Compatibility Conditions for Discrete Planar Structures

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- Beamer slides available at

  `www.math.utah.edu/~treiberg/ApplMath_9_30_19.pdf`

- Taken from the article

  Andrejs Treibergs, Andrej Cherkaev and Predrag Krtolica, Compatibility conditions for discrete planar structures, International Journal of Solids and Structures, (Article in press. Electronic version available July 19, 2019.)

3. Outline.

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- Four models
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    - Vanishing curvature compatibility condition
  - (ND) Prescribed Length
    - Geometric interpretation
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    - (ND) approximates (NC) in sense of Alexandrov
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    - Genericity of BTP Trusses
    - Hexagons are basis of compatibility conditions
- Asymptotic compatibility density
- Boundary integral of compatibility condition for all four problems.

We study of compatibility conditions on discrete structures. A structure or truss in $\mathbf{R}^d$ with $d = 2$ or $d = 3$ consists of a finite number of vertices (nodes) connected by straight edges ("bars," or "links") which form a connected graph. We often consider rigid substructures of the triangular grid in $\mathbf{R}^2$, *e.g.*,

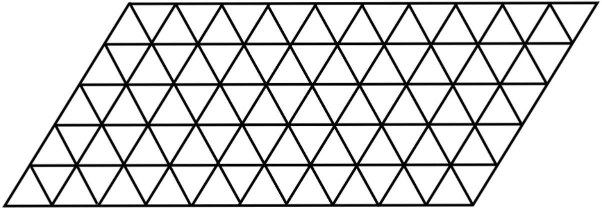


Figure: Truss.

If there is a surplus of edges in the truss, then it has fault tolerance or resilience to damage. Edges may be removed (damaged) without the truss losing rigidity. We try to quantify the fault tolerance.

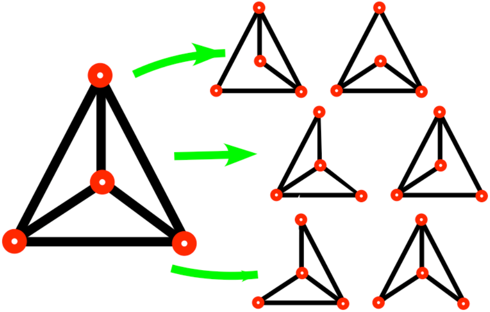


Figure: Over-rigid truss. Removing any edge leaves a rigid truss.

Suppose we specify lengths of edges and try to solve for positions of the vertices. If the truss has more edges than necessary to determine these positions, this over-specification means that to solve, data must satisfy compatibility conditions in length data. We explore how compatibility conditions as a measure of excess rigidity.

We consider four problems, two continuum models and two discretizations approximating the continuum models.

(NC) The Prescribed Nonlinear Strain problem,
(ND) its discrete approximation the Prescribed Length problem,
(LC) the linearization of (NC), the Prescribed Linearized Strain problem
(LD) the linearization of (ND), the Prescribed Elongations problem.

The continuum problems are overdetermined PDE's. The discretized problems are overdetermined equations. Each problem requires Compatibility Conditions (also called integrability conditions) on their data to be solvable. The compatibility conditions for (NC), (ND) and (LC) are fairly well understood. The compatibility conditions for (LD) are less well understood and are investigated in this work.

(NC) Prescribed Nonlinear Strain

Let $\mathcal{B} \subset \mathbb{E}^2$ be a Euclidean material disk domain with with piecewise smooth boundary and coordinates (X^1, X^2), and $\mathcal{S} \subset \mathbb{E}^2$ the target domain with coordinates (x^1, x^2). A configuration is an in-plane displacement

$$\phi : \mathcal{B} \to \mathcal{S}.$$

Its material (Lagrangian) strain tensor measuring the distortion of ϕ is

$$E[\phi] = \tfrac{1}{2}\left(F_A \bullet F_B - \delta_{AB}\right) = \tfrac{1}{2}\left(\zeta - I\right), \qquad \text{where } {F^a}_A = \frac{\partial \phi^a}{\partial X^A}$$

where ζ is the 2×2 Green Deformation tensor (Right Cauchy-Green Deformation Tensor) and I is the identity matrix.

Note that if ϕ is a rigid motion then $E = 0$.

Associated to a configuration is the energy of deformation, whose positive definite energy density W depends on the nonlinear strain

$$\text{Energy} = \int_{\mathcal{B}} W(E[\phi](x))\, dx.$$

Study of variational problems for energy minimizing configurations under prescribed boundary conditions is a major theme of nonlinear elasticity.

- Minimizing energy over all configurations ϕ with appropriate boundary conditions results in elliptic systems.
- Equivalently, we may minimize the energy over all strains E satisfying the compatibility conditions.

But our focus in this study are the prescribed strain equations and their discretizations.

10. (NC) Prescribed Green Deformation (Nonlinear Strain).

(NC) may be interpreted as a geometric problem. The equation for configurations ϕ with prescribed Green tensor is just the equation for a mapping to Euclidean space $\phi : \mathcal{B} \to \mathbb{E}^2$ with prescribed pull-back metric

$$\phi^*(ds^2_{\mathbb{E}^2}) = \zeta.$$

In coordinates, this is the overdetermined system

$$\text{(NC)} \qquad F_A \bullet F_B = \zeta_{AB}. \quad \Longleftrightarrow \quad \begin{cases} \dfrac{\partial \phi}{\partial X^1} \bullet \dfrac{\partial \phi}{\partial X^1} = \zeta_{11} \\ \dfrac{\partial \phi}{\partial X^1} \bullet \dfrac{\partial \phi}{\partial X^2} = \zeta_{12} \\ \dfrac{\partial \phi}{\partial X^2} \bullet \dfrac{\partial \phi}{\partial X^2} = \zeta_{22} \end{cases}$$

The compatibility condition for (NC) to be soluble is that curvature of Euclidean metric being pulled back by the mapping equals the curvature of the prescribed metric, namely, the prescribed metric ζ has vanishing Riemannian curvature.

The compatibility condition is the vanishing of Riemannian Curvature. In two dimensions, this is the same as the vanishing the Gauss curvature. Putting $D^2 = \zeta_{11}\zeta_{22} - \zeta_{12}^2$, the Gauss curvature of ζ is

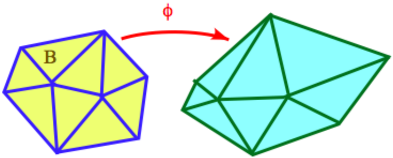
$$\begin{aligned} K = &\frac{1}{2D^2}\left(-\zeta_{22,11} + 2\zeta_{12,12} - \zeta_{11,22}\right) \\ &+ \frac{\zeta_{22}}{4D^4}\left(\zeta_{11,1}\zeta_{22,1} - 2\zeta_{11,1}\zeta_{12,2} + \zeta_{11,2}^2\right) \\ &+ \frac{\zeta_{12}}{4D^4}\left(-2\zeta_{12,1}\zeta_{22,1} - 2\zeta_{11,2}\zeta_{12,2} + 4\zeta_{12,1}\zeta_{12,2} - \zeta_{11,2}\zeta_{22,1} + \zeta_{11,1}\zeta_{22,2}\right) \\ &+ \frac{\zeta_{11}}{4D^4}\left(\zeta_{11,2}\zeta_{22,2} - 2\zeta_{12,1}\zeta_{22,2} + \zeta_{22,1}^2\right) \end{aligned}$$

$K = 0$ is the integrability condition for the local solvability of the differential system

$$\text{(NC)} \qquad F_A \bullet F_B = \zeta.$$

(ND) Prescribed Lengths

Discretizing the domain by a piecewise linear triangulation $\mathcal{T}$ of $\mathcal{B}$, let V_i be its vertices, E_{ij} its edges and T_{ijk} its triangular faces. Its 1-skeleton is a truss approximating the material $\mathcal{B}$. The lengths of the edges are computed from the metric on $\mathcal{B}$, *e.g.*, let L_{ij} be the distance from V_i to V_j in the ζ metric. The L_{ij} satisfy the triangle inequality on triangles.



We seek an immersion of $\mathcal{B}$ with vertices $X_i = \phi(V_i)$ which realizes the prescribed lengths of edges

$$\text{(ND)} \qquad |X_i - X_j| = L_{ij} \qquad \text{for all edges } ij$$

It is not immediately clear why this discretization approximates (NC). It is not the result of a finite difference scheme nor discrete differential forms.

(ND) $$|X_i - X_j| = L_{ij} \qquad \text{for all edges } ij$$

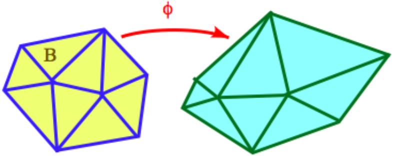
For a solution $X_i \in \mathbb{R}^2$ to exist, then the total of angles of triangles at the vertex has to be 2π. Suppose V_0 is an interior vertex and $V_1, \ldots, V_n$ are adjacent vertices going around V_0. The angles of adjacent edges may be computed using the cosine law

$$\alpha_i = \cos^{-1}\left(\frac{L_{i,i+1}^2 - L_{0,i}^2 - L_{0,i+1}^2}{2L_{0,i}L_{0,i+1}}\right)$$

The curvature at the vertex is defined to be the angle excess

$$K(V_0) = 2\pi - \textstyle\sum_{i=1}^n \alpha_i.$$

Compatibility for (ND) is that $K(V_i) = 0$ for all interior vertices.



The prescribed data for (NC) is a Riemannian metric ζ on $\mathcal{B}$. To campare it to (ND), we associate a metric $\zeta_{\mathcal{T}}$ to $(\mathcal{T}, L_{ij})$, the data for the discrete problem (ND). By filling in triangles with Euclidean triangles with the same side lengths, we let $\zeta_{\mathcal{T}}$ be the trianglewise Euclidean metric defined on all edges and triangles of $\mathcal{B}$. By extending the map on vertices to triangles, (ND) becomes the geometric problem to find an isometric immersion

$$\phi : (\mathcal{B}, \zeta_{\mathcal{T}}) \to (\mathbb{E}^2, ds^2_{\mathbb{E}^2})$$

that pulls back the Euclidean metric to the trianglewise metric

$$\phi^*(ds^2_{\mathbb{E}^2}) = \zeta_{\mathcal{T}}.$$

The curvature of the trianglewise Euclidean (polyhedral) metric is concentrated at the vertices of the tringulation. The curvature may be viewed as an atomic measure that contributes $K(V_i)$ at each vertex.

For $\mathcal{C}^2$ metrics, the curvature measure of a subset $G \subset \mathcal{B}$ is the integral of Gauss curvature

$$\omega_\zeta(G) = \int_G K\, dA \qquad \text{for measurable } G \subset \mathcal{B}.$$

For polyhedral surfaces, the right side is sum of atomic curvatures concentrated at the vertices in G.

Both metrics ζ of (NC) and $\zeta_\mathcal{T}$ of (ND) give the lengths of curves in $\mathcal{B}$, and thus induce distance functions $\rho_\zeta(x, y)$ and $\rho[\mathcal{T}, L_{ij}](x, y)$ which are the minimal length of curves between points x and y on the surface corresponding to each metric.

Let $\mathcal{T}_n$ be a sequence of triangulation of $\mathcal{B}$ which get finer and finer such that the diameters of triangles tend uniformly to zero. Let L_{ij} be the ρ_ζ distance between vertices. Let $\rho_n = \rho[\mathcal{T}_n, L^n_{ij}]$ be the corresponding PL distance functions.

Theorem (A. D. Alexandrov 1962)

Let $\mathcal{T}_n$ be a sequence of PL triangulations of $\mathcal{B}$ such that the ρ-diameter of triangles tend uniformly to zero as $n \to \infty$. Let ρ_n be the corresponding polyhedral distance. Then $\rho_n \to \rho_\zeta$ uniformly on $\mathcal{B} \times \mathcal{B}$. Moreover, the integral curvature measures $\omega_{\mathcal{T}_n}$ converge weakly to ω_ζ.

Thus, in this weak sense of Alexandrov, this sequence of discrete problems (ND) approximates the continuum problem (NC). Moreover, the compatibility conditions of (ND) approximate those of (NC).

In fact this theorem holds for the much more general Alexandrov Spaces of bounded curvature, which are, roughly speaking, some completion of $\mathcal{C}^2$ Riemannian surfaces and polyhedral surfaces.

By a theorem of Alexandrov, if $K(V_i) = 0$ at all interior vertices, then there exists an isometric immersion $\phi_{\mathcal{T}} : (\mathcal{B}, \zeta_{\mathcal{T}}) \to \mathbb{E}^2$ so that $\zeta_{\mathcal{T}} = \phi^*_{\mathcal{T}}(ds^2_{\mathbb{E}^2})$. (One pastes together triangles in turn and checks that there is a full Euclidean neighborhood surrounding every vertex.)

If a flat prescribed metric ζ is approximated by a polyhedral metric $\zeta_{\mathcal{T}}$, then the curvature at vertices of the PL metric $\zeta_{\mathcal{T}}$ vanishes.

Theorem (Solving (NC) by approximation by solutions of (ND))

Let $\mathcal{B}$ be a bounded open topological disk in $\mathbb{E}^2$ with polygonal boundary and ζ be a prescribed $\mathcal{C}^2$ Riemannian metric defined in a neighborhood of $\overline{\mathcal{B}}$ with induced distance ρ. There is a sequence of PL triangulations, $\mathcal{T}_n$ such that the largest ζ-diameter of the triangles of $\mathcal{T}_n$ tends to zero. Let ϕ_n be an isometric immersion of $(\mathcal{B}, \zeta_n)$. Then for each $\mathcal{T}_n$ there is a rigid motions m_n such that $m_n \circ \phi_n \to \phi : \mathcal{B} \to \mathbb{E}^2$ converges uniformly to a map such that $\rho(x, y) = |\phi(x) - \phi(y)|$ for all x, y. Moreover, $\phi \in \mathcal{C}^1$ and satisfies (NC).

Let $X_i \in \mathbf{R}^d$, $i = 1, \ldots, n$ denote position of the vertex. Let $\{i, j\} \in E$ be pairs of distinct indices connected by a straight edge. $e = \#E$. Suppose that the length L_{ij} is prescribed. Then for each edge $\{i, j\} \in E$, we get an equation, yielding a system of e equations in dn unknowns

$$\text{(ND)} \qquad |X_i - X_j|^2 = (X_i - X_j) \bullet (X_i - X_j) = L_{ij}^2$$

Because rigid motions preserve lengths, a rigid motion of a solution is also a solution.

If there is only one solution of (ND) up to rigid motion, we say that the truss is rigid. A smooth one-parameter family of solutions is called a flex.

There may be several noncongruent configurations that solve (ND), however they may not allow nontrivial flexes.

(LC) Prescribed Linearized Strain

21. Linearized Strain (LC) and its Compatibility

Linearizing (NC) around $\phi = \mathrm{Id}$ yields the equation of prescribed linearized strain. If we consider a variation $\phi(t)$ for $t \in (-\epsilon, \epsilon)$ with $\phi(0) = \mathrm{Id}$ then an infinitesimal deformation $u : \mathcal{B} \to \mathbb{E}^2$ given by

$$u = \left.\frac{\partial \phi}{\partial t}\right|_{t=0}$$

satisfies the equation of prescribed linear strain

$$\text{(LC)} \qquad \frac{1}{2}\left(\frac{\partial u^i}{\partial X_j} + \frac{\partial u^j}{\partial X_i}\right) = \epsilon_{ij}$$

where $\epsilon_{ij} = \epsilon_{ji}$ is the strain field. Were u to exist, since $u : \mathbf{R}^2 \to \mathbf{R}^2$, the strain field satisfies the *linearized continuum compatibility condition* in $\mathcal{B}$,

$$\mathrm{Ink}(\epsilon) = \epsilon_{11,22} - 2\epsilon_{12,12} + \epsilon_{22,11} = 0$$

where $\epsilon_{ij,pq} = \frac{\partial^2 \epsilon_{ij}}{\partial x_p\, \partial x_q}$. Note that $\mathrm{Ink}(\epsilon)$ agrees with the "linear part" of Gauss curvature.

Mechanically, compatibility conditions follow from the requirement that deformations of neighboring infinitesimal rectangles don't overlap. Thus satisfying the compatibility condition is a property of a material point.

(LD) Prescribed Elongations

Now let vertex positions $X_i \in \mathbf{R}^d$ and lengths L_{ij} depend on time t. To deduce the linearized equations, let the structure be deformed from its $t = 0$ position. Differentiating with respect to time, at $t = 0$,

$$\text{(LD)} \qquad (X_i - X_j) \bullet (u_i - u_j) = \lambda_{ij}$$

for all $\{i, j\} \in E$. Here the unknown displacements and prescribed elongations are

$$u_i = \dot{X}_i(0), \qquad \lambda_{ij} = L_{ij}(0)\dot{L}_{ij}(0)$$

We denote the system (LD) with e equations, dn unknowns as

$$Au = \Lambda$$

$\ker A$ denotes the velocities of vertices which preserve the lengths of bars up to first order. $\ker A$ always contain the velocity fields of rigid motions which are r_d dimensional. For $d = 2$ this corresponds to velocities of translations and rotations, so $r_2 = 3$. In $d = 3$, translations and rotations are each 3 dimensional so $r_3 = 6$.

If $\ker A$ only contains velocity fields of rigid motions, then the truss is said to be infintesimally rigid. If $\ker A$ admits other vector fields, then we say the truss is infinitesimally flexible. The system

$$Au = \Lambda$$

has e equations and dn unknowns. To be solvable, the right side must satisfy $\mathcal{C} = e - \operatorname{rank} A$ independent linear compatibility equations.

James Clerk Maxwell observed that if there are more unknowns than equations, then the truss could not possibly be rigid. There are at most $dn - r_d$ pivot variables and e equations. Thus there are at least

$$\mathcal{M} = e - dn + r_d$$

compatibility conditions. We call $\mathcal{M}$ the Maxwell Number. If $\mathcal{M} < 0$ then $\dim \ker A > r_d$ and the structure is infinitesimally flexible. For degenerate systems, it may happen that the structure is infinitesimally flexible but have $\mathcal{M} \geq 0$.

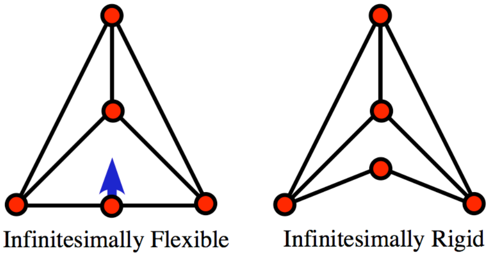


Figure: Both trusses have $e = 7$, $n = 5$ and $\mathcal{M} = 7 - 10 + 3 = 0$.

If the truss is infinitesimally rigid, then $e \geq dn - r_d = \operatorname{rank} A$. To solve

$$Au = \Lambda$$

for u, a general Λ will have to satisfy

$$\mathcal{C} = e - \operatorname{rank} A \geq e - dn + r_d = \mathcal{M}.$$

compatibility equations.

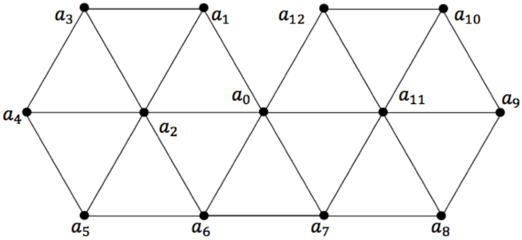
M. F. Thorpe and his collaborators have studied the underdetermined case of $Au = \Lambda$. They studied the onset of flexibility in random structures as network models for solidification of glass.

A. Cherkaev & L. Zhornitskaya and A. Cherkaev, V. Vinogradov & S. Leelavanichkul studied trusses made up of "waiting links" for damage wave propagation and impact protection.

A. Cherkaev, A. Kouznetsov and A. Panchenko looked at still states (no stress) in networks that allow two lengths for each edge in (ND). They also looked at traveling waves in bistable lattices (Nonincreasing Hook's Law springs).

Theorem (Counting Formula for Triangulated Trusses)

Let B be a PL domain embedded in the plane with triangular faces and $g+1$ disjoint simple boundary curves. Then the truss consisting of the one-skeleton $B^{(1)}$ has Maxwell Number $\mathcal{M}(B^{(1)}) = 3g + v_i$, where v_i is the number of interior vertices of X.



a_2 and a_{11} are the only interior nodes and $g=0$ so $\mathcal{M}=2$. Note that for this truss, one can remove as many as two edges, e.g. a_0a_1 and a_0a_{12}, and keep rigidity. But removing, e.g., the single edge a_6a_7 makes the figure flexible.

Proof. We shall suppose that the truss is a triangulated domain, embedded in the plane and bounded by $g+1$ pairwise disjoint simple closed curves. Let f be the number of triangular faces. The Euler Characteristic for a triangulated domain in the plane is given by the formula

$$\chi = 1 - g = f - e + v.$$

If v_b and v_i denote the number of interior and boundary nodes, and e_b and e_i the number of boundary and interior edges, we have for disjoint simple boundary curves

$$e = e_b + e_i, \qquad v = v_b + v_i, \qquad e_b = v_b, \qquad 3f = e_b + 2e_i \tag{1}$$

Substituting Euler's formula it follows that

$$3\chi = e_b - e_i + 3v_i.$$

Hence the Maxwell Dimension

$$\mathcal{M} = e - 2v + 3 = 3 + e_i - 2v_i - v_b = 3 - 3\chi + v_i = 3g + v_i \geq 0. \qquad \square$$

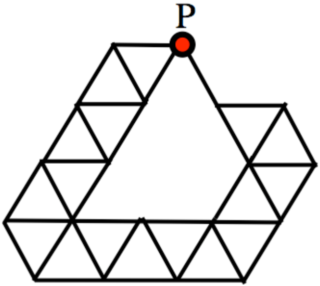


Figure: P is over-determined from two sides giving a compatibility equation.

Compatibility conditions occur in a sub-truss because there are more than two bars attached to a vertex whose elongations have to be consistent.

The number of compatibility conditions $\mathcal{C}$ corresponds to the number of dependent rows in A. Equations correspond to edges of the truss.

Theorem

The number of compatibility conditions $\mathcal{C}$ is the maximal number of edges that can be removed from the truss without losing infinitesimal rigidity.

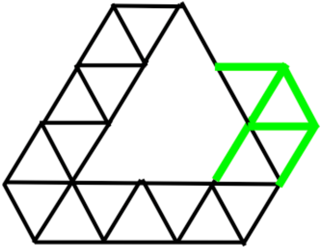


Figure: Removing one green edge will destroy infinitesimal rigidity.

However, not every subset of $\mathcal{C}$ edges can be removed. The truss in the figure has $\mathcal{C} = 1$ but the removal of any one of the green edges results in immediate loss of infinitesimal rigidity.

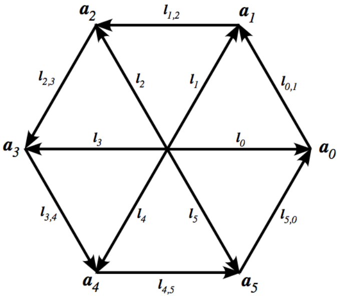


Figure: Smallest triangular sub-truss supporting compatibility equation in triangular grid.

We can compute the compatibility condition for the hexagon in two ways. Formulate the equations

$$Au = \Lambda.$$

Gaussian Elimination yields a compatibility (solvability) equation on Λ.

The second method uses plane geometry. If $\ell_i = |a_i|$ and $\ell_{i,i+1} = |a_{i+1} - a_i|$, where $i = 0, \ldots, 5$ taken mod 6, then by the cosine law the sum of the central angles must be

$$2\pi = \sum_{i=0}^{5} \cos^{-1}\left(\frac{\ell_{i+1}^2 + \ell_i^2 - \ell_{i,i+1}^2}{2\ell_{i+1}\ell_i}\right)$$

Differentiating

$$0 = -\sum_{i=0}^{5} \frac{\left\{\begin{array}{c} 2\ell_{i+1}\ell_i(2\ell_{i+1}\dot{\ell}_{i+1} + 2\ell_i\dot{\ell}_i - 2\ell_{i,i+1}\dot{\ell}_{i,i+1}) \\ -(\ell_{i+1}^2 + \ell_i^2 - \ell_{i,i+1}^2)(2\ell_{i+1}\dot{\ell}_i + 2\ell_i\dot{\ell}_{i+1}) \end{array}\right\}}{4\ell_{i+1}^2\ell_i^2\sqrt{1 - \frac{\ell_{i+1}^2 + \ell_i^2 - \ell_{i,i+1}^2}{2\ell_{i+1}\ell_i}}}$$

For the regular unit hexagon, $\ell_i = \ell_{i,i+1} = 1$. Hence

$$0 = -\frac{1}{2\sqrt{3}} \sum_{i=0}^{5} \left\{2(2\dot{\ell}_{i+1} + 2\dot{\ell}_i - 2\dot{\ell}_{i,i+1}) - (2\dot{\ell}_i + 2\dot{\ell}_{i+1})\right\}$$

which reduces to the Wagon Wheel Condition:

$$\mathcal{W} = \sum_{i=0}^{5} \dot{\ell}_i - \sum_{i=0}^{5} \dot{\ell}_{i,i+1} = 0 \tag{2}$$

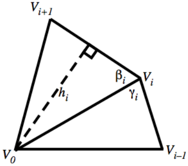
For affine hexagons, the compatibility equation is the wagon wheel condition weighted by the respective side lengths

$$0 = \sum_{i=0}^{5} \ell_i \, \dot{\ell}_i - \sum_{i=0}^{5} \ell_{i,i+1} \dot{\ell}_{i,i+1}$$

A general wagon-wheel condition holds for stars (unions of adjacent triangles) about interior vertices of any valence.

Suppose that V_0 is an interior vertex of valence n in a triangulated truss, and that $V_1, \ldots, V_n$ are the adjacent vertices going around in order. Let $\alpha_i = \angle V_i V_0 V_{i+1}$. It turns out that the compatibility equation is again that a weighted sum of the radial elongations L_i equals a weighted sum of the concentric elongations, $L_{i,i+1}$ for $i = 1, \ldots, n$ taken modulo n. By regrouping the sum, this becomes

$$0 = \sum_{i=1}^{n} \frac{\ell_{i,i+1}}{\ell_i \ell_{i+1} \sin \alpha_i} L_{i,i+1} - \sum_{i=1}^{n} \left\{ \frac{\ell_i - \ell_{i+1} \cos \alpha_i}{\ell_i \ell_{i+1} \sin \alpha_i} + \frac{\ell_i - \ell_{i-1} \cos \alpha_{i-1}}{\ell_{i-1} \ell_i \sin \alpha_{i-1}} \right\} L_i \tag{3}$$



The wagon wheel condition may be rewritten in a simpler form. If we denote $\beta_i = \angle V_0 V_i V_{i+1}$ and $\gamma_i = \angle V_0 V_i V_{i-1}$, then the area of the triangle

$$\begin{aligned} 2\,\mathrm{A}(\triangle V_0 V_i V_{i+1}) &= \ell_i \ell_{i+1} \sin \alpha_i = \ell_i \ell_{i,i+1} \sin \beta_i \\ &= \ell_{i+1} \ell_{i,i+1} \sin \gamma_{i+1}. \end{aligned}$$

It follows that

$$\frac{\ell_{i,i+1}}{\ell_i \ell_{i+1} \sin \alpha_i} = \frac{\ell_{i,i+1}}{\ell_i \ell_{i,i+1} \sin \beta_i} = \frac{1}{\ell_i \sin \beta_i} = \frac{1}{h_i} \tag{4}$$

where

$$h_i = \ell_i \sin \beta_i = \ell_{i+1} \sin \gamma_{i+1}$$

is the support distance, the distance of the of line through the $\ell_{i,i+1}$ side to V_0. Also, subtracting the projection of ℓ_{i+1} on ℓ_i we obtain

$$\ell_i - \ell_{i+1} \cos \alpha_i = \ell_{i,i+1} \cos \beta_i.$$

Then

$$\frac{\ell_i - \ell_{i+1} \cos \alpha_i}{\ell_i \ell_{i+1} \sin \alpha_i} = \frac{\ell_{i,i+1} \cos \beta_i}{\ell_i \ell_{i,i+1} \sin \beta_i} = \frac{\cos \beta_i}{h_i}$$

and

$$\frac{\ell_i - \ell_{i-1} \cos \alpha_{i-1}}{\ell_{i-1} \ell_i \sin \alpha_{i-1}} = \frac{\ell_{i-1,i} \cos \gamma_i}{\ell_i \ell_{i-1,i} \sin \gamma_i} = \frac{\cos \gamma_i}{h_{i-1}}.$$

This gives the final wagon wheel condition.

Theorem

In a triangulated truss, the compatibility condition at an interior vertex has the form

$$0 = \sum_{i=1}^{n} \frac{L_{i,i+1}}{h_i} - \sum_{i=1}^{n} \left\{ \frac{\cos(\beta_i)}{h_i} + \frac{\cos(\gamma_i)}{h_{i-1}} \right\} L_i, \tag{5}$$

where h_i is defined by (4)*.*

Rearranging the wagon wheel condition (3) into a triangle-wise sum

$$0 = \sum_{i=1}^{n} \frac{1}{h_i} \left\{ L_{i,i+1} - \cos(\beta_i) L_i - \cos(\gamma_{i+1}) L_{i+1} \right\}.$$

In other words, on average, the projected elongations of the radial components onto the circumferential line cancels the circumferential component of that line.

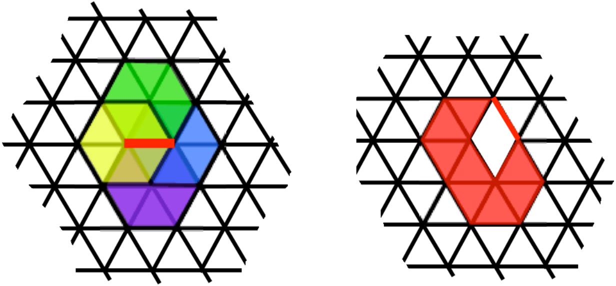


Figure: Compatibility cells around undamaged and damaged edge.

Four independent compatibility hexagons involve a given interior edge in an undamaged triangular grid. If an edge is damaged (edge removed from the grid) then compatibility condition surrounding the damaged edge involves more hexagons and the compatibility region is larger. More edges are involved to overdetermine the vertices near the damage so the material is weakened.

38. Relate Compatibility of (LD) to Compatibility of Linearized Strain (LC)

Suppose $D \subset \mathbf{R}^d$ is a domain. Recall the problem determining an infinitesimal deformation $u : D \to \mathbf{R}^d$ by prescribing the strains

$$\text{(LC)} \qquad \frac{1}{2}\left(\frac{\partial u^i}{\partial x_j} + \frac{\partial u^j}{\partial x_i}\right) = \epsilon_{ij}$$

where $\epsilon_{ij} = \epsilon_{ji}$ is a given symmetric strain field. Were such u to exist, because it is a map of Euclidean Spaces the strain field must necessarily satisfy the continuum compatibility condition in D,

$$\epsilon_{ij,pq} - \epsilon_{jp,qi} + \epsilon_{pq,ij} - \epsilon_{qi,jp} = 0$$

for all indices i, j, p, q where $\epsilon_{ij,pq} = \frac{\partial^2 \epsilon_{ij}}{\partial x_p \, \partial x_q}$.

This is the linearized equivalent of saying that the pulled back metric of a map between Euclidean Spaces must have vanishing Riemann curvature.

In $d = 2$ this boils down to one equation

$$\mathrm{Ink}(\epsilon) = \epsilon_{11,22} - 2\epsilon_{12,12} + \epsilon_{22,11} = 0.$$

The infinitesimal deformations equations of a hexagon, $Au = \Lambda$ is a discretization of the continuum equations for prescribed strain

$$(LC) \qquad \frac{1}{2}\left(\frac{\partial u^i}{\partial x_j} + \frac{\partial u^j}{\partial x_i}\right) = \epsilon_{ij}$$

Its compatibility equation approximates continuum compatibility.

Theorem (Krtolica's Expansion of Wagon Wheel Condition)

Let $\mathcal{B}_{3r} \subset \mathbf{R}^2$ be a disk radius $3r$ about 0 and $\mathcal{H} \subset \overline{\mathcal{B}_{2r}}$ be a regular hexagon with side length $\delta \le r$ containing 0. Let $u \in \mathcal{C}^4(\mathcal{B}_{3r}, \mathbf{R}^2)$ be an infinitesimal deformation satisfying the strain equation (LC). The wagon-wheel condition (2) for the u-induced rates of change of distances between vertices of $\mathcal{H}$ has the Taylor expansion about the origin

$$\mathcal{W} = -\tfrac{3}{4}\left(\epsilon_{11,22} - 2\epsilon_{12,12} + \epsilon_{22,11}\right)\delta^2 + 0 \cdot \delta^3 + \mathrm{o}(\delta^3)$$

as $\delta \to 0$ uniformly in $\overline{\mathcal{B}_{2r}}$ depending on $\|u\|_{\mathcal{C}^4(\overline{\mathcal{B}_{2r}}, \mathbf{R}^2)}$. So if the discrete compatibility condition $\mathcal{W} = 0$ holds for all δ, then the continuum compatibility conditions $\operatorname{Ink}(\epsilon) = 0$ holds.

Proof of the Theorem depends on expressing the rate of change of distance in terms of strains.

Lemma

Let $\mathcal{B}_{3r} \subset \mathbf{R}^2$ be a disk radius $3r$ about the origin and $a_i, a_j \in \mathcal{B}_{3r}$. Let $u \in \mathcal{C}^4(\mathcal{B}_{3r}, \mathbf{R}^2)$ be an infinitesimal deformation with strains given by (39). If $\phi(x,t)$ is a deformation such that $\phi(x,0) = x$ and $\dot{\phi}(x,0) = u(x)$, then

$$\left.\frac{d}{dt}\right|_{t=0} |\phi(a_i,t) - \phi(a_j,t)| = \frac{1}{|a_i - a_j|} \int_0^1 (a_i - a_j)^T \epsilon(\gamma(s))(a_i - a_j)\, ds$$

where $\gamma(s) = a_i + s(a_j - a_i)$ for $0 \le s \le 1$ is a parameterization of the line segment from a_i to a_j.

Proof.

$$\begin{aligned}
&\left.\frac{d}{dt}\right|_{t=0} |\phi(a_i,t)-\phi(a_j,t)| \\
&= \left.\frac{d}{dt}\right|_{t=0} \sqrt{(\phi(a_i,t)-\phi(a_j,t))^T(\phi(a_i,t)-\phi(a_j,t))} \\
&= \left.\frac{(\phi(a_i,t)-\phi(a_j,t))^T(\dot\phi(a_i,t)-\dot\phi(a_j,t))}{\sqrt{(\phi(a_i,t)-\phi(a_j,t))^T(\phi(a_i,t)-\phi(a_j,t))}}\right|_{t=0} \\
&= \frac{(a_i-a_j)^T(u(a_i)-u(a_j))}{\sqrt{(a_i-a_j)^T(a_i-a_j)}} \\
&= \frac{(a_i-a_j)^T(u(a_i)-u(a_j))}{|a_i-a_j|} \\
&= \frac{1}{|a_i-a_j|}\int_0^1 (a_i-a_j)^T\frac{d}{ds}u(\gamma(s))\,ds
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{|a_i - a_j|} \int_0^1 (a_i - a_j)^T \nabla u(\gamma(s))\, \dot{\gamma}(s)\, ds \\
&= \frac{1}{|a_i - a_j|} \int_0^1 (a_i - a_j)^T \nabla u(\gamma(s))\, (a_i - a_j)\, ds \\
&= \frac{1}{|a_i - a_j|} \int_0^1 (a_i - a_j)^T \epsilon(\gamma(s))\, (a_i - a_j)\, ds
\end{aligned}
$$

where $v^T(\nabla u)\, v = \frac{1}{2} v^T(\nabla u + (\nabla u)^T)\, v = v^T \epsilon\, v$, proving the lemma. □

To prove the theorem, the strains are expressed in Taylor Series about zero. The elongations of the edges of the hexagon are computed by integrating the Taylor Series in their expressions. The twelve elongations are put into the wagon wheel condition and coefficients are collected (using MAPLE!) □

A truss is generic if the number of compatibility conditions equals the Maxwell Count. Equivalently, if and only if it is infinitesimally rigid.

Instead of determining which trusses are generic, we define a class of generic trusses, the Bigon-Triangle-Prism (BTP) Trusses, that include the structures we wish to deal with such as triangulated structures.

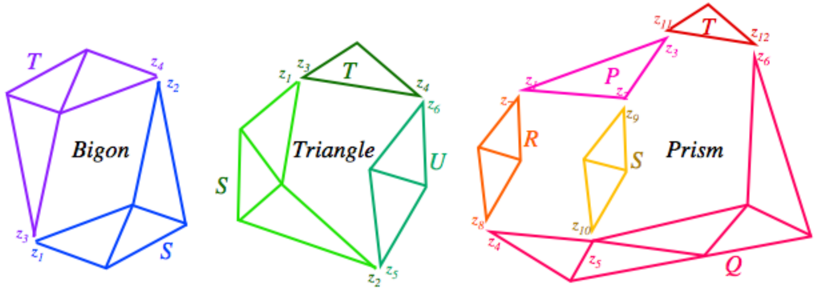


Figure: BTP Constructions. P, Q, R, S, T and U are rigid subtrusses with labeled nodes identified. z_1, z_2, z_4, z_5 would flex if not for z_3, z_6.

The BTP-Trusses trusses are finite trusses built by assembling subunits of smaller BTP-Trusses according to some rules. The basic BTP trusses:

- A single edge with two ending vertices is the basic BTP truss.
- A pair of edges attached to the same two vertices form a *bigon*, which is also rigid.
- Three edges connected in a triangle also make a rigid truss.

A rigid truss with two labeled vertices behaves like a single edge: two or three rigid trusses may be attached bigon or triangle fashion to make a larger rigid truss. Two distinct nodes at the same coordinates may be pinned together to make a single node. Two rigid trusses may be connected by three edges prism fashion.

Since the third connecting edge may be far from the other edges, determining the rigidity of a truss is not a local problem.

The composition rules of BTP trusses are as follows.

- Single links.
- Bigons. Suppose S and T are two BTP-Trusses, each containing at least two distinct points $z_1, z_2 \in S$ and $z_3, z_4 \in T$ such that the coordinates $z_1 = z_3$ and $z_2 = z_4$. The bigon is the disjoint union "$\amalg$" of S and T whose two points are identified.

$$T_{\text{bigon}} = (S \amalg T)/\{z_1 \sim z_3, z_2 \sim z_4\}$$

- Triangles. Suppose S, T and U are three BTP-Trusses, each containing at least two distinct points $z_1 \neq z_2$ in S, $z_3 \neq z_4$ in T and $z_5 \neq z_6$ in U such that the coordinates $z_2 = z_3$, $z_4 = z_5$ and $z_6 = z_1$ and such that $z_1 z_2 z_4$ is non-degenerate (the three points are not collinear.) The triangle is the disjoint union of three sides whose three points are identified pairwise.

$$T_{\text{triangle}} = (S \amalg T \amalg U)/\{z_1 \sim z_3, z_2 \sim z_5, z_4 \sim z_6\}$$

The triangle is assembled by pinning two points together in each of the three subassemblies to form a triangle.

- Prisms. Suppose P, Q, R, S, T are BTP-Trusses with at least three distinct points $z_1, z_2, z_3 \in P$ and $z_4, z_5, z_6 \in Q$ satisfying a non-degeneracy condition and at least two distinct points $z_7 \neq z_{10}$ in R, $z_8 \neq z_{11}$ in S and $z_9 \neq z_{12}$ in T such that $z_i \sim z_{i+6}$ for $i = 1 \ldots 6$. The prism is the disjoint union with points identified

$$T_{\text{prism}} = (P \amalg \cdots \amalg T)/\{z_i \sim z_{i+6} \text{ for } i = 1 \ldots 6.\}$$

- Pin a vertex. Suppose that T is a truss that has two distinct vertices $z_1, z_2 \in T$ with the same coordinates. The new truss is built by pinning the vertices

$$T_{\text{pin}} = T/\{z_1 \sim z_2\}.$$

For example, three single links may be assembled to a simple nondegenerate triangle. Another identical copy of this triangle may be attached to the first at two vertices and overlapping the first, forming a "bigon." The third vertices from each triangle are distinct nodes but have the same coordinates. Finally, these vertices may be pinned together. The BTP-Truss structure is not unique. The same double triangle truss also results from attaching the second edge to each of the three original edges of a triangle.

The triangle and prism constructions require a nondegeneracy condition. *e.g.* in a triangle, the three edges cannot be collinear. In a prism, if the upper and lower triangles are connected by three parallel line segments, then the resulting truss is not infinitesimally rigid because it has a shearing flex. Similarly, if the line segments have a common point of intersection then the prism isn't infinitesimally rigid it will have a rotational flex about the common point.

Put $z_i = (x_i, y_i)$. The nondegeneracy condition for the prism is

$$\begin{vmatrix} x_1 - x_4 & y_1 - y_4 & x_1 y_4 - x_4 y_1 \\ x_2 - x_5 & y_2 - y_5 & x_2 y_5 - x_5 y_2 \\ x_3 - x_6 & y_3 - y_6 & x_3 y_6 - x_6 y_3 \end{vmatrix} \neq 0.$$

If the legs were parallel, then the first two columns are multiples of one another and the determinant vanishes. If the lines determined by the legs meet at the origin, then the areas of the parallelogram determined by the endpoints of the legs all vanish, so the last column is zero and a nontrivial flex is given by a rotation about the origin for P and zero for Q. The determinant is invariant under translation so any point may be the meeting point.

Theorem

Nondegenerate BTP-Trusses are infinitesimally rigid, hence generic. The number of compatibility conditions under a BTP combination is determined from the compatibility conditions of its parts. Let c_i be the number of compatibility conditions for the part T_i.

- *Segments have $c = 0$.*
- *Bigons have $c_{\text{bigon}} = c_1 + c_2 + 1$.*
- *Triangles have $c_{\text{triangle}} = c_1 + c_2 + c_3$.*
- *Prisms have $c_{\text{prism}} = c_1 + \cdots + c_5$.*
- *Pinning a vertex has $c_{\text{pin}} = c_1 + 2$.*

It is unknown to the authors whether all infinitesimally rigid trusses are BTP-trusses.

An immediate consequence is that the trusses of triangulated domains are infinitesimally rigid.

Corollary

Let T be a triangulated truss such that all triangles are non-degenerate. Then T is generic. Suppose that T is built up starting from a single edge one step at a time by attaching two connected edges to form a triangle, such as gluing on a triangle to an outer edge, or by attaching a single boundary edge to two existing vertices, such as gluing on a triangle to two existing edges, or such as connecting two vertices to surround a hole. The number of compatibility conditions is n_b, the number of times a single edge is glued to two vertices.

Proof. The process of building the truss is just the BTP construction where triangles are made from the previous stage and two segments, and bigons are made from the previous stage and one segment. Each bigon increases the compatibility count by one. □

For simply connected subtrusses of the standard triangular lattice we know more: the basis for the compatibity conditions are just the wagon wheel conditions centered at interior vertices.

Theorem (Basis for compatibility conditions in a hexagonal trusses)

Let X be the union of finitely many 2-triangles of the hexagonal lattice. Suppose that the boundary ∂X consists of a $g+1$ disjoint simple closed curves. Then the truss X is generic: the number of compatibility conditions equals the Maxwell number. Moreover, a basis for the compatibility conditions consits of one condition for each hexagon about an interior vertex and three for each ring-girder around every hole.

$$\mathcal{C} = \mathcal{M} = 3g + v_i.$$

The interior vertices may be regarded as material points. The additional compatibility from each hole is a discrtization feature.

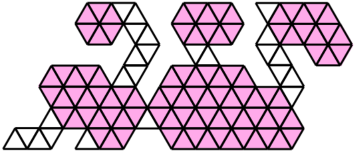


Figure: Decompose Simply Connected Truss into Plates and Girders.

Begin with simply connected trusses. Let $H(V_i)$ denote an open hexagon about an interior vertex. Decompose the union of hexagons about interior vertices into connected components, called plates

$$\coprod P_j = \bigcup_{\text{interior vertex } V_i} H(V_i)$$

The connected components of the remainder are called girders

$$\coprod G_i = X - \overline{\cup_j P_j}$$

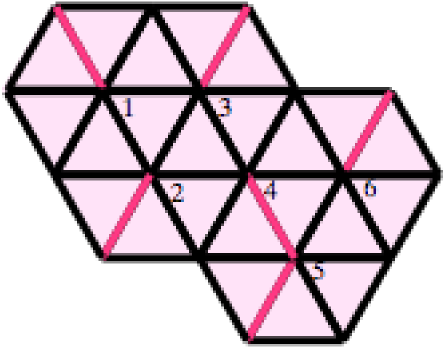


Figure: Order vertices and remove one edge per hexagon, maintaining rigidity.

Argue that each plate is generic. Order the vertices from one end to the other. Remove on edge of each hexagon in turn, maintaining rigidity as you build up the hexagons. Thus v_i edges may be removed.

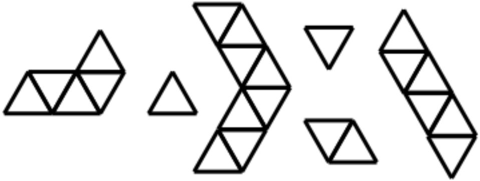


Figure: Girders are statically determined.

Argue that each girder is statically determined: it is rigid but without compatibility conditions. Removing any edge from a girder results in a flexible (hence infinitesimally flexible) structure.

Then argue that a simply connected truss made up of girders and plates is generic. Removing an edge from each hexagon in the plates results in a statically determined truss.

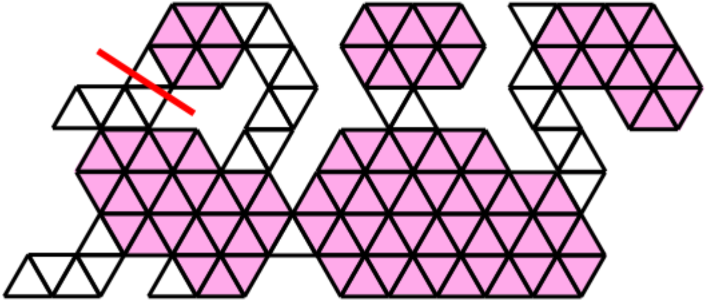


Figure: Taking out a "branch cut" reduces the Maxwell count by three.

For multiply connected domains, argue by induction on the number of holes. Removing a "branch cut" reduces the number compatibility conditions by the number of interior vertices along the cut plus three.

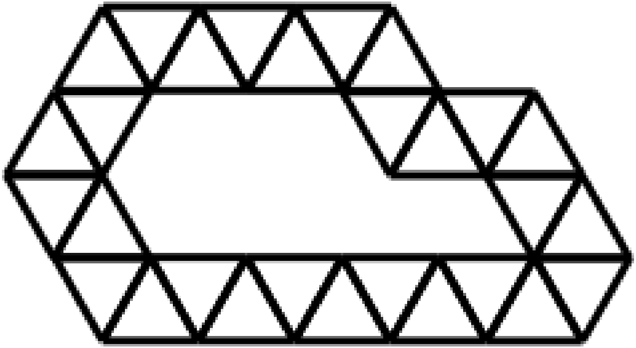


Figure: About each hole is a "ring-girder" which contributes three compatibility conditions.

Asymptotic Compatibility Density measures strength of a truss.

How much do holes weaken a material? Assume that the material is periodic. Lets compute the large-scale average compatibility condition density for damaged material relative to the undamaged material.

For simplicity, let the basic cell Υ by a $k \times k$ union of hexagons centered on $ae_1 + be_2$ where $a, b = 1, \ldots, k$ and $e_1 = (1, 0)$ and $e_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$. Suppose there are h holes per cell and m interior vertices taken by each hole. Assuming that cells are bounded by $h + 1$ pairwise disjoint simple closed curves, let Ω_n be the union consisting of $n \times n$ cells slightly overlapping, centered on $ake_1 + bke_2$ where $a, b = 1, \ldots, n$.
The asymptotic compatibility number is

$$AC = \lim_{n\to\infty} \frac{c(\Omega_n)}{\mathrm{A}(\Omega_n)}.$$

The total number of holes is $g = n^2 h$. The total number of interior vertices is

$$v_i = k^2 n^2 - hmn^2$$

The area is base times height minus corner triangles, thus

$$AC = \lim_{n\to\infty} \frac{v_1 + 3g}{\mathrm{A}(\Omega_n)} = \lim_{n\to\infty} \frac{[k^2 n^2 - hmn^2] + 3n^2 h}{nk(nk+1)\frac{\sqrt{3}}{2}} = \frac{k^2 - h(m-3)}{\frac{\sqrt{3}}{2}k^2}.$$

Note that removing a single edge reduces the number of interior vertices by four, but introduces a ring girder which supports three compatibility conditions. Thus $m - 3 \geq 1$ compatibility conditions are lost for each hole.

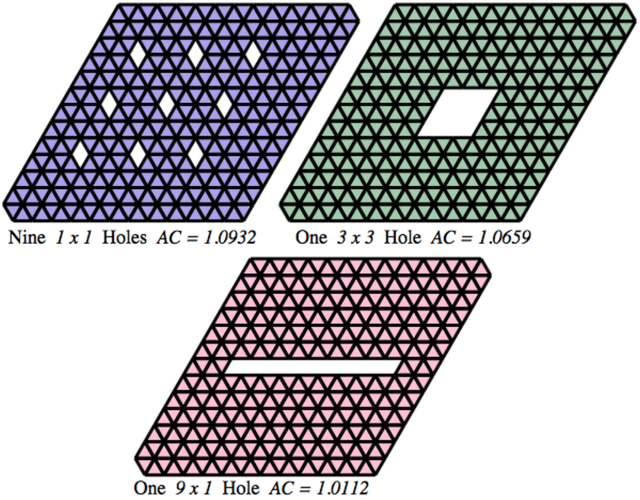


Figure: 13×13 Period Cell with Holes of Area 18 Triangles ($k = 13$, $p = 4$).

$$AC = \frac{k^2 - h(m-3)}{\frac{\sqrt{3}}{2}k^2}, \qquad AC_{\text{many 1-link holes}} = \frac{k^2 - p^2 + 2p - 1}{\frac{\sqrt{3}}{2}k^2}.$$

The asymptotic compatibility depends not just on the total area removed from the cell. Taking out more holes of the same total area has larger AC, a proxy for material resilience.

For example if one link is removed, $m = 4$ and triangle has area 2 triangles. Removing $h = (p-1)^2$ one-link removes $m = 4$ interior vertices per hole and has the same area $2(p-1)^2$ triangles as the $(p-1) \times (p-1)$ rhombus, which removes $h = 1$ hole and $m = p^2$ interior vertices. If the hole is a $(p-1)^2 \times 1$ trapezoid, it also has the same number of triangles, $h = 1$ but removes $m = 2p^2 - 4p + 2$ interior vertices.

$$AC_{\text{rhombus}} = \frac{k^2 - p^2 + 3}{\frac{\sqrt{3}}{2}k^2}, \qquad AC_{\text{trapezoid}} = \frac{k^2 - 2p^2 + 4p - 5}{\frac{\sqrt{3}}{2}k^2}.$$

For simplicity, all edges of the truss have unit length. The infinitesimal deformations u are related to the elongations of the edges via $Au = \Lambda$ where A is a $e \times dn$ matrix of rank r. Let us denote the compatibility conditions $B\Lambda = 0$ where B is an $(e - r) \times e$ matrix. Hooke's Law says the forces $C\Lambda$ along the edges are proportional to the elongations where $C = \operatorname{diag}(c_1, \ldots, c_e)$ is the $e \times e$ diagonal of positive spring constants matrix. $A^T C\Lambda$ are forces at the vertices. $K = A^T CA$ is the stiffness matrix which is nonnegative definite with rank r.

Then the force balance is $A^T C\Lambda = F$ where F is the vector of tractions applied at the e vertices. It has a unique solution if infinitesimal flexes are eliminated by fixing $dn - r$ unknowns.

The equation for balanced forces may be solved for elongations or displacements.

$$\begin{aligned} A^T C\Lambda &= F; \\ B\Lambda &= 0. \end{aligned} \qquad\qquad \begin{aligned} A^T C\Lambda &= F \\ \Lambda &= Au \end{aligned}$$

The analagous equations for linearized elastostatics are in terms of strains ϵ or infinitesimal displacements u are

$$\begin{aligned} \mathrm{div}\cdot\mathbf{c}\cdot\epsilon &= \rho f; \\ \nabla\times(\nabla\times\epsilon) &= 0. \end{aligned} \qquad\qquad \begin{aligned} \mathrm{div}\cdot\mathbf{c}\cdot\epsilon &= \rho f \\ \epsilon &= \frac{1}{2}(\nabla^T u + \nabla u) \end{aligned}$$

where ρ is mass density, f is an external body force and $\mathbf{c}(x)$ is the elasticity tensor.

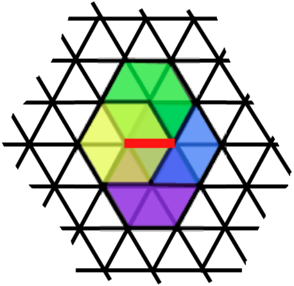
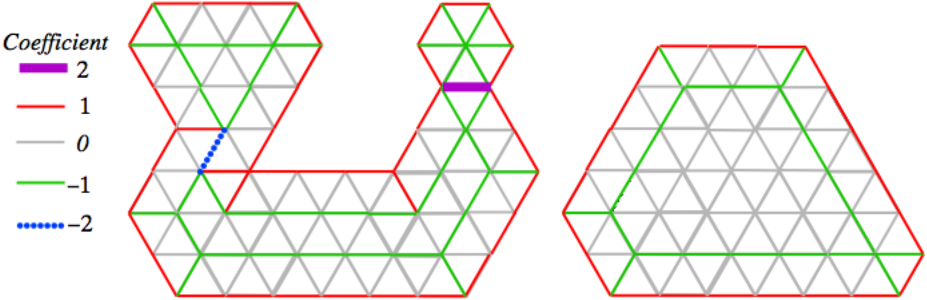


Figure: Total contribution from an interior edge cancels in the sum of WW's

Thus the sum of all wagon wheel conditions $\sigma(L)$ as a functional of elongations in a triangular truss is supported near the boundary of the domain.

Consider the union of hexagons $\mathcal{P}$ in a triangular truss whose boundary curve is a single simple closed curve. Let $\sigma(L)$ be the sum of the wagon wheels conditions for the hexagons of $\mathcal{P}$.

For those edges E_{ij} that are included in four hexagons, as a radial edge for the hexagons centered at the endpoints and as a circumferential edge for those hexagons centered on the opposite vertices of triangles containing the edge, the sum cancels and the coefficient of L_{ij} is zero in $\sigma(L)$. Thus, only the edges whose endpoints are in a double layer, at most one unit from ∂P, contribute to $\sigma(L)$.



In the case that the $\mathcal{P}$ is a convex union of regular hexagons $\sigma(L) = 0$ simplifies because there are only boundary edges, a single incoming edge at corners and boundary parallel interior edges.

$$\sum_{E_{ij} \text{ is boundary edge}} L_{ij} = \sum_{E_{ij} \text{ is boundary parallel edge}} L_{ij} + \sum_{E_{ij} \text{ is incoming edge}} L_{ij}.$$

As a simple application, we can conclude that if there are no elongations on the boundary of a domain, then there cannot be only positive elongations in the neighboring edges of the boundary layer.

Theorem

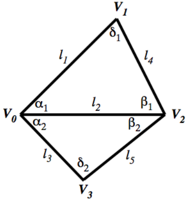
Let $\mathcal{P}$ be a convex union of hexagons structure with interior points. Suppose that the elongations L_{ij} are zero on the boundary $\partial\mathcal{P}$ and positive on edges within one link of the boundary. Then L cannot satisfy the compatibility conditions at all interior points of $\mathcal{P}$.

Proof. The weights $\sigma(E_{ij})$ are positive on boundary edges and nonpositive and somewhere negative on the rest of the edges in the unit boundary layer. Thus $\sigma(L) = 0$ cannot hold. □

For a triangulated structure, the sum of all wagon wheel compatibility conditions vanishes on an interior edge.

Lemma

Let E_{02} be an interior edge that is bounded on opposite sides by two nondegenerate triangles $V_0V_1V_2$ and $V_0V_2V_3$. Suppose further that all four V_0, V_1, V_2 and V_3 are interior vertices. Then the sum of the four wagon wheel conditions that involve E_{02}, the ones centered on V_0, V_1, V_2 and V_3, has zero L_{02} coefficient.



Proof. Denote the lengths $\ell_1 = |V_0V_1|$, $\ell_2 = |V_0V_2|$, $\ell_3 = |V_0V_3|$, $\ell_4 = |V_1V_2|$ and $\ell_5 = |V_2V_3|$. Denote the angles $\alpha_1 = \angle V_1V_0V_2$, $\alpha_2 = \angle V_2V_0V_3$, $\beta_1 = \angle V_0V_1V_2$, $\beta_2 = \angle V_2V_3V_0$, $\delta_1 = \angle V_1V_2V_0$ and $\delta_2 = \angle V_0V_2V_3$.

The wagon wheel conditions that involve E_{01} are the ones centered at V_0 and V_1 where E_{01} is a radial edge and those centered on V_2 and V_3 where E_{01} is a concentric edge. The sum of coefficients of L_{01} is

$$\frac{\ell_2}{\ell_2\ell_3\sin\beta_1}+\frac{\ell_2}{\ell_3\ell_5\sin\beta_2}-\left\{\frac{\ell_2-\ell_3\cos\alpha_2}{\ell_2\ell_3\sin\alpha_2}+\frac{\ell_2-\ell_1\cos\alpha_1}{\ell_1\ell_2\sin\alpha_1}\right\}$$
$$-\left\{\frac{\ell_2-\ell_4\cos\delta_1}{\ell_2\ell_4\sin\delta_1}+\frac{\ell_2-\ell_5\cos\delta_2}{\ell_2\ell_5\sin\delta_2}\right\}$$

Twice the areas of triangle $V_0V_2V_1$ and $V_0V_2V_3$ are, respectively,

$$2A_1=\ell_1\ell_2\sin\alpha_1=\ell_1\ell_4\sin\beta_1=\ell_2\ell_4\sin\delta_1$$
$$2A_2=\ell_2\ell_3\sin\alpha_2=\ell_3\ell_5\sin\beta_2=\ell_2\ell_5\sin\delta_2.$$

The sum of coefficients of L_{01} becomes

$$\frac{-\ell_2+\ell_1\cos\alpha_1+\ell_4\cos\delta_1}{2A_1}+\frac{-\ell_2+\ell_3\cos\alpha_2+\ell_5\cos\delta_2}{2A_2}=0.$$

This is because the sum of the lengths of the projections of the sides V_0V_1 and V_2V_1 onto the side V_0V_2 equals the length of V_0V_2, namely, $\ell_1\cos\alpha_1+\ell_4\cos\delta_1=\ell_2$. A similar equation holds for triangle $V_0V_2V_3$. $\square$

The value of σ on different types of edges depends on which interior stars contain the edge.

Theorem

For the triangulated structure, let $\mathcal{P}$ be the union of closed star neighborhoods of all interior points. Let

$$\sigma(L) = \sum_{ij} \sigma(E_{ij})\, L_{ij} \tag{6}$$

be the sum of the WW conditions corresponding to the interior points. $\sigma(E_{ij})$, the E_{ij} coefficient vanishes except for edges that either touch the boundary of $\mathcal{P}$ or both endpoints are one link away from the boundary. For such edges, $\sigma(E_{ij})$ is the sum of WW conditions whose star neighborhoods contain the edge E_{ij} as either radial or circumferential edge. They have expressions in terms of the geometry of the triangulation.

Proof. By the Lemma, σ vanishes for edges with one endpoint farther than one link from $\partial\mathcal{P}$. There are seven combinatorial types of non-vanishing conditions: of the four vertices, there are several possibilities. (1) One vertex is interior which corresponds to a boundary edge or a unique incoming edge. (2) two vertices are interior which corresponds to an isthmus edge, an extreme of multiple incoming edges or a spine edge; (3) three vertices are interior which corresponds to the middle of multiple incoming edges or a parallel boundary edge.

For example, in case both side triangles touch boundary, V_0 is interior but V_1, V_2 and V_3 are not, then E_{02} is radial and

$$\sigma(E_{02}) = -\frac{\cos\beta_1}{h_1} - \frac{\cos\beta_2}{h_2}$$

where

$$h_1 = \ell_2 \sin\beta_1, \qquad h_2 = \ell_2 \sin\gamma_1. \tag{7}$$

The other cases are similar. □

70. Compatibility line integral for (ND).

A compatibility line integral also holds for (ND). Let γ be a contractible closed curve that bounds the subdomain $\mathcal{P}$. There is a boundary equation that holds for the double layer near the boundary that amounts to saying that the total angle change going around the outer boundary is 2π.

Theorem (Compatibility line integral for (ND))

In a triangulated structure, suppose that the union of stars $\mathcal{P}$ is bounded by a single simple curve γ. Then the total turning angle of the γ may be expressed in terms of the prescribed lengths of edges on or within one link of the boundary edge $2\pi = \sum_{V_i\in\gamma}\left[\pi - \sum_{\triangle(V_j,V_i,V_k)\in\mathcal{F}}\alpha(V_j,V_i,V_k)\right]$ *where $\mathcal{F}$ are triangular faces and* $\alpha(V_j,V_i,V_k) = \cos^{-1}\left(\frac{{\ell_{ij}}^2+{\ell_{ik}}^2-{\ell_{jk}}^2}{2\ell_{ij}\ell_{jk}}\right)$. *is the angle of the triangle at V_i.*

Proof. The inner sum is the interior angle, the sum of the angles of triangles adjacent to the boundary vertex V_i. Thus the bracket is the outer turning angle of γ at V_i. The outer sum is the total over boundary vertices of the turning angles, which adds up to 2π for planar domains. □

For (NC), the curvature is an exact differential

$$\mathcal{K}\, dA = d\omega,$$

where ω is the connection form which is a derivative of the metric. Using Stokes' Theorem, the integral around the boundary of a region Ω gives the Gauss Bonnet Formula

$$\iint_\Omega \mathcal{K}\, dM + \int_{\partial\Omega} \kappa_g\, ds + \sum_i \alpha_i = 2\pi, \tag{8}$$

where $\mathcal{K}$, κ_g, dM, ds and α_i are the Gauss curvature, the geodesic curvature of the boundary curve, the area form, the arclength and angle changes at the corners expressed in terms of the ζ metric. The identical vanishing of the Gauss curvature $\mathcal{K} = 0$ compatibility condition for prescribed deformation tensor ζ gives compatibility integral around a curve.

Expressing this in terms of ambient derivatives of ζ_{ij} gives the compatibility boundary integral. Let $\tilde{\zeta}_{ij}$ denote the metric expressed in a rotated frame adapted to the boundary.

Theorem (Compatibility line integral for (ND))

Let Ω be a simply connected domain with $\mathcal{C}^2$ boundary and let ζ_{ij} be $\mathcal{C}^2$ satisfying the local compatibility conditions on the closure $\bar{\Omega}$. Then

$$2\pi = \int_0^L \left[-\frac{1}{2}\tilde{\zeta}_{11,2} + \tilde{\zeta}_{12,1} - \frac{\tilde{\zeta}_{12}\tilde{\zeta}_{11,1}}{2\tilde{\zeta}_{11}} + \tilde{\zeta}_{11}\tilde{\kappa}_g \right] \frac{ds}{\sqrt{\zeta_{11}\zeta_{22}-\zeta_{12}^2}}.$$

Near each boundary point, $\tilde{\zeta}_{ij}$ expressed in a frame where $\tilde{\mathbf{e}}_1 = \mathbf{t}$ and $\tilde{\mathbf{e}}_2 = \boldsymbol{\nu}$ are the unit tangent and inner normal vectors, κ_g is the geodesic curvature of the boundary and ds is the arclength.

The integral involves the tangential and normal derivatives of components of the metric $\tilde{\zeta}_{ij}$.

This expression may be extended to domains with corners.

Let the one form

$$\beta = \beta_i \, dx^i$$

where

$$\beta_i = \epsilon_{1i,2} - \epsilon_{2i,1}.$$

Then

$$\mathrm{Ink}(\epsilon) = 0 \qquad \Longleftrightarrow \qquad d\beta = 0.$$

The equivalence of the closedness of the one form β and the vanishing of $\mathrm{Ink}(\epsilon)$ was observed in 1901 by Weingarten to study dislocations along cracks.

Applying Stokes's Theorem gives a boundary integral formula for (LC).

Theorem (Compatibility line integral for (LC))

Let Ω be a simply connected domain with $\mathcal{C}^2$ boundary and let ϵ_{ij} be $\mathcal{C}^2$ and satisfy the local compatibility conditions on the closure $\bar{\Omega}$. Then

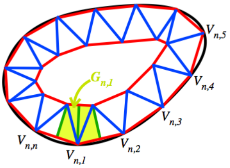
$$0 = \int_{\partial\Omega} \left(\tilde{\epsilon}_{11} - \tilde{\epsilon}_{22}\right) \kappa_g - \frac{\partial}{\partial \boldsymbol{\nu}} \tilde{\epsilon}_{11} \, ds.$$

Near boundary points, $\tilde{\epsilon}_{ij}$ is expressed in a frame where $\tilde{\mathbf{e}}_1 = \mathbf{t}$ and $\tilde{\mathbf{e}}_2 = \boldsymbol{\nu}$ are the unit tangent and inner normal vectors.

The compatibility line integral for (LC) involves the metric and only the normal derivative of the tangential component.

There is an extension of this formula to domains with corners.

A Krtolica expansion also yields the boundary integrand. Let Ω be a simply connected subdomain with $\mathcal{C}^2$ boundary. We can build an approximation Ω_n by approximating $\partial\Omega$ by a piecewise linear curve that passes through n equally distant points $V_{n,1}, V_{n,2}, \ldots, V_{n,n} \in \partial\Omega$ taken in order around $\partial\Omega$, attaching inward facing equilateral triangles to each of the segments, connecting their interior vertices with edges forming a ring girder G_n along the boundary, and then filling the remainder with an arbitrary triangulation.



Then the compatibility sum for (LD) gives an equation $\mathcal{V}(G_n, L) = 0$ on the prescribed elongations L which is a weighted sum involving all edges of the double layer, the edges in the girder G_n at most one link from $\partial\Omega_n$. We may partition the girder into n pieces $G_{n,i}$ localized near each of the rim vertices $V_{n,i}$ and split the sum

$$G_n = \bigcup_{i=1}^n G_{n,i}; \qquad \mathcal{V}(G_n, L_n) = \sum_{i=1}^n \mathcal{V}(G_{n,i} L_n)$$

It turns out, that if we fix a vertex $V_{n,1} = X \in \partial\Omega$ and take an arbitrary strain field ϵ near X, and consider its induced elongations L_n, for the constructed triangulations, then the boundary strain compatibility for (LD) of each localized piece converges to the (LC) boundary integrand

$$\mathcal{V}(G_{n,1}, L_n)\,\Delta_n \to \beta(e_1(X)) = \left[-\frac{\partial \epsilon_{11}}{\partial \boldsymbol{\nu}}(X) + (\epsilon_{11}(X) - \epsilon_{22}(X))\kappa(X)\right] ds$$

as $n \to \infty$, where $r = \Delta_n = |V_{n,i+1} - V_{n,i}|$ for all i is the common distance between boundary vertices at the n-th stage and $\boldsymbol{\nu}$ is the inward normal.

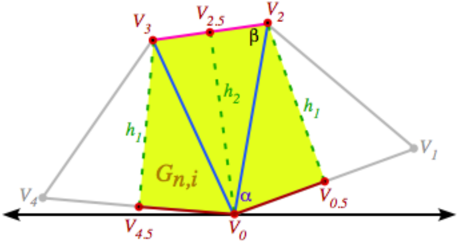


Figure: Piece of a boundary girder $G_{n,i}$.

Since we suppose that the boundary is $\mathcal{C}^3$ we perform the computation for a specific boundary curve that agrees up to the third order to any given boundary curve.

Theorem (Expansion of compatibility condition along a curve)

Let Ω be a subdomain and $\partial\Omega$ a $\mathcal{C}^3$ curve through the origin $V_0 = 0$ and tangent to the x-axis such that at the origin, its curvature is κ, and its derivative of curvature with respect to arclength is b. Let Ω be the region above the curve. Let $B_\delta \subset \mathbf{R}^2$ be a disk radius δ about the origin. Let $V_1, V_4 \in \partial\Omega$ be vertices on both sides of the origin such that $|V_1 - V_0| = |V_4 - V_0| = r$ and let V_2 and V_3 be interior vertices above $\partial\Omega$ such that $\triangle V_0V_1V_2$ and $\triangle V_0V_3V_4$ are equilateral triangles. We suppose that $r > 0$ is so small that $V_1, \ldots, V_4$ are in B_δ. Let $\mathcal{T}_r$ be a truss such that V_0 is adjacent only to vertices V_1, V_2, V_3 and V_4. Let $u \in \mathcal{C}^4(B_\delta, \mathbf{R}^2)$ be an infinitesimal deformation satisfying the strain equation (LC).

Theorem (Continued.)

If $V_{0.5}$ and $V_{3.5}$ are the midpoints of the sides V_0V_1 and V_4V_0, resp., then let the localized piece of boundary girder $G_{n,1}$ near the origin be the $V_0V_{0.5}V_2V_3V_{4.5}$ part of the truss $\mathcal{T}_r = \{E_{01}, E_{02}, E_{03}, E_{04}, E_{12}, E_{23}, E_{34}\}$. The curve compatibility condition for $\partial\Omega$ of (LD), where we take half of the contributions from sides E_{01} and E_{04}, for the u-induced rates of change of distances of $\mathcal{T}_r$ has the Taylor expansion about the origin

$$\begin{aligned}\mathcal{V}(G_{n,1}, L_n) &= -\epsilon_{11,2} + (\epsilon_{11} - \epsilon_{22})\kappa \\ &+ \left[\frac{\sqrt{3}}{12}\epsilon_{11,11} - \frac{\sqrt{3}}{4}\epsilon_{22,22} + \left(\frac{3\sqrt{3}}{4}\epsilon_{11,2} + \frac{\sqrt{3}}{6}\epsilon_{12,1} - \frac{\sqrt{3}}{4}\epsilon_{22,2}\right)\kappa\right] r \\ &+ \Big[\left(\tfrac{1}{8}\epsilon_{11,11} - \tfrac{1}{6}\epsilon_{12,12} - \tfrac{1}{24}\epsilon_{22,11} - \tfrac{1}{6}\epsilon_{22,22}\right)\kappa + \left(\tfrac{1}{2}\epsilon_{12,1} - \tfrac{9}{8}\epsilon_{22,2}\right)\kappa^2 \\ &\quad + \frac{b}{3}\epsilon_{11} + \frac{b\kappa^2}{3}\epsilon_{12} + \left(\frac{\kappa^3}{8} - \frac{b}{3}\right)\epsilon_{22}\Big] r^2 + o(r^2)\end{aligned} \tag{9}$$

as $r \to 0$. Hence, in the limit, the discrete curve sum compatibility condition as $r \to 0$ tends to the continuum curve integral compatibility conditions of (LC).

The distance between V_2 and V_3 will be smaller or larger than r, depending on whether $\kappa > 0$ or $\kappa < 0$. Note that the third derivative of the boundary influences only the r^2 term.

Proof. The proof is similar to Krtolica's Theorem. We express the coordinates of the vertices in terms of r and the induced elongations L_n in terms of ϵ_{ij}. The boundary compatibility equations of (LD) expressed as power series in r. The result pops out using the computer algebra system ©MAPLE. Here are some details.

For convenience, parameterize the curve in terms of r, the length of the segment from the origin to the point on the curve

$$(x(r), y(r)) = r\Big(\cos\alpha(r), \sin(\alpha(r))\Big)$$

where $\sin\alpha(r) = \frac{\kappa}{2}r + \frac{b}{6}r^2$. Approximating $\cos\alpha(r) = \sqrt{1 - \sin^2\alpha(r)}$ by the binomial series, and truncating to fourth degree,

$$\cos\alpha(r) \approx 1 - \frac{\kappa^2}{8}r^2 - \frac{\kappa b}{12}r^3 - \left(\frac{\kappa^4}{128} + \frac{b^2}{72}\right)r^4$$

The curvature is $\kappa(r) = \kappa + br + \frac{3}{4}\kappa^3 r^2 + \frac{37}{24}\kappa^2 b r^3 + \mathbf{O}(r^4)$.

The points $V_1 = (x(r), y(r))$ and $V_4 = (x(-r), y(-r))$ are on the curve such that $|V_1| = |V_4| = r + \mathbf{O}(r^6)$. V_2 and V_3 are found by rotating V_1 and V_4 by $\pm 60^\circ$. In the curve sum compatibility condition, $\ell_{23} = |V_3 - V_2|$. $(\ell_{23})^2 = \frac{3}{4}[1 + X(r)]r^2$ may be computed from the Pythagorean formula for $V_3 - V_2$ and then the power series of ℓ_{23} and ℓ_{23}^{-1} may be computed from the binomial series in $X(r)$. The support distance is $h_1 = \frac{\sqrt{3}}{2} r$ from V_2 to $V_0 V_1$ and the angle $\alpha = \angle V_1 V_0 V_2 = \frac{\pi}{3}$. If $V_{3.5} = \frac{1}{2}(V_2 + V_3)$ is the midpoint, then the support distance $h_2 = |V_{3.3}|$. h_2^{-1} is also found using a binomial expansion. The angle $\beta = \angle V_3 V_2 V_0 = \angle V_2 V_3 V_0$. A series is deduced from $\cos\beta = \frac{\ell_{23}}{2r}$. The compatibility condition from Theorem 13., taking half the contribution of the $V_0 V_1$ and $V_0 V_4$ sides, is

$$\mathcal{V}(G_{n,1}) r = \frac{1}{2h_1}\left(L_{01} + L_{40}\right) - \frac{1}{h_2} L_{2,3} + \left(\frac{\cos\beta}{h_2} - \frac{\cos\frac{\pi}{3}}{h_1}\right)\left(L_{03} + L_{04}\right).$$

The lowest order terms of this expression are given by (9). $\square$

Note that this approximation requires a second order Taylor approximation. The first order terms cancel and that second order terms limit to the compatibility condition.

Thanks!