - p_j(0)|}.$$

If we hold $p_i(t)$ fixed and consider the succession of edges $p_i p_{j_k}$ sequentially around p_i for $k = 1, \ldots, m$, then $w(i, j_k)$ is the velocity of triangle $p_i p_{j_k} p_{j_{k+1}}$ around $p_i p_{j_{k-1}} p_k$. It follows that

$$\textstyle\sum_{k=1}^{\ell} w(i, j_k)$$

is the velocity of triangle $p_i p_{j_\ell} p_{j_{\ell+1}}$ around $p_i p_{j_m} p_{j_1}$. Because after m edges we return back to the triangle $p_i p_m p_1$ which does not turn relative to itself, for each i,

$$0 = \sum_{k=1}^{m} w(i, j_k) = \sum_{j:(i,j)\in\mathcal{E}} \alpha'_{ij}(0) \frac{p_i(0) - p_j(0)}{|p_i(0) - p_j(0)|}.$$

Setting $\omega_{ij} = \dfrac{\alpha'_{ij}(0)}{|p_i(0) - p_j(0)|}$ we see that (1) and (2) holds for any infinitesimal deformation.

Under rigid motion, no face moves relative to its neighbor so $\omega_{ij} = 0$.

Since $\omega_{ij} = \omega_{ji}$, we consider it as a function on the set of edges. Denote the map $M : \mathbf{R}^{3V-6} \to \mathbf{R}^{3V}$ by

$$M(\omega) = \left(\cdots , \sum_{j:(i,j)\in\mathcal{E}} \omega_{ij}(p_i - p_j), \cdots \right).$$

Lemma 2.

$L : \mathbf{R}^{3V} \to \mathbf{R}^{3V-6}$ and $M : \mathbf{R}^{3V-6} \to \mathbf{R}^{3V}$ are dual:

$$z \bullet M(\omega) = L(z) \bullet \omega$$

for all $z \in \mathbf{R}^{3V}$ and all $\omega \in \mathbf{R}^{3V-6}$.

Proof of Lemma.

$$
\begin{aligned}
L(z) \bullet \omega &= \sum_{i<j} (p_i - p_j) \bullet (z_i - z_j)\, \omega_{ij} \\
&= \sum_{i<j} (p_i - p_j) \bullet z_i\, \omega_{ij} - \sum_{i<j} (p_i - p_j) \bullet z_j\, \omega_{ij}
\end{aligned}
$$

Interchanging i and j in the second sum

$$
\begin{aligned}
&= \sum_{i,j} (p_i - p_j) \bullet z_i\, \omega_{ij} \\
&= \sum_{i} z_i \bullet \left\{ \sum_{j} \omega_{ij} (p_i - p_j) \right\} \\
&= z \bullet M(\omega).
\end{aligned}
$$

□

Theorem 3.

P is infinitesimally rigid if and only if P is ω-rigid.

Proof of Theorem.

We assume that P does not degenerate to a subset of a line. Then

$$\begin{aligned} P \text{ is infinitesimally rigid.} &\iff \dim(\ker(L)) = 6 \\ &\iff L \text{ is onto.} \\ &\iff \ker(M) = \{0\} \\ &\iff P \text{ is } \omega\text{-rigid.} \end{aligned}$$

$\square$

Theorem 4.

If P is infinitesimally rigid then it is rigid.

Proof.

We assume that P is infinitesimally rigid so it does not degenerate to a subset of a line. Consider smooth $f : \mathbf{R}^{3V} \to \mathbf{R}^{3V-6}$ for $i < j$, $(i,j) \in \mathcal{E}$

$$f(p) = (\cdots, (p_i - p_j) \bullet (p_i - p_j), \cdots)$$

Its differential

$$df_p(h_1, \ldots, h_V) = (\cdots, 2(p_i - p_j) \bullet (h_i - h_j), \cdots) = 2L(h)$$

has 6-dimensinal kernel so df_p is onto. Thus p is a regular value and by the Implicit Function Theorem, near p the solution $f(q) = f(p)$ is a six dimensional surface, and so must coincide with six dimensions of conjugates of p. In other words, a deformation of p near p through isometries is a rigid motion applied to p. □

A polyhedron P is strictly convex if at each vertex there is a supporting plane that touches P at that vertex but at no other point.

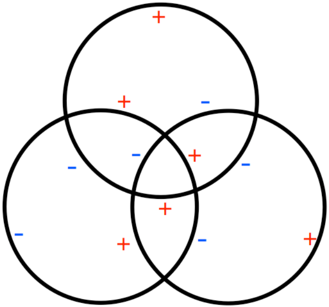
Theorem 5. (Cauchy, Dehn, Weyl, Alexandrov)

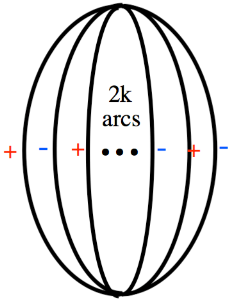
Assume that P strictly convex polyhedron with triangular faces that does not degenerate to a subset of a line. Then P is infinitesimally rigid.

Cauchy proved in 1813 that a convex polyhedron whose whose faces are rigid must be rigid. He made an error that was corrected by Steinitz in 1916. Max Dehn gave the first proof of infinitesimal rigidity of convex polyhedra in 1916. Weyl gave another in 1917. The current proof is a simplification of Alexandrov, explained in his magnificent 1949 opus, *Convex Polyhedra.* We follow Gluck, 1974.

Lemma 6. (Cauchy)

Let L be a finite graph on the two-sphere $\mathbb{S}^2$ with no circular edges and no region of $\mathbb{S}^2 - L$ bounded by just two edges of L. Mark the edges of L randomly with $+$ or $-$, and let N_ν be the number of sign changes as one circles around the vertex ν. Let $N = \sum_\nu N_\nu$ be the total number of sign changes. Let V be the number of vertices of L. Then $N \leq 4V - 8$.





N and V for Octohedron and Longitudes

Proof.

Let G be a region of $\mathbb{S}^2 - L$ and N_G the number of sign changes around G. Then also

$$N = \sum_G N_G.$$

Let F_n denote the number of regions with n boundary edges, where an edge is counted twice if G lies on both sides. Note N_G is an even number $\leq n$. By assumption $F_1 = F_2 = 0$. Hence

$$N = \sum_G N_G \leq 2F_3 + 4F_4 + 4F_5 + 6F_6 + 6F_7 + \cdots$$

$\square$

Let V, E, F and C denote the number of vertices, edges, regions and components of L. Then

$$V - E + F = 1 + C \geq 2$$
$$2E = \textstyle\sum_n nF_n$$
$$F = \textstyle\sum_n F_n$$

Hence

$$\begin{aligned} 4V - 8 &\geq 4E - 4F \\ &= \textstyle\sum_n (2n-4)F_n \\ &= 2F_3 + 4F_4 + 6F_5 + 8F_6 + 10F_7 + \cdots \\ &\geq 2F_3 + 4F_4 + 4F_5 + 6F_6 + 6F_7 + \cdots \\ &\geq N. \end{aligned}$$

$\square$

Lemma 7. (Alexandrov)

Let p_0 be a strictly convex vertex of P and $p_1, \ldots, p_m$ be the vertices taken in sequence around p_0 that are joined to p_0 by an edge. Let $\omega_1, \ldots, \omega_m$ be numbers such that

$$0 = \sum_{j=1}^{m} \omega_j (p_0 - p_j). \tag{3}$$

Then either $\omega_j = 0$ for all j or the sign of ω_j changes at least four times going around p_0, ignoring any zeros.

Proof.

Let l_0 be the number of sign changes around p_0. It is even and ≥ 0. We show first that if $l_0 = 0$ then all $\omega_j = 0$ and second that $l_0 \neq 2$.

If $l_0 = 0$ then all vectors $p_j - p_0$ lie in the same open half space at p_0 by the strict convexity of P at p_0. But then (3) with all ω_j of the same sign implies all $\omega_j = 0$.

If $l_0 = 2$, then the vertices can be shifted so that in circling p_0 we have $\omega_1, \ldots, \omega_k \geq 0$ and $\omega_{k+1}, \ldots, \omega_m \leq 0$. By strict convexity, there is a plane through p_0 separating $p_1, \ldots, p_k$ from $p_{k+1}, \ldots, p_m$. Therefore the vectors $\omega_j(p_j - p_0)$ which are nonzero all lie in the same corresponding half space. But then (3) implies all $\omega_j = 0$, so $l_0 = 0$ contrary to the assumption. $\square$

Proof.

We show that P is ω-rigid. Let ω be a bending of P. Mark each edge $(i, j) \in \mathcal{E}$ with $+$ if $\omega_{ij} > 0$ and with $-$ if $\omega_{ij} < 0$. Let L be the subgraph consisting of the marked edges where ω_{ij} is nonzero.

Assuming $L \neq \emptyset$, by Alexandrov's Lemma, there are at least four sign changes around every vertex of L. Thus the total number of sign changes around L is at least $4V(L)$. By Cauchy's Lemma, the total number of sign changes around L satisfies $N \leq 4V(L) - 8$ which is a contradiction.

It follows that $L = \emptyset$ and so all $\omega_{ij} = 0$. Thus P is infinitesimally rigid. $\square$

Corollary 8. (Cauchy)

Assume that P strictly convex polyhedron with triangular faces that does not degenerate to a subset of a line. Then P is rigid.

Proof.

P is infinitesimally rigid by the Infinitesimal Rigidity Theorem. But infinitesimal rigidity implies rigidity by Theorem 4.. $\square$

The rigidity theorem required that P be convex at each vertex. But the triangulation may include vertices on the natural edges of P or interior to a natural face. A natural face of a convex two dimensional surface is the two dimensional intersection of a supporting plane with the surface. A natural edge is a one dimensional intersection. Alexandrov proved rigidity for triangulations whose vertices may be on the natural edge. This was generalized to allow vertices on the natural faces as well.

Theorem 9. (Connelly, 1978)

A triangulated convex polyhedral surface is second order infinitesimally rigid, hence rigid.

We have seen that such surfaces (e.g. the square pyramid, Fig. 5.) may not be (first order) infinitesimally rigid.

Thanks!