

Final Exam Practice Problems

Math 2210-001: Calculus III – Fall 2009

This problem set should not be considered comprehensive. It should be regarded instead as a guide for your own studying. The best way to use it is to run through the problems here, then once you have figured out where the gaps in your knowledge lie, study those sections in the book.

1. Dot/Cross Product Prove that $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$

Hint: Write \mathbf{u} , \mathbf{v} , and \mathbf{w} as components, then just plug in and verify.

Solution:

$$\begin{aligned}\mathbf{u} &= u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k} \\ \mathbf{v} &= v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k} \\ \mathbf{w} &= w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}\end{aligned}$$

$$\begin{aligned}(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} &= ((u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}) \cdot \mathbf{w} \\ &= (u_2v_3 - u_3v_2)w_1 - (u_1v_3 - u_3v_1)w_2 + (u_1v_2 - u_2v_1)w_3\end{aligned}$$

$$\begin{aligned}\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) &= \mathbf{u} \cdot ((v_2w_3 - v_3w_2)\mathbf{i} - (v_1w_3 - v_3w_1)\mathbf{j} + (v_1w_2 - v_2w_1)\mathbf{k}) \\ &= u_1(v_2w_3 - v_3w_2) - u_2(v_1w_3 - v_3w_1) + u_3(v_1w_2 - v_2w_1)\end{aligned}$$

It's easy to see that if you multiply all the terms out, the two sides are equal.

2. Lines and Tangent Lines

Find an equation (or equations) for the line through $(4, 0, 6)$ and perpendicular to the plane $x - 5y + 2z = 10$.

Solution: The line will be of the form

$$\mathbf{r} = \mathbf{a}t + \mathbf{r}_0$$

Since it has to be perpendicular to the plane, we know that its direction must be

$$\mathbf{a} = \mathbf{i} - 5\mathbf{j} + 2\mathbf{k}$$

Similarly, since it starts at the point $(4, 0, 6)$, its starting point must be

$$\mathbf{r}_0 = 4\mathbf{i} + 6\mathbf{j}$$

Then the final equation is

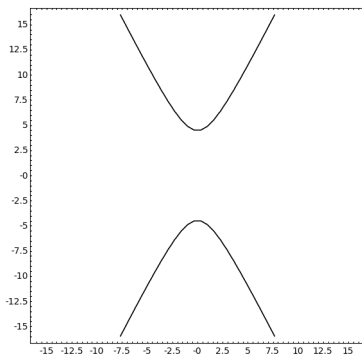
$$\mathbf{r} = (\mathbf{i} - 5\mathbf{j} + 2\mathbf{k})t + (4\mathbf{i} + 6\mathbf{j})$$

3. Curvature and Acceleration

(a) Sketch the following curve in the xy -plane, then compute its curvature.

$$y^2 - 4x^2 = 20$$

Solution: You can see that the graph is a hyperbola, opening upward and downward.



There are a few ways to solve this problem; I'm going to use the clever one. The curvature will actually be the same on both parts of this graph – the upper one and the lower one – so I'm just going to ditch the upper one. Then

$$y = \sqrt{20 + 4x^2}$$

$$y' = \frac{4x}{(20 + 4x^2)^{1/2}}$$

$$y'' = \frac{80}{(20 + 4x^2)^{3/2}}$$

and plugging into the curvature formula we get

$$\begin{aligned} \kappa(x) &= \frac{|y''(x)|}{[1 + (y')^2]^{3/2}} \\ &= \frac{|80(20 + 4x^2)^{-3/2}|}{\left[1 + \left(4x(20 + 4x^2)^{-1/2}\right)^2\right]^{3/2}} \\ &= \frac{|80(20 + 4x^2)^{-3/2}|}{\left[1 + 16x^2(20 + 4x^2)^{-1}\right]^{3/2}} \\ &= \frac{80}{\left[1 + 16x^2(20 + 4x^2)^{-1}\right]^{3/2} (20 + 4x^2)^{3/2}} \\ &= \frac{80}{[20 + 4x^2 + 16x^2]^{3/2}} \\ &= \frac{80}{[20 + 20x^2]^{3/2}} \\ &= \frac{10}{[5 + 5x^2]^{3/2}} \end{aligned}$$

(b) Find the curvature κ , the unit tangent vector \mathbf{T} , and the unit normal vector \mathbf{N} for the curve

$$\mathbf{r}(t) = \frac{1}{2}t^2\mathbf{i} + t\mathbf{j} + \frac{1}{3}t^3\mathbf{k}$$

when $t = 2$.

Solution: I've left off quite a bit of the algebra for this one. You may be able to solve this faster if you use different formulas, but it doesn't matter so much. Any problem I give you on the test will be more reasonable than this.

Again, there are a thousand ways to do this.

$$\begin{aligned}\mathbf{r}(t) &= \frac{1}{2}t^2\mathbf{i} + t\mathbf{j} + \frac{1}{3}t^3\mathbf{k} \\ \mathbf{r}'(t) &= t\mathbf{i} + \mathbf{j} + t^2\mathbf{k} \\ \mathbf{r}''(t) &= \mathbf{i} + 2t\mathbf{k}\end{aligned}$$

The easiest way to compute the curvature is thus:

$$\begin{aligned}\kappa &= \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3} \\ &= \frac{|(2t - 0)\mathbf{i} - (2t^2 - t^2)\mathbf{j} + (0 - 1)\mathbf{k}|}{(t^2 + 1^2 + (t^2)^2)^{3/2}} \\ &= \frac{\sqrt{t^4 + 4t^2 + 1}}{(t^4 + t^2 + 1)^{3/2}}\end{aligned}$$

Now the unit tangent vector

$$\begin{aligned}\mathbf{T} &= \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \\ &= \frac{t\mathbf{i} + \mathbf{j} + t^2\mathbf{k}}{\sqrt{t^4 + t^2 + 1}}\end{aligned}$$

Finally the unit normal vector

$$\begin{aligned}\mathbf{T}' &= \frac{(t^4 - 1)\mathbf{i} - (2t^3 + t)\mathbf{j} + (t^3 + 2t)\mathbf{k}}{(t^4 + t^2 + 1)^{3/2}} \\ \|\mathbf{T}'\| &= \frac{\sqrt{t^4 + 4t^2 + 1}}{t^4 + t^2 + 1} \\ \mathbf{N} &= \frac{\mathbf{T}'}{\|\mathbf{T}'\|} \\ &= \frac{(t^4 - 1)\mathbf{i} - (2t^3 + t)\mathbf{j} + (t^3 + 2t)\mathbf{k}}{\sqrt{(t^4 + 4t^2 + 1)(t^4 + t^2 + 1)}}\end{aligned}$$

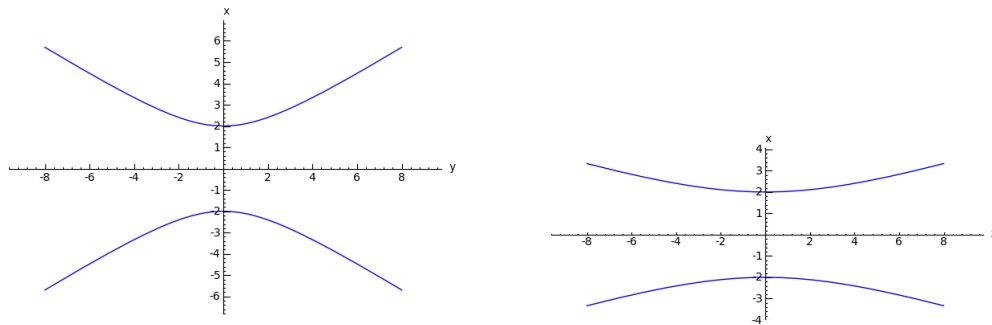
4. Surfaces in Three Dimensions

Given the surface defined by the equation

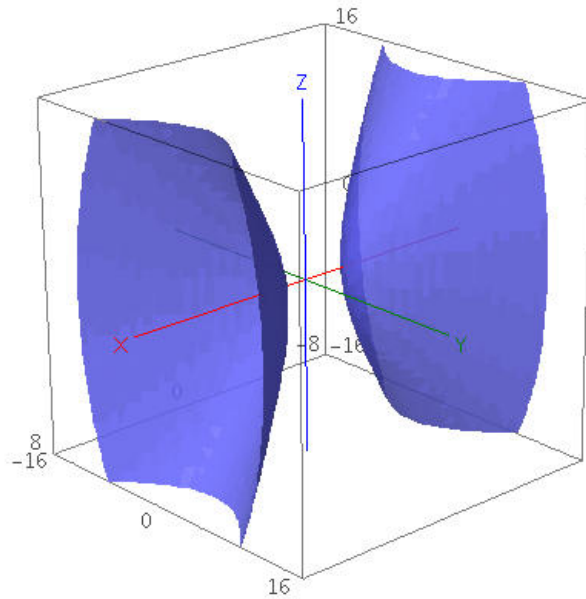
$$9x^2 - 4y^2 - z^2 = 36$$

draw at least three cross-sections *on separate graphs*, then graph the surface in three dimensions.

Solution: I'm going to give you two traces and the final picture; you can figure the rest out for yourself.



You can see from the graphs above that the yz trace would actually have nothing in it. If you want to do any more traces, the best ones to do would be something like the $x = 8$ and $x = -8$ cross-sections.



The surface turns out to be a hyperboloid of two sheets.

5. Functions of Multiple Variables

Draw a contour plot for the function

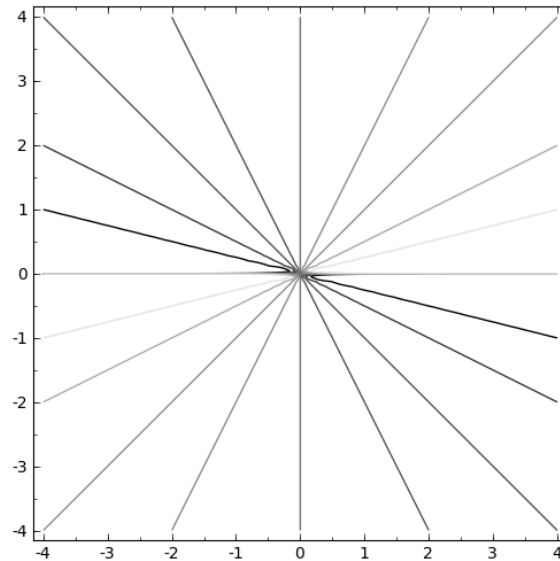
$$f(x, y) = \frac{x}{y}$$

I'm not going to tell you what level sets to draw, or how many; your job is to give me a nice picture of

the function. It's up to you to figure out what are the best contours to draw.

Solution: The process here is to plot the lines

$$\frac{x}{y} = k$$
$$x = ky$$



I've used $k = -4, -2, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, 2, 4$ to generate this plot. Lighter colors represent higher values of k .

6. Limits

Show that the function defined by

$$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$$

has no limit at the origin.

Solution: This is a matter of showing that it has different limits from two different directions. Let's try going toward $(0, 0)$ along the positive x -axis and along the positive y -axis.

Along the positive y -axis, $x = 0$:

$$\begin{aligned} \lim_{y \rightarrow 0^+} \frac{x^2 - y^2}{x^2 + y^2} &= \lim_{y \rightarrow 0^+} \frac{0^2 - y^2}{0^2 + y^2} \\ &= \lim_{y \rightarrow 0^+} -\frac{y^2}{y^2} \\ &= \lim_{y \rightarrow 0^+} -1 \\ &= -1 \end{aligned}$$

Along the positive x -axis, $y = 0$:

$$\begin{aligned}\lim_{x \rightarrow 0^+} \frac{x^2 - y^2}{x^2 + y^2} &= \lim_{x \rightarrow 0^+} \frac{x^2 - 0^2}{x^2 + 0^2} \\ &= \lim_{x \rightarrow 0^+} \frac{x^2}{x^2} \\ &= \lim_{x \rightarrow 0^+} 1 \\ &= 1\end{aligned}$$

7. Directional Derivatives

Given the function

$$f(x, y) = x^4 - y^3$$

- (a) Find a unit vector in the direction of fastest increase at the point $(2, 1)$. What is the rate of increase at this point?

Solution: The vector in part is in the direction of the gradient. First note that

$$\nabla f = 4x^3 \mathbf{i} - 3y^2 \mathbf{j}$$

and

$$\nabla f(2, 1) = 32\mathbf{i} - 3\mathbf{j}$$

The rate of increase in this direction is

$$\|\nabla f(2, 1)\| = \|32\mathbf{i} - 3\mathbf{j}\| = \sqrt{32^2 + (-3)^2} = \sqrt{1033}$$

The unit vector in the appropriate direction will then be

$$\frac{\nabla f(2, 1)}{\|\nabla f(2, 1)\|} = \frac{32}{\sqrt{1033}} \mathbf{i} - \frac{3}{\sqrt{1033}} \mathbf{j}$$

- (b) What is the rate of increase in the direction of \mathbf{i} at $(2, 1)$?

Solution: The rate of increase in the direction of \mathbf{i} is simply the directional derivative in that direction, or

$$D_{\mathbf{i}}(2, 1) = \mathbf{i} \cdot \nabla f(2, 1) = \mathbf{i} \cdot (32\mathbf{i} - 3\mathbf{j}) = 32$$

8. Chain Rule

Suppose

$$w = x^2 + x \sin y; \quad x = se^t, \quad y = st$$

Find $\frac{\partial w}{\partial t}$ using the chain rule and express your answer in terms of s and t .

Solution: First, the chain rule tells us that

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial t}$$

I'll compute the various derivatives here:

$$\frac{\partial w}{\partial x} = 2x + \sin y$$

$$\frac{\partial x}{\partial t} = se^t$$

$$\frac{\partial w}{\partial y} = x \cos y$$

$$\frac{\partial y}{\partial t} = s$$

Now just plug in.

$$\begin{aligned}\frac{\partial w}{\partial t} &= (2x + \sin y)se^t + (x \cos y)s \\ &= (2se^t + \sin st)se^t + (se^t \cos st)s\end{aligned}$$

You can simplify this if you like, but it won't end up looking much simpler.

9. Tangent Planes and Approximations

Show that the surfaces $x^2 + 4y^2 + z^2 = 6$ and $\frac{1}{2}x^4 + 3y^4 - 2y^2 + 4z^2 - 6z = 11$ are tangent at $(1, 1, 1)$. That is, the two surfaces must be touching, and their tangent planes must be parallel.

Solution: First, note that *there is an error in this problem*. The second surface doesn't actually pass through the point $(1, 1, 1)$. I'll just prove that their tangent planes are parallel.

To prove that the tangent planes are parallel, we simply need to find the (perpendicular) direction vectors that define them. Those are found by viewing the surface as being a level set of a function, then taking the gradient. In other words:

$$\begin{aligned}\nabla(x^2 + 4y^2 + z^2)(1, 1, 1) &= (2x\mathbf{i} + 8y\mathbf{j} + 2z\mathbf{k})(1, 1, 1) \\ &= 2\mathbf{i} + 8\mathbf{j} + 2\mathbf{k} \\ \nabla\left[\frac{1}{2}x^4 + 3y^4 - 2y^2 + 4z^2 - 6z\right](1, 1, 1) &= [2x^3\mathbf{i} + (12y^3 - 4y)\mathbf{j} + (8z - 6)\mathbf{k}](1, 1, 1) \\ &= 2\mathbf{i} + 8\mathbf{j} + 2\mathbf{k}\end{aligned}$$

Since these vectors are multiples of one another, the tangent planes are parallel.

10. Maxima and Minima

Find and classify the critical points of the function

$$f(x, y) = x^3 + y^3 - xy$$

Solution: First, find the critical points.

$$\begin{aligned}\nabla f &= 0 \\ (3x^2 - y)\mathbf{i} + (3y^2 - x)\mathbf{j} &= 0\end{aligned}$$

That gives the system of equations

$$\begin{array}{ll} 3x^2 - y = 0 & 3y^2 - x = 0 \\ 3x^2 = y & 3y^2 = x \end{array}$$

Now solve them, by whatever method is convenient.

$$\begin{array}{l} 3x^2 = y \\ y^2 = 9x^4 \end{array}$$

Substitute into the other equation to get

$$\begin{array}{l} 3(9x^4) - x = 0 \\ 27x^4 - x = 0 \\ x(27x^3 - 1) = 0 \end{array}$$

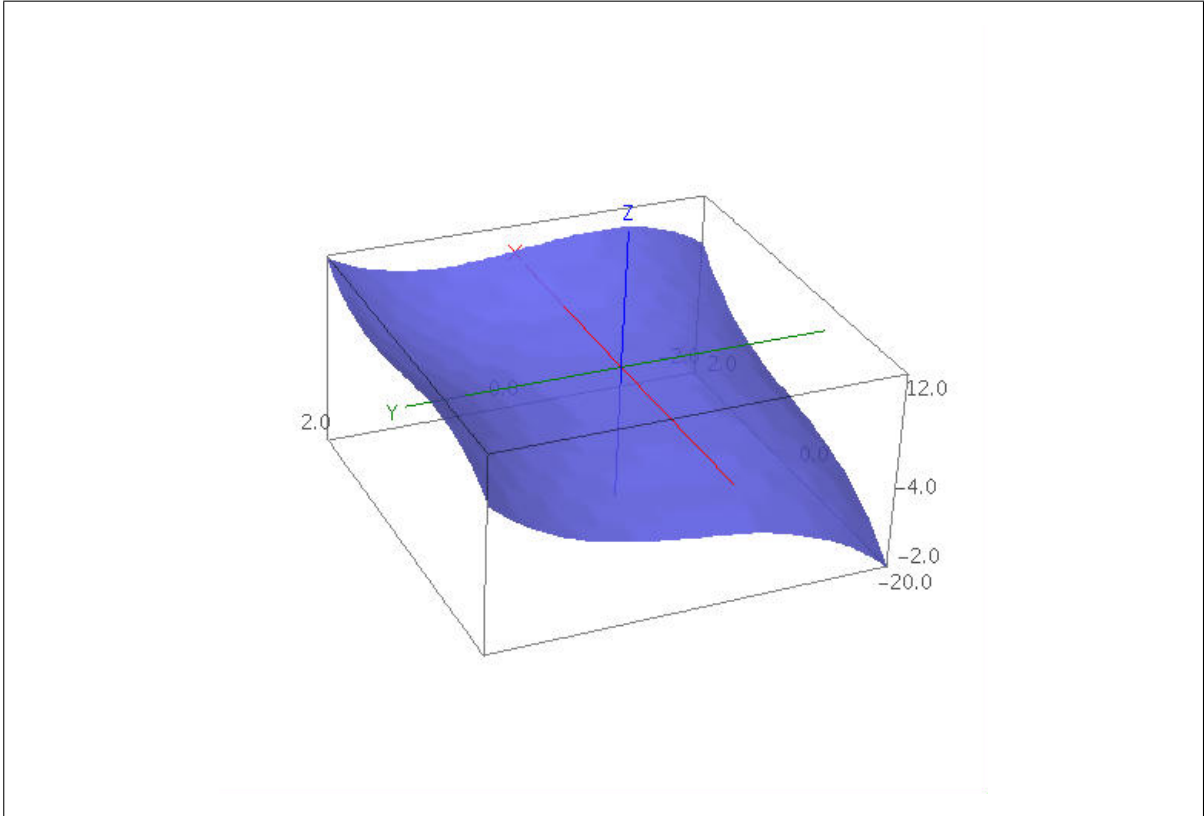
$$\begin{array}{ll} x = 0 & 27x^3 - 1 = 0 \\ x = 0 & x^3 = \frac{1}{27} \\ x = 0 & x = \frac{1}{3} \\ y = 3(0)^2 & y = 3\left(\frac{1}{3}\right)^2 \\ y = 0 & y = \frac{1}{3} \end{array}$$

Our two points, then, are $(0, 0)$ and $(\frac{1}{3}, \frac{1}{27})$. Now to classify them.

$$\begin{aligned} D(x, y) &= f_{xx}f_{yy} - f_{xy}^2 \\ &= (6x)(6y) - (-1)^2 \\ &= 36xy - 1 \end{aligned}$$

Since $D(0, 0) = -1$, that point is a saddle; since $D(\frac{1}{3}, \frac{1}{27}) = -\frac{5}{9}$, that point is also a saddle.

These answers seem pretty reasonable if you know what the surface looks like. And you do, if you can see the graph below.



11. Lagrange Multipliers

- (a) You have a deep and abiding need to build boxes. Today, you have two types of wood to work with. Birch weighs 2 lb per square foot, while spruce weighs 1 lb per square foot. The bottom of the box will be made of birch, while the sides and lid will be made of spruce. If the box must hold 2 cubic feet, what is the lightest box you can build?

Solution: I'm going to call the height of the box h , the width w , and the length l .

The first step is to figure out what our function is, and what our constraint is. The function we are trying to minimize is the weight function, which is

$$\begin{aligned} f(l, w, h) &= 2lw + lw + 2wh + 2lh \\ &= 3lw + 2wh + 2lh \end{aligned}$$

The constraint equation is given by the volume, which is

$$g(l, w, h) = lwh = 2$$

Now set up the equations and solve.

$$\begin{aligned} \nabla f &= \lambda \nabla g \\ 3w + 2h &= \lambda wh \\ 3l + 2h &= \lambda lh \\ 2w + 2l &= \lambda lw \\ lwh &= 2 \end{aligned}$$

This is a bit easier to deal with after a bit of monkeying around. Here I've multiplied each equation by a single variable.

$$3lw + 2lh = \lambda lwh$$

$$3lw + 2wh = \lambda lwh$$

$$2wh + 2lh = \lambda lwh$$

Combine the equations to get

$$3lw + 2lh = 3lw + 2wh = 2wh + 2lh$$

Pick any two of these and start solving.

$$3lw + 2lh = 3lw + 2wh$$

$$2lh = 2wh$$

$$lh - wh = 0$$

$$h(l - w) = 0$$

So then $h = 0$ or $l = w$. But if $h = 0$, then $lwh = 0$, which violates the constraint that $lwh = 2$. So say $l = w$. Then

$$3lw + 2lh = 2wh + 2lh$$

$$3l^2 + 2lh = 2lh + 2lh$$

$$3l^2 - 2lh = 0$$

$$l(3l - 2h) = 0$$

For the same reasons as above, $l \neq 0$, so we now have that $3l = 2h$, or $h = \frac{3}{2}l$.

$$lwh = 2$$

$$\frac{3}{2}l^3 = 2$$

$$l^3 = \frac{4}{3}$$

$$l = \sqrt[3]{\frac{4}{3}}$$

So then

$$l = \sqrt[3]{\frac{4}{3}}$$

$$w = \sqrt[3]{\frac{4}{3}}$$

$$h = \frac{3}{2} \sqrt[3]{\frac{4}{3}} = \sqrt[3]{\frac{9}{2}}$$

(b) Find the maxima and minima of the function

$$f(x, y) = x^2 + y^2 - 4x$$

in the region $2x^2 + y^2 \leq 9$.

Hint: Because this region includes its boundary, it actually represents two separate problems. First find the maximum and minimum on the inside, then find the maximum and minimum on the boundary.

Solution: It's really two problems, as the hint says. One is to find the extrema inside the region.

$$\begin{aligned}\nabla f &= 0 \\ (2x - 4)\mathbf{i} + 2y\mathbf{j} &= 0\end{aligned}$$

So the extremum inside the region is at

$$\begin{aligned}2x - 4 &= 0 & 2y &= 0 \\ x &= 2 & y &= 0\end{aligned}$$

We don't need to test this for now. Just remember that we have the point $(2, 0)$. Now test for extrema on the boundary. To do that, we use Lagrange Multipliers.

$$\begin{aligned}\nabla(x^2 + y^2 - 4x) &= \lambda \nabla(2x^2 + y^2) \\ 2x - 4 &= \lambda 4x \\ 2y &= \lambda 2y \\ 2x^2 + y^2 &= 9\end{aligned}$$

From the second equation, we know that either $y = 0$ or $\lambda = 1$. I'll handle each of these cases individually.

When $y = 0$, plugging into the constraint yields

$$\begin{aligned}2x^2 + 0^2 &= 9 \\ 2x^2 &= 9 \\ x^2 &= \frac{9}{2} \\ x &= \pm \frac{3}{\sqrt{2}} = \pm \frac{3\sqrt{2}}{2}\end{aligned}$$

Now we have the points $(\pm \frac{3\sqrt{2}}{2}, 0)$.

When $\lambda = 1$, plugging into the first equation yields

$$\begin{aligned}2x - 4 &= 4x \\ -2x &= 4 \\ x &= -2\end{aligned}$$

Then plug into the constraint and get

$$\begin{aligned}2(-2)^2 + y^2 &= 9 \\ 8 + y^2 &= 9 \\ y^2 &= 1 \\ y &= \pm 1\end{aligned}$$

This adds one last pair of points, $(-2, \pm 1)$.
 Now plug in all of our points.

$$f(2, 0) = -4 \quad \leftarrow \text{minimum}$$

$$f\left(\frac{3\sqrt{2}}{2}, 0\right) = \frac{9}{2} - 6\sqrt{2} \approx -3.985$$

$$f\left(-\frac{3\sqrt{2}}{2}, 0\right) = \frac{9}{2} + 6\sqrt{2} \approx 12.985$$

$$f(-2, \pm 1) = 13 \quad \leftarrow \text{maximum}$$

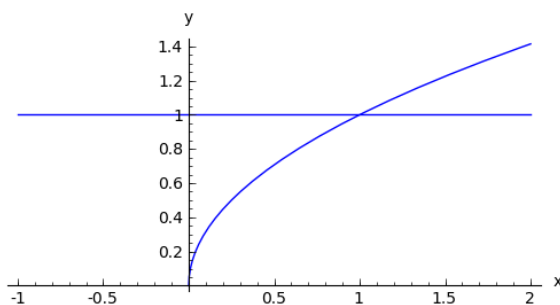
12. Iterated Integrals (changing order of integration)

Evaluate the integral

$$\int_0^1 \int_{\sqrt{x}}^1 2ye^x \, dydx$$

by first exchanging the order of integration.

Solution: The region is the enclosed part of the graph below.



Exchanging the order of integration yields

$$\int_0^1 \int_{\sqrt{x}}^1 2ye^x \, dydx = \int_0^1 \int_0^{y^2} 2ye^x \, dx dy$$

which you shouldn't have too much trouble integrating.

13. Integrals in Polar Coordinates

Find the volume of the solid under the surface $z = x^2 + y^2$, above the xy -plane, and inside the cylinder $x^2 + (y - 1)^2 = 1$.

Hint: If you're having trouble with the equation of the cylinder, try starting with the equation $x^2 + y^2 = 2y$ instead.

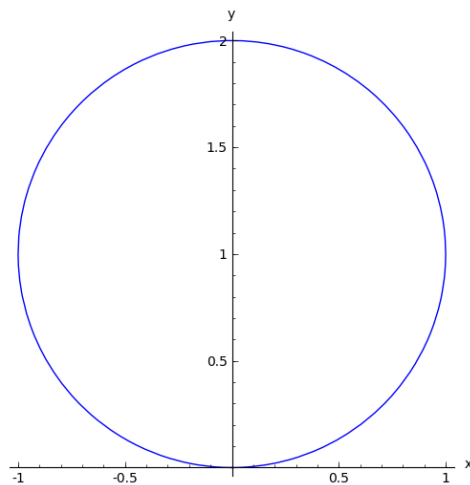
Solution: In polar, these conditions become

$$z = r^2$$

$$z = 0$$

$$r = 2 \sin \theta$$

The region looks like this:



with $0 \leq \theta < \pi$. The integrals turn out to be

$$\int_0^\pi \int_0^{2 \sin \theta} r^2 r dr d\theta$$

I'll let you integrate that yourself.

14. Center of Mass

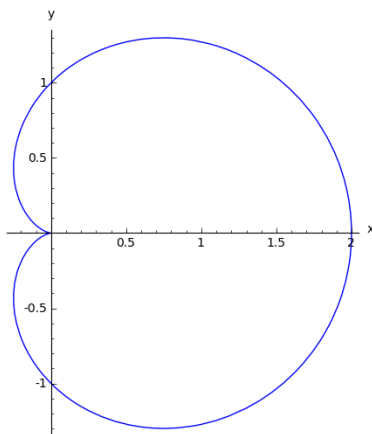
Find the center of mass of the region inside the curve

$$r = 1 + \cos \theta$$

where the density function is $\delta(r, \theta) = r$.

Hint: You know how to do this. The trick is to not panic. It's best to do this in polar, but all else fails, you can just convert everything to Cartesian coordinates.

Solution: The region looks like this:



Clearly $\bar{y} = 0$, by symmetry. The formula for \bar{x} is

$$\bar{x} = \frac{M_y}{m}$$

so now we need to find the mass, and the moment about the y -axis. Here are the integrals, all set up for you.

$$M_y = \iint_S xr \, dA = \int_0^{2\pi} \int_0^{1+\cos\theta} r^2 \cos\theta \, r \, dr \, d\theta$$

$$m = \iint_S r \, dA = \int_0^{2\pi} \int_0^{1+\cos\theta} r \, r \, dr \, d\theta$$

I'll let you evaluate those yourself. They're a bit beastly; it's more important to understand the set-up than how to actually integrate these monsters.

15. Surface Area

Find the area of the surface $z = x^2 - y^2$ inside the region $x^2 + y^2 \leq 9$.

Solution: We're trying to find the surface area of the surface defined by the function $f(x, y) = x^2 - y^2$. This looks horrible, but it isn't. Let's just plug into the surface area formula

$$SA = \iint_S \sqrt{f_x^2 + f_y^2 + 1} \, dA$$

Here are the partials:

$$f_x = 2x$$

$$f_y = -2y$$

Plugging in, we get

$$SA = \iint_S \sqrt{(2x)^2 + (-2y)^2 + 1} \, dA$$

$$= \iint_S \sqrt{4x^2 + 4y^2 + 1} \, dA$$

This looks terrible, and it is. Unless, of course, you convert to polar, in which case it becomes

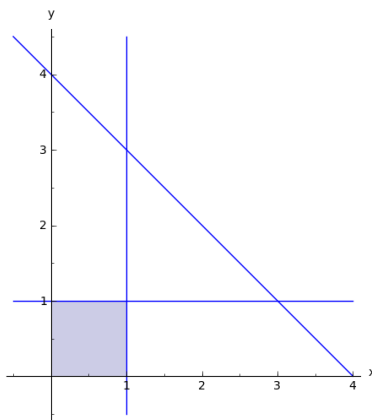
$$SA = \int_0^{2\pi} \int_0^3 \sqrt{4r^2 + 1} r dr d\theta$$

which is almost a vacation compared to the stuff I normally make you integrate.

16. Triple Integrals

Find the volume of the region in the first octant bounded by the planes $x + y + z = 4$, $y = 1$, and $x = 1$.

Solution: The xy -trace looks like this:



I've shaded the region in question. The x and y bounds are fairly straightforward — just a square — while the lower z -bound is 0 and the upper z -bound is given by the equation of the plane, i.e.,

$$z = 4 - x - y$$

Then the integrals turn out to be

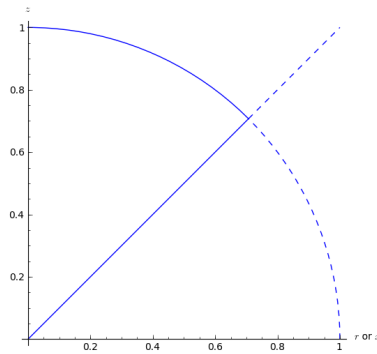
$$\int_0^1 \int_0^1 \int_0^{4-x-y} 1 dz dx dy$$

Again I will allow you to evaluate this yourself. Let me know if you run into any trouble.

17. Triple Integrals in Cylindrical and Spherical

Find the volume of the solid bounded above by the sphere $x^2 + y^2 + z^2 = 1$ and below by the cone $x^2 + y^2 = z^2$

Solution: First the cross-section in the rz -plane, i.e. $y = 0$, common to both parts:



I've dashed the irrelevant parts of the cross-section to make it clear what the solid actually is.

(a) Using cylindrical coordinates

Solution: In cylindrical, the equations become

$$z = \sqrt{1 - r^2}$$

$$z = r$$

The shadow of this region on the xy -plane is a circle of radius one, so the integrals turn out to be

$$\int_0^{2\pi} \int_0^1 \int_r^{\sqrt{1-r^2}} 1 r dz dr d\theta$$

(b) Using spherical coordinates

Solution: In spherical, the equations become

$$\rho = 1$$

$$\phi = \frac{\pi}{4}$$

The second equation — the equation for the cone — is a little hard to see, unless you look at the cross-section diagram above. It's pretty clear that the line corresponding to the cone is at an angle $\frac{\pi}{4}$ below the z -axis. This time the integrals turn out to be

$$\int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^1 1 \rho^2 \sin \phi d\rho d\phi d\theta$$

18. Change of Variables Write down the formulas for a change of variables from Cartesian to Spherical (or vice versa), then prove that the Jacobian of this transformation is $\rho^2 \sin \phi$.

Solution: The formulas, which you should know, are

$$x = \rho \sin \phi \cos \theta$$

$$y = \rho \sin \phi \sin \theta$$

$$z = \rho \cos \phi$$

The Jacobian is given by

$$\begin{aligned} J(\rho, \theta, \phi) &= \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix} \\ &= \begin{vmatrix} \sin \phi \cos \theta & \rho \cos \phi \cos \theta & -\rho \sin \phi \sin \theta \\ \sin \phi \sin \theta & \rho \cos \phi \sin \theta & \rho \sin \phi \cos \theta \\ \cos \phi & -\rho \sin \phi & 0 \end{vmatrix} \\ &= (0 + \rho^2 \sin^3 \phi \cos^2 \theta) + (\rho^2 \cos^2 \phi \sin \phi \cos^2 \theta + 0) + (\rho^2 \sin^3 \phi \sin^2 \theta + \rho^2 \cos^2 \phi \sin \phi \sin^2 \theta) \\ &= \rho^2 \sin^3 \phi \cos^2 \theta + \rho^2 \cos^2 \phi \sin \phi \cos^2 \theta + \rho^2 \sin^3 \phi \sin^2 \theta + \rho^2 \cos^2 \phi \sin \phi \sin^2 \theta \\ &= \rho^2 \sin \phi \cos^2 \theta (\cos^2 \phi + \sin^2 \phi) + \rho^2 \sin \phi \sin^2 \theta (\sin^2 \phi + \cos^2 \phi) \\ &= \rho^2 \sin \phi \cos^2 \theta + \rho^2 \sin \phi \sin^2 \theta \\ &= \rho^2 \sin \phi (\cos^2 \theta + \sin^2 \theta) \\ &= \rho^2 \sin \phi \end{aligned}$$

No sweat.

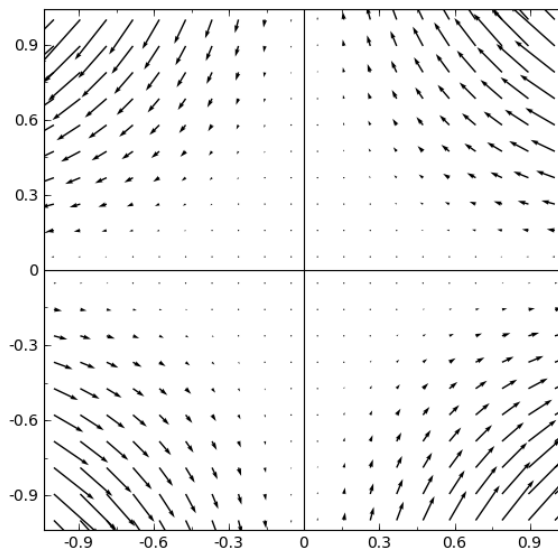
19. Vector Fields

Given the vector field

$$\mathbf{F}(x, y, z) = -x^2 y \mathbf{i} + y^2 x \mathbf{j}$$

- (a) Plot the vector field. The lengths of the vectors get large quite quickly, so you probably won't be able to plot too many.

Solution: The vector field looks like this:



(b) Make a conjecture about the curl and divergence of the vector field.

Solution: I'm going to guess that there's positive curl, because it's kind of generally spiraling counterclockwise. What about the divergence? Who can possibly know? Let's find out!

(c) Calculate the curl and divergence of the vector field.

Solution: Divergence:

$$\begin{aligned}\nabla \cdot \mathbf{F} &= \frac{\partial}{\partial x}(-x^2y) + \frac{\partial}{\partial y}(y^2x) \\ &= -2xy + 2xy \\ &= 0\end{aligned}$$

Curl – this is a 2-dimensional vector field, so we can use the quick formula:

$$\begin{aligned}\nabla \times \mathbf{F} &= \left(\frac{\partial}{\partial x}(y^2x) - \frac{\partial}{\partial y}(-x^2y) \right) \mathbf{k} \\ &= (y^2 + x^2) \mathbf{k}\end{aligned}$$

20. Line Integrals

Suppose the force of gravity on your satellite in orbit of the Earth is given by the vector field

$$\mathbf{F}(x, y, z) = -\frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}$$

Calculate the work required to move your satellite along a straight line from its launch position of $(0, 10, 0)$ to orbit at $(1, 10, 0)$.

Solution: This is actually a conservative vector field, but let's forget about that for a moment and do this the old-fashioned way.

First, let's parameterize the curve.

$$\mathbf{r}(t) = t\mathbf{i} + 10\mathbf{j} \quad 0 \leq t \leq 1$$

Now the integral is

$$\begin{aligned}
 \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C Mdx + Ndy + Pdz \\
 &= \int_C -\frac{x}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} dx - \frac{y}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} dy - \frac{z}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} dz \\
 &= \int_0^1 -\frac{t}{(t^2 + 10^2 + 0^2)^{\frac{3}{2}}} dt - \frac{10}{(t^2 + 10^2 + 0^2)^{\frac{3}{2}}} 0dt - \frac{0}{(t^2 + 10^2 + 0^2)^{\frac{3}{2}}} 0dt \\
 &= \int_0^1 -\frac{t}{(t^2 + 10^2 + 0^2)^{\frac{3}{2}}} dt \\
 &= \frac{1}{(t^2 + 100)^{\frac{1}{2}}} \Big|_0^1 \\
 &= \frac{1}{(1^2 + 100)^{\frac{1}{2}}} - \frac{1}{(0^2 + 100)^{\frac{1}{2}}} \\
 &= \frac{1}{(101)^{\frac{1}{2}}} - \frac{1}{(100)^{\frac{1}{2}}} \\
 &\approx -0.000496
 \end{aligned}$$

This is the amount of work that the vector field did on *you*; the amount of work you do is 0.000496.

21. Independence of Path

Show that

$$\mathbf{F}(x, y, z) = yze^{xy}\mathbf{i} + xze^{xy}\mathbf{j} + (e^{xy} + 1)\mathbf{k}$$

is a conservative force, and then find the work required to move a particle from $(0, 0, 0)$ to $(\ln(2), 1, 5)$.

Solution: First, show that it's a conservative force. That's simply a matter of proving that the curl is zero:

$$\begin{aligned}
 \nabla \times \mathbf{F} &= \left\| \begin{array}{ccc} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yze^{xy} & xze^{xy} & e^{xy} + 1 \end{array} \right\| \\
 &= (xe^{xy} - xe^{xy})\mathbf{i} + (ye^{xy} - ye^{xy})\mathbf{j} + (ze^{xy} + xye^{xy} - ze^{xy} - xye^{xy})\mathbf{k} \\
 &= 0
 \end{aligned}$$

Great. Now we can find f such that $\nabla f = \mathbf{F}$, which will make evaluating the integral quite a bit easier.

$$f(x, y, z) = \int yze^{xy} dx = ze^{xy} + C(y, z)$$

$$f_y = \frac{\partial}{\partial y}(ze^{xy} + C(y, z))$$

$$xze^{xy} = xze^{xy} + C_y(y, z)$$

$$C_y(y, z) = 0$$

So C is really a function of z alone.

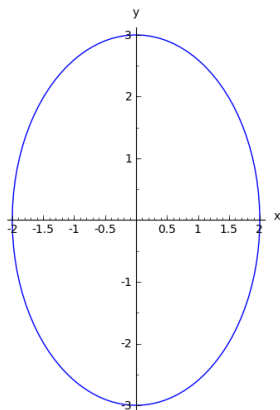
$$\begin{aligned}f(x, y, z) &= ze^{xy} + C_y(y, z) \\f_z &= \frac{\partial}{\partial z}(ze^{xy} + C(z)) \\e^{xy} + 1 &= e^{xy} + C_z(z) \\C_z(z) &= 1\end{aligned}$$

So $C(z) = z$, and $f(x, y, z) = ze^{xy} + z$.

22. Greene's Theorem in the Plane

Use Greene's Theorem to find the area of the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ by evaluating a line integral around the outside. This will require you to parameterize the curve representing the boundary of the ellipse.

Solution: First let's parameterize the curve. It looks like this:



It's an ellipse, i.e. a stretched circle, with radius 3 in the y direction and 2 in the x direction. That means that a pretty good parameterization would be a stretched version of the parameterization of a circle, i.e.

$$\mathbf{r}(t) = 2 \cos t \mathbf{i} + 3 \sin t \mathbf{j} \quad 0 \leq t < 2\pi$$

Now we want to re-write the area integral using Greene's Theorem:

$$\iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA = \oint_C M dx + N dy$$

We can choose, for instance, $M = -\frac{1}{2}y$, $N = \frac{1}{2}x$.

$$\begin{aligned} \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA &= \oint_C M dx + N dy \\ \iint_S 1 dA &= \oint_C -\frac{1}{2}y dx + \frac{1}{2}x dy \\ &= \int_0^{2\pi} -\frac{1}{2}3 \sin t(-2 \sin t) dt + \frac{1}{2}2 \cos t(3 \cos t) dt \\ &= \int_0^{2\pi} 3 \sin^2 t + 3 \cos^2 t dt \\ &= \int_0^{2\pi} 3 dt \\ &= 3t \Big|_0^{2\pi} \\ &= 6\pi \end{aligned}$$

23. Surface Integrals

Find the mass of a hemisphere of radius 4, if its density is given by $\delta(x, y) = z$.

Hint: Evaluating this integral will require you to be immensely clever about how you simplify it, but once you succeed, the integral is quite easy.

Solution: The hemisphere would have equation

$$x^2 + y^2 + z^2 = 4$$

and sits above a circle of radius 2 in the xy -plane. It is described by the function

$$f(x, y) = \sqrt{4 - x^2 - y^2}$$

The formula for the mass of the hemisphere is

$$\begin{aligned} \iint_S \delta(x, y) \sqrt{f_x^2 + f_y^2 + 1} dA &= \iint_S z \sqrt{\left(\frac{-x}{\sqrt{4 - x^2 - y^2}} \right)^2 + \left(\frac{-y}{\sqrt{4 - x^2 - y^2}} \right)^2 + 1} dA \\ &= \iint_S \sqrt{4 - x^2 - y^2} \sqrt{\frac{x^2}{4 - x^2 - y^2} + \frac{y^2}{4 - x^2 - y^2} + 1} dA \\ &= \iint_S \sqrt{4 - x^2 - y^2} \sqrt{\frac{x^2 + y^2 + 4 - x^2 - y^2}{4 - x^2 - y^2}} dA \\ &= \iint_S \sqrt{4 - x^2 - y^2} \frac{2}{\sqrt{4 - x^2 - y^2}} dA \\ &= \iint_S 2 dA \end{aligned}$$

Now, the area of S is $\pi r^2 = 4\pi$, so the result is $2(4\pi) = 8\pi$.