Final Exam Practice Problems

Math 2210-001: Calculus III – Fall 2009

This problem set should not be considered comprehensive. It should be regarded instead as a guide for your own studying. The best way to use it is to run through the problems here, then once you have figured out where the gaps in your knowledge lie, study those sections in the book.

Chapter 11

1. Dot/Cross Product

   Prove that $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$

   Hint: Write $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *as components, then just plug in and verify.*

2. Lines and Tangent Lines

   Find a equation for the line through $(4, 0, 6)$ and perpendicular to the plane $x - 5y + 2z = 10$.

3. Curvature and Acceleration

   (a) Sketch the following curve in the xy-plane, then compute its curvature.

   $$y^2 - 4x^2 = 20$$

   (b) Find the curvature κ, the unit tangent vector $\mathbf{T}$, and the unit normal vector $\mathbf{N}$ for the curve

   $$\mathbf{r}(t) = \frac{1}{2}t^2\mathbf{i} + t\mathbf{j} + \frac{1}{3}t^3\mathbf{k}$$

   when $t = 2$.

4. Surfaces in Three Dimensions

   Given the surface defined by the equation

   $$9x^2 - 4y^2 - z^2 = 36$$

   draw at least three cross-sections *on separate graphs*, then graph the surface in three dimensions.

Chapter 12

5. Functions of Multiple Variables

   Draw a contour plot for the function

   $$f(x, y) = \frac{x}{y}$$

   I'm not going to tell you what level sets to draw, or how many; your job is to give me a nice picture of the function. It's up to you to figure out what are the best contours to draw.

6. Limits

   Show that the function defined by
   $$f(x,y) = \frac{x^2 - y^2}{x^2 + y^2}$$
   has no limit at the origin.

7. Directional Derivatives

   Given the function
   $$f(x,y) = x^4 - y^3$$
   (a) Find a unit vector in the direction of fastest increase at the point $(2,1)$. What is the rate of increase at this point?

   (b) What is the rate of increase in the direction of $\mathbf{i}$ at $(2,1)$?

8. Chain Rule Suppose
   $$w = x^2 + x\sin y;\ x = se^t,\ y = st$$
   Find $\frac{\partial w}{\partial t}$ *using the chain rule* and express your answer in terms of s and t.

9. Tangent Planes and Approximations

   Show that the surfaces $x^2 + 4y^2 + z^2 = 6$ and $\frac{1}{2}x^4 + 3y^4 - 2y^2 + 4z^2 - 6z = 11$ are tangent at $(1,1,1)$. That is, the two surfaces must be touching, and their tangent planes must be parallel.

10. Maxima and Minima

    Find and classify the critical points of the function
    $$f(x,y) = x^3 + y^3 - xy$$

11. Lagrange Multipliers

    (a) You have a deep and abiding need to build boxes. Today, you have two types of wood to work with. Birch weighs 2 lb per square foot, while spruce weighs 1 lb per square foot. The bottom of the box will be made of birch, while the sides and lid will be made of spruce. If the box must hold 2 cubic feet, what is the lightest box you can build?

    (b) Find the maxima and minima of the function
    $$f(x,y,z) = x^2 + y^2 - 4x$$
    in the region $2x^2 + y^2 \leq 9$.

    Hint: Because this region includes its boundary, it actually represents two separate problems. First find the maximum and minimum on the inside, then find the maximum and minimum on the boundary.

Chapter 13

12. Iterated Integrals (changing order of integration)

    Evaluate the integral
    $$\int_0^1 \int_{\sqrt{x}}^1 2ye^x\, dydx$$
    by first exchanging the order of integration.

13. Integrals in Polar Coordinates

    Find the volume of the solid under the surface $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + (y-1)^2 = 1$.

    Hint: If you're having trouble with the equation of the cylinder, try starting with the equation $x^2 + y^2 = 2y$ instead.

14. Center of Mass

    Find the center of mass of the region inside the curve

    $$r = 1 + \cos\theta$$

    where the density function is $\delta(r, \theta) = r$.

    Hint: You know how to do this. The trick is to not panic. It's best to do this in polar, but all else fails, you can just convert everything to Cartesian coordinates.

15. Surface Area

    Find the area of the surface $z = x^2 - y^2$ inside the region $x^2 + y^2 \leq 9$.

16. Triple Integrals

    Find the volume of the region in the first octant bounded by the planes $x + y + z = 4$, $y = 1$, and $x = 1$.

17. Triple Integrals in Cylindrical and Spherical

    Find the volume of the solid bounded above by the sphere $x^2 + y^2 + z^2 = 1$ and below by the cone $x^2 + y^2 = z^2$

    (a) Using cylindrical coordinates

    (b) Using spherical coordinates

18. Change of Variables

    Write down the formulas for a change of variables from Cartesian to Spherical (or vice verse), then prove that the Jacobian of this transformation is $\rho^2 \sin\phi$.

Chapter 14

19. Vector Fields

    Given the vector field

    $$\mathbf{F}(x, y, z) = -x^2 y\mathbf{i} + y^2 x\mathbf{j}$$

    (a) Plot the vector field. The lengths of the vectors get large quite quickly, so you probably won't be able to plot too many.

    (b) Make a conjecture about the curl and divergence of the vector field.

    (c) Calculate the curl and divergence of the vector field.

20. Line Integrals

    Suppose the force of gravity on your satellite in orbit of the Earth is given by the vector field

    $$\mathbf{F}(x, y, z) = -\frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}$$

Calculate the work required to move your satellite along a straight line from its launch position of $(0, 10, 0)$ to orbit at $(1, 10, 0)$.

21. Independence of Path

    Show that
    $$F(x, y, z) = yze^{xy}\mathbf{i} + xze^{xy}\mathbf{j} + (e^{xy} + 1)\mathbf{k}$$
    is a conservative force, and then find the work required to move a particle from $(0, 0, 0)$ to $(\ln(2), 1, 5)$.

22. Greene's Theorem in the Plane

    Use Greene's Theorem to find the area of the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ by evaluating a line integral around the outside. This will require you to parameterize the curve representing the boundary of the ellipse.

23. Surface Integrals

    Find the mass of a hemisphere of radius 4, if its density is given by $\delta(x, y) = z$.

    Hint: Evaluating this integral will require you to be immensely clever about how you simplify it, but once you succeed, the integral is quite easy.