Exam 3 Practice Problems

Math 2210-001: Calculus III – Fall 2009

1. Find the extreme values of $f(x,y) = x^3 + 2y^3$ given the constraint $x^2 + y^2 = 1$. *(Hint: You should find six candidate points.)*
2. Find the point on the sphere of radius 1, centered at the origin, which is closest to the point $(4,2,1)$.
3. Find the volume of the region in the first octant bounded by the plane $x + 2y + 3z = 6$.
4. Evaluate the integral
$$\int_1^e \int_0^{\ln x} y \, dy \, dx$$
(Hint: It is entirely possible that this is not integrable in its present form.)
5. Find the area of one leaf of a the three-leafed rose defined by the curve $r = 2\sin 3\theta$.
6. Evaluate the integral
$$\int_0^1 \int_0^{\sqrt{1-x^2}} e^{-(x^2+y^2)} \, dy \, dx$$
(Hint: It is entirely possible that this is not integrable in its present form.)
7. Find the surface area of the piece of the sphere $x^2 + y^2 + z^2 = 4$ that is inside the cylinder $x^2 + y^2 = 1$.
8. Find the surface area of the portion of the parabolic cylinder $z = x^2$ that is directly above the triangle with vertices $(0,0,0)$, $(1,0,0)$, and $(1,1,0)$.
9. Evaluate the integral
$$\iiint_D xy \, dV$$
where D is the solid region in the first octant bounded above by the hemisphere $z = \sqrt{4 - x^2 - y^2}$.
10. Find the mass of the cube C with sides of length 2, given that the density at a point is equal to the square of the distance from the origin.
11. Find the center of mass of the tetrahedron in the first octant, bounded by the plane $x + y + z = 1$, if the density at a point is equal to the sum of the point's coordinates.
12. Using an integral in cylindrical coordinates, find the volume of the region bounded above by the sphere $x^2 + y^2 + z^2 = 2$ and below by the paraboloid $z = x^2 + y^2$.
13. Using an integral in spherical coordinates, find the volume of the portion of the sphere $x^2 + y^2 + z^2 = 4$ above the xy-plane and inside the cone $z^2 = 3(x^2 + y^2)$.
14. Evaluate the integral
$$\iint_R e^{(y-x)^2} \, dA$$
where R is the region bounded by $y = \frac{3}{2}x$, $y = 2x$, and $y = x + 1$. Use the coordinate transformation $x = u + v$, $y = u + 2v$.
15. Evaluate the integral
$$\iint_R 49x^2 y \, dA$$
where R is the region bounded by $2x - y = 1$, $2x - y = -2$, $x + 3y = 0$, $x + 3y = 1$. Use a clever change of variables.

1. $f(x,y) = x^3 + 2y^3$ $\qquad$ $g(x,y) = x^2 + y^e$ since $x^2+y^2=1$

$f_x = 3x^2$ $\qquad$ $g_x = 2x$

$f_y = 6x^2$ $\qquad$ $g_y = 2y$

So
$$3x^2 = \lambda \cdot 2x$$
$$6y^2 = \lambda \cdot 2y$$
$$x^2+y^2=1$$

$3x^2 = \lambda \cdot 2x$:
$$3x^2 - \lambda \cdot 2x = 0$$
$$x(3x-2\lambda)=0$$
$x=0$ or $x = \frac{2}{3}\lambda$

$$6y^2 = \lambda \cdot 2y$$
$$6y^2 - \lambda \cdot 2y = 0$$
$$2y(\quad - \lambda) = 0$$
$y=0$ or $\cancel{x}y = \frac{1}{3}\lambda$

$x=0$:
$0^2+y^2=1 \rightsquigarrow (0,1)$
$y = \pm 1 \quad (0,-1)$

$y=0$:
$x^2+0^2=1 \rightsquigarrow (1,0)$
$x = \pm 1 \quad (-1,0)$

Ok. If $y \neq 0$, then
$$3y = \lambda$$
But $x=0 \Rightarrow \lambda=0 \Rightarrow y=0$, which fails the constraint. So:
$$3y = \lambda \qquad x = 2/3\,\lambda$$
$$\frac{3}{2}x = \lambda$$
$$3y = \frac{3}{2}x$$
$$2y = x$$
So:
$$(2y)^2 + y^2 = 1$$
$$5y^2 = 1$$
$$y^2 = \frac{1}{5}$$
$$y = \pm\frac{1}{\sqrt{5}}$$
$$2\left(\pm\frac{1}{\sqrt{5}}\right) = x$$
$$\pm\frac{2}{\sqrt{5}} = x$$
$\rightsquigarrow \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$, $\left(-\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right)$

$f(0,1) = 0^3 + 2\cdot 1^3 = 2 \leftarrow$ Max
$f(0,-1) = 0^3 + 2(-1)^3 = -2 \leftarrow$ Min
$f(1,0) = 1^3 + 2\cdot 0^3 = 1$
$f(-1,0) = (-1)^3 + 2\cdot 0^3 = -1$
$f\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right) = \frac{8}{5^{3/2}} + \frac{2}{5^{3/2}} = \frac{10}{5\sqrt{5}} = \frac{2}{\sqrt{5}}$
$f\left(-\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right) = -\frac{8}{5^{3/2}} - \frac{2}{5^{3/2}} = -\frac{10}{5\sqrt{5}} = -\frac{2}{\sqrt{5}}$

Maximum: $\boxed{2}$
Minimum: $\boxed{-2}$

2. $x^2+y^2+z^2=1$

$\rightsquigarrow g(x,y,z)=x^2+y^2+z^2$

$f(x,y,z)=\cancel{\sqrt{(x-4)^2+(y-2)^2+(z-1)^2}}$ &

$f(x,y,z) = (x-4)^2+(y-2)^2+(z-1)^2$

$$\nabla f = \lambda \nabla g$$

$$\cancel{2}(x-4)=\lambda\cdot\cancel{2}x$$
$$\cancel{2}(y-2)=\lambda\cdot\cancel{2}y$$
$$\cancel{2}(z-1)=\lambda\cdot\cancel{2}z$$
$$x^2+y^2+z^2=1$$

$x-4=\lambda x$
$(1-\lambda)x=4$
$x=\frac{4}{1-\lambda}$

$y-2=\lambda y$
$(1-\lambda)y=2$
$y=\frac{2}{1-\lambda}$

$z-1=\lambda z$
$(1-\lambda)z=1$
$z=\frac{1}{1-\lambda}$

$$x=2y=4z$$
$$(4z)^2+(2z)^2+z^2=1$$
$$16z^2+4z^2+z^2=1$$
$$21z^2=1$$
$$z^2=1/21$$
$$z=\pm 1/\sqrt{21} \rightsquigarrow x=\pm 4/\sqrt{21},\ y=\pm 2/\sqrt{21}$$

so: $(4/\sqrt{21}, 2/\sqrt{21}, 1/\sqrt{21})$ & $(-4/\sqrt{21}, -2/\sqrt{21}, -1/\sqrt{21})$

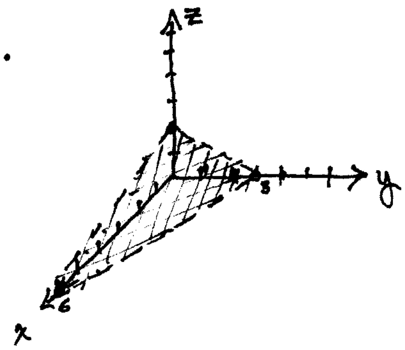
$f(4/\sqrt{21}, 2/\sqrt{21}, 1/\sqrt{21})=$ It's tempting to plug in & check, but it's easier than this.

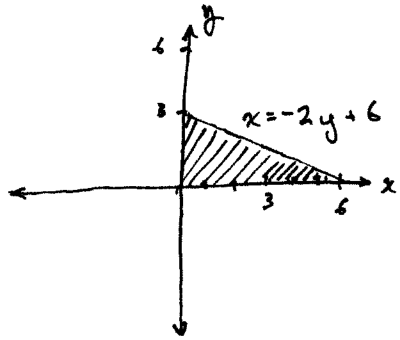
One of these is closest to $(4,2,1)$, & one of these is farthest from $(4,2,1)$.

Clearly the closest point is

$$\boxed{(4/\sqrt{21}, 2/\sqrt{21}, 1/\sqrt{21})}$$

3.





← This is the plane $z=0$, so the curve on top is

$$x + 2y + 3(0) = 6$$

$$x = -2y + 6$$

The HEIGHT of the solid is

$$z = 2 - \frac{2}{3}y - \frac{1}{3}x$$

THUS:

$$\int_0^3 \int_0^{-2y+6} 2 - \frac{2}{3}y - \frac{1}{3}x \, dx \, dy = \int_0^3 2x - \frac{2}{3}xy - \frac{1}{6}x^2 \Big|_0^{-2y+6} dy$$

$$= \int_0^3 2(-2y+6) - \frac{2}{3}(-2y+6) - \frac{1}{6}(-2y+6)^2 \, dy$$

$$= \int_0^3 -4y + 12 + \frac{4}{3}y - 4 - \frac{1}{6}(4y^2 + {}^{-}24y + 36) \, dy$$

$$= \int_0^3 -\frac{8}{3}y + 8 - \frac{2}{3}y^2 + 4y - 6 \, dy$$

$$= \int_0^3 -\frac{2}{3}y^2 + \frac{4}{3}y + 2 \, dy$$

$$= -\frac{2}{9}y^3 + \frac{2}{3}y^2 + 2y \Big|_0^3$$

$$= (-6 + 6 + 6) - (0 + 0 + 0)$$

$$= \boxed{6}$$

4. $\int_1^e \int_0^{\ln x} y \, dy \, dx$

switch orders.

$\int_0^1 \int_0^{e^y} y \, dx \, dy$

$= \int_0^1 xy \Big|_0^{e^y} dy$

$= \int_0^1 y e^y \, dy$ $\quad u = y \quad du = 1$ $\quad dv = e^y \quad v = e^y$ Integration by parts

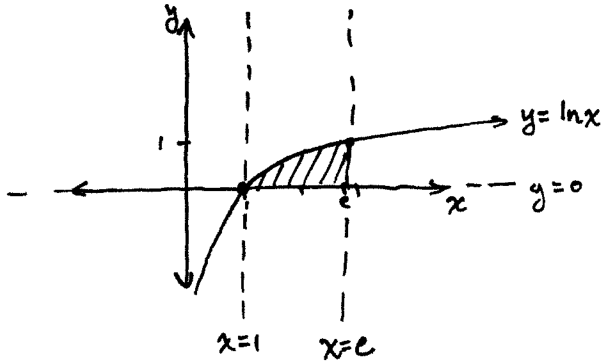
$= y e^y \Big|_0^1 - \int_0^1 e^y \, dy$

$= (1 \cdot e^1 - 0 \cdot e^0) - e^y \Big|_0^1$

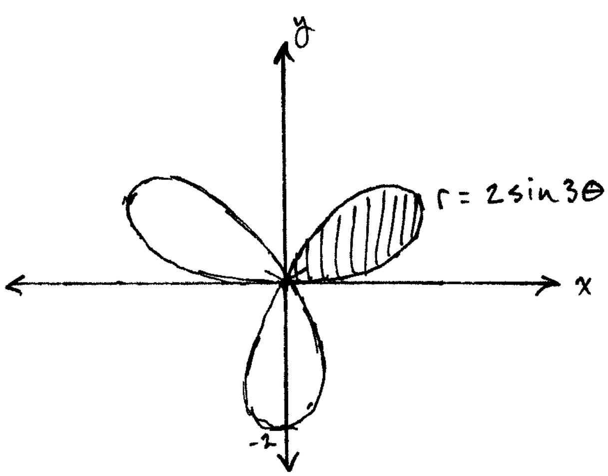
$= e - (e^1 - e^0)$

$= e - e + 1$

$= \boxed{1}$



5.



$\int_0^{\pi/3} \int_0^{2\sin 3\theta} 1 \cdot r \, dr \, d\theta$

$= \int_0^{\pi/3} \frac{1}{2} r^2 \Big|_0^{2\sin 3\theta} d\theta$

$= \int_0^{\pi/3} \frac{1}{2} (2\sin 3\theta)^2 \, d\theta$

$= 2\int_0^{\pi/3} \sin^2 3\theta \, d\theta$

$= 2\int_0^{\pi/3} \cdot \frac{1}{2} (1 - \sin 6\theta) \, d\theta$

$= \int_0^{\pi/3} 1 - \sin 6\theta \, d\theta$

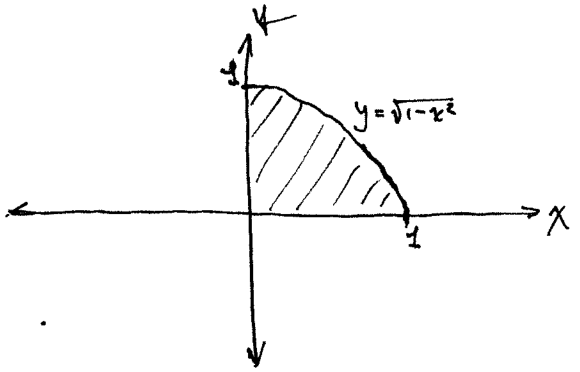
$= \left(\theta + \frac{1}{6}\cos 6\theta\right)\Big|_0^{\pi/3}$

$= \pi/3 + \frac{1}{6}\cos 2\pi - 0 - \frac{1}{6}\cos 0$

$= \pi/3 + \frac{1}{6} - \frac{1}{6}$

$= \boxed{\pi/3}$

6.



$$\int_0^1 \int_0^{\sqrt{1-x^2}} e^{-(x^2+y^2)}\, dy\, dx$$

$$= \int_0^{\pi/2} \int_0^1 e^{-r^2}\, r\, dr\, d\theta$$

$$= \int_0^{\pi/2} -\tfrac{1}{2} e^{-r^2} \Big|_0^1 d\theta$$

$$= \int_0^{\pi/2} -\tfrac{1}{2}\left[e^{-1} - e^0\right] d\theta$$

$$= \int_0^{\pi/2} -\tfrac{1}{2}\left[e^{-1} - 1\right] d\theta$$

$$= -\tfrac{1}{2}\left[e^{-1} - 1\right]\theta \Big|_0^{\pi/2}$$

$$= -\tfrac{1}{2}\left[e^{-1} - 1\right]\tfrac{\pi}{2} = \boxed{-\tfrac{\pi}{4}\left[e^{-1} - 1\right]}$$

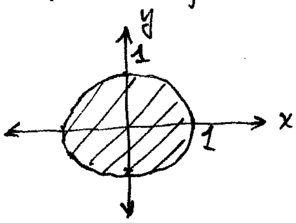
Note If this answer troubles you, good! But it's OK - the result is positive, since $e^{-1} - 1 < 0$.

7. So the function is really:

$$f(x,y) = z = \sqrt{4 - x^2 - y^2}$$

What is the region? It's over $x^2+y^2=1$:

So:



$$f_x = -\frac{x}{\sqrt{4-x^2-y^2}}$$

$$f_y = -\frac{y}{\sqrt{4-x^2-y^2}}$$

&:

$$\sqrt{\left(\frac{-x}{\sqrt{4-x^2-y^2}}\right)^2 + \left(\frac{y}{-\sqrt{4-x^2-y^2}}\right)^2 + 1}$$

$$= \sqrt{\frac{x^2+y^2+4-x^2-y^2}{4-x^2-y^2}}$$

$$= \sqrt{\frac{4}{4-x^2-y^2}}$$

And we want to integrate:

$$\iint_R \sqrt{\frac{4}{4-x^2-y^2}}\, dA \xrightarrow[\text{polar!}]{\text{in}} \int_0^{2\pi}\int_0^1 \sqrt{\frac{4}{4-r^2}}\, r\, dr\, d\theta$$

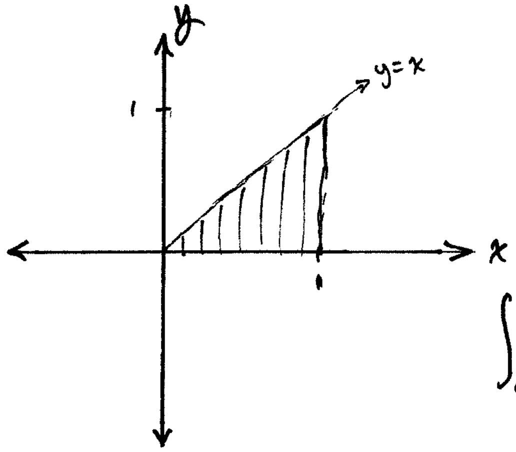
$$= \int_0^{2\pi}\int_0^1 \frac{2r}{(4-r^2)^{1/2}}\, dr\, d\theta = -2\int_0^{2\pi} (4-r^2)^{1/2}\Big|_0^1 d\theta$$

$$= -2\int_0^{2\pi} (4-1)^{1/2} - (4-0)^{1/2}\, d\theta = -2\int_0^{2\pi} \sqrt{3} - 2\, d\theta$$

$$= -2(\sqrt{3}-2)\theta\Big|_0^{2\pi} = \boxed{-4(\sqrt{3}-2)\pi}$$

This is also actually positive.

8.



$$f(x,y) = x^2$$
$$f_x = 2x$$
$$f_y = 0$$

$$\sqrt{f_x^2 + f_y^2 + 1} = \sqrt{(2x)^2 + 0^2 + 1} = \sqrt{4x^2+1}$$

$$\int_0^1 \int_0^x \sqrt{4x^2+1}\, dy\, dx = \int_0^1 y\sqrt{4x^2+1} \Big|_0^x dx$$

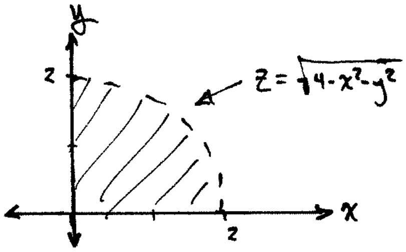
$$= \int_0^1 x\sqrt{4x^2+1}\, dx$$

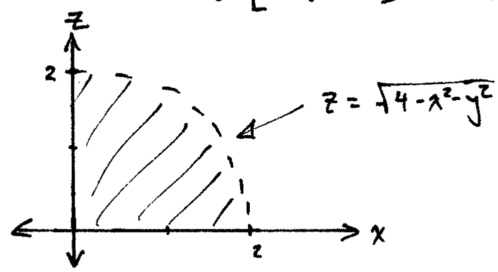
$$= \frac{1}{6}(4x^2+1)^{3/2}\Big|_0^1$$

$$= \frac{1}{6}\left[(4(1)^2+1) - (4(0)^2+1)\right]$$

$$= \frac{1}{6}[5-1] = \frac{1}{6}[4] = \boxed{\frac{2}{3}}$$

9.





$$\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} xy\, dz\, dy\, dx = \int_0^2 \int_0^{\sqrt{4-x^2}} xy\sqrt{4-x^2-y^2}\, dy\, dx$$

$$= -\frac{1}{3}\int_0^2 x(4-x^2-y^2)\Big|_0^{\sqrt{4-x^2}} dx$$

$$= -\frac{1}{3}\int_0^2 x(4-x^2-(4-x^2))^{3/2} - x(4-x^2)^{3/2} dx$$

$$= -\frac{1}{3}\int_0^2 0 - x(4-x^2)^{3/2} dx$$

$$= +\frac{1}{3}\int_0^2 x(4-x^2)^{3/2} dx$$

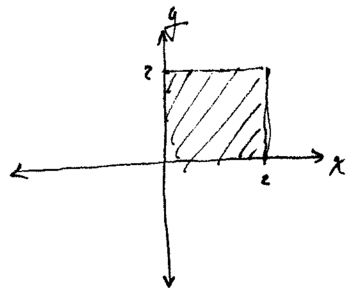
$$= -\frac{1}{5}\cdot\frac{1}{3}\quad (4-x^2)^{5/2}\Big|_0^2$$

$$= -\frac{1}{15}\cdot\left[(4-2^2)^{5/2} - (4-0)^{5/2}\right]$$

$$= -\frac{1}{15}\cdot -4^{5/2}$$

$$= \frac{1}{15}\cdot 32 = \boxed{\frac{32}{15}}$$

10.



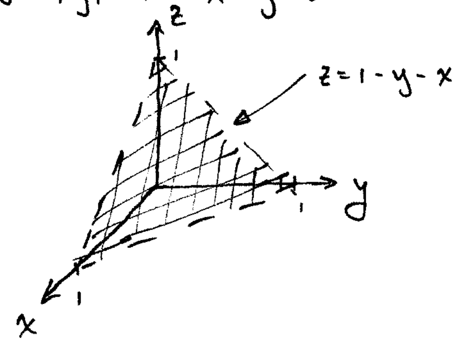
Note: There are no tricky computations for these two problems, so I've skipped some steps.

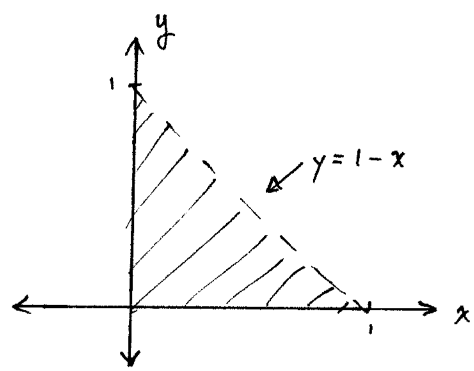
$$\int_0^2\int_0^2\int_0^2 x^2+y^2+z^2 \, dx\,dy\,dz = \int_0^2\int_0^2 2y^2+2z^2+8/3 \, dy\,dz$$

$$= \int_0^2 4z^2 + 32/3 \, dz$$

$$= \boxed{32}$$

11. $\delta(x,y,z) = x+y+z$





Important: This shape is very symmetric about the line $x=y=z$. So the center of mass has the property that $\bar{x} = \bar{y} = \bar{z}$.

$$m = \int_0^1\int_0^{1-x}\int_0^{1-y-x} x+y+z \, dz\,dy\,dx = \int_0^1\int_0^{1-x} \tfrac{1}{2} - xy - \tfrac{1}{2}x^2 - \tfrac{1}{2}y^2 \, dy\,dx$$

$$= \int_0^1\int_0^{1-x} \tfrac{1}{2} - xy - \tfrac{1}{2}x^2 - \tfrac{1}{2}y^2 \, dy\,dx$$

$$= \int_0^1 \tfrac{1}{6}x^3 - \tfrac{1}{2}x + \tfrac{1}{3} \, dx$$

$$= \boxed{1/8} \longleftarrow \text{MASS}$$

$$M_{yz} = \int_0^1\int_0^{1-x}\int_0^{1-y-x} x(x+y+z) \, dz\,dy\,dx$$

$$= \int_0^1\int_0^{1-x} x\left(\tfrac{1}{2} - xy - \tfrac{1}{2}x^2 - \tfrac{1}{2}y^2\right) dy\,dx$$

$$= \int_0^1 x\left(\tfrac{1}{6}x^3 - \tfrac{1}{2}x + \tfrac{1}{3}\right) dx$$

$$= \int_0^1 \tfrac{1}{6}x^4 - \tfrac{1}{2}x^2 + \tfrac{1}{3}x \, dx$$

$$= \boxed{\tfrac{1}{30}} \longleftarrow M_{yz}$$

$$\bar{z} = \bar{y} = \bar{x} = \frac{M_{yz}}{m} = \frac{1/30}{1/8} = 8/30 = 4/15$$

So $\boxed{(4/15,\ 4/15,\ 4/15)}$

12. Want volume of region under

$$x^2+y^2+z^2=2$$

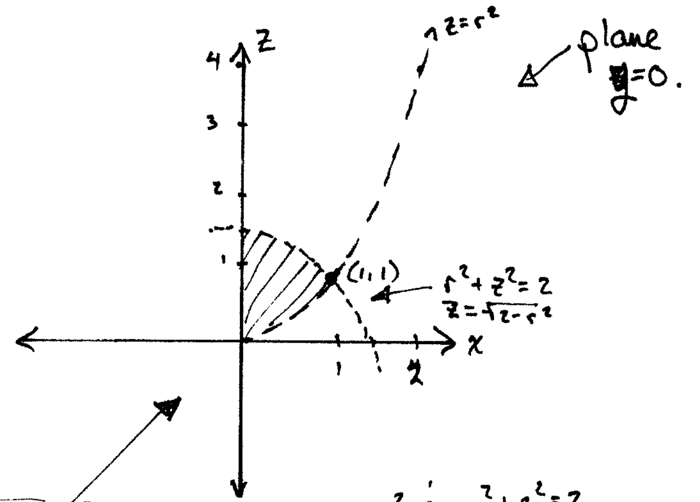
~~above~~ above

$$x^2+y^2=z$$

In other words:

$$r^2+z^2=2$$

in cylindrical. $r^2=z$



The shape looks like this, rotated a.b. z-axis.

$z=r^2$ & $r^2+z^2=2$ intersect at

$$z+z^2=2$$
$$z^2+z-2=0$$
$$(z-2)(z+1)=0$$
$$\cancel{z=-2},\ \underline{z=1,\ r=1}$$

$$\int_0^{2\pi}\int_0^1\int_{r^2}^{\sqrt{2-r^2}} 1\cdot r\,dz\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^1 r\sqrt{2-r^2}-r^3\,dr\,d\theta$$

$$=\int_0^{2\pi}\left[-\frac{1}{3}(2-r^2)^{3/2}-\frac{1}{4}r^4\right]_0^1 d\theta$$

$$=\int_0^{2\pi}-\frac{1}{3}-\frac{1}{4}-\left(-\frac{1}{3}(2)^{3/2}\right)d\theta$$

$$=\int_0^{2\pi}\frac{2}{3}\sqrt{2}-\frac{7}{12}\,d\theta$$

$$=\boxed{\frac{4}{3}\pi\sqrt{2}-\frac{7}{6}\pi}$$

3. under $x^2+y^2+z^2=4$

Above $z=0$

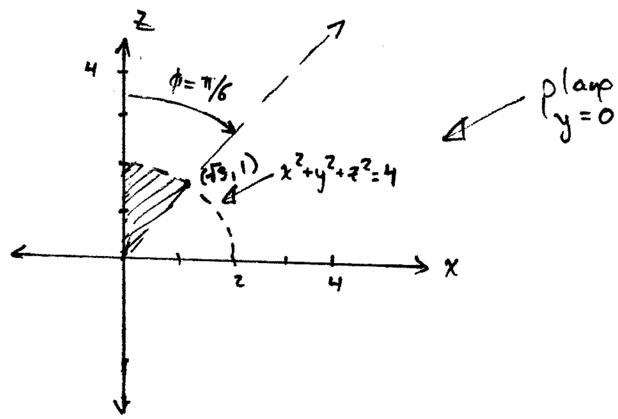
Inside $z^2=3(x^2+y^2)$

spherical:

$\rho=2$

$\phi\le\pi/2$

$\phi=\pi/6$



$$\int_0^{\pi/6}\int_0^{2\pi}\int_0^2 1\cdot\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

$$=\int_0^{\pi/6}\int_0^{2\pi}\frac{1}{3}\rho^3\sin\phi\Big|_0^2\,d\theta\,d\phi$$

$$=\frac{8}{3}\int_0^{\pi/6}\int_0^{2\pi}\sin\phi\,d\theta\,d\phi$$

$$=\frac{8}{3}\int_0^{\pi/6}2\pi\sin\phi\,d\phi$$

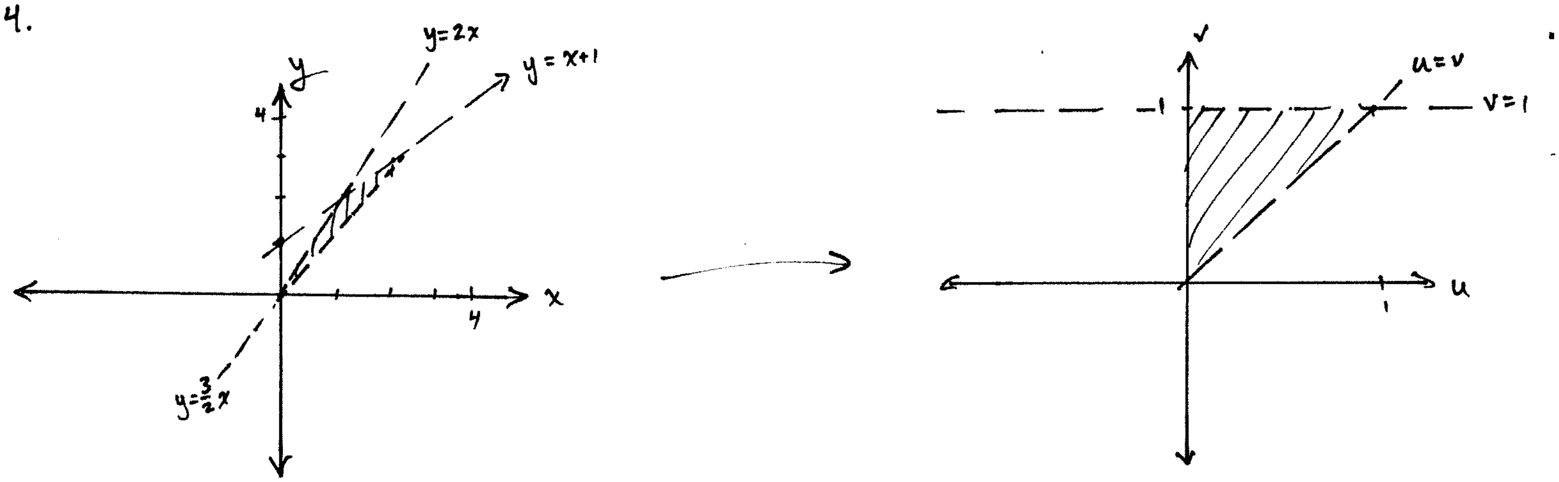
$$=-\frac{16\pi}{3}\cos\phi\Big|_0^{\pi/6}$$

$$=-\frac{16\pi}{3}\left[\cos\pi/6-\cos 0\right]$$

$$=\boxed{-\frac{16\pi}{3}\left[\frac{\sqrt{3}}{2}-1\right]}$$

This is actually positive.

4.



$$y = \frac{3}{2}x$$
$$u + 2v = \frac{3}{2}(u+v)$$
$$u + 2v = \frac{3}{2}u + \frac{3}{2}v$$
$$\frac{1}{2}v = \frac{1}{2}u$$
$$u = v$$

$$y = 2x$$
$$u + 2v = 2(u+v)$$
$$u + 2v = 2u + 2v$$
$$u = 0$$

$$y = x+1$$
$$u + 2v = u + v + 1$$
$$v = 1$$

$$e^{(y-x)^2} = e^{(u+2v-u-v)^2}$$
$$= e^{v^2}$$

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 2 - 1 = 1$$

$$\iint_R e^{(y-x)^2}\,dA = \int_0^1\int_0^v e^{v^2}\,du\,dv = \int_0^1 v e^{v^2}\,dv = \frac{1}{2}\cancel{\int} e^{v^2}\Big|_0^1 = \frac{1}{2}\left[e^1 - e^0\right]$$

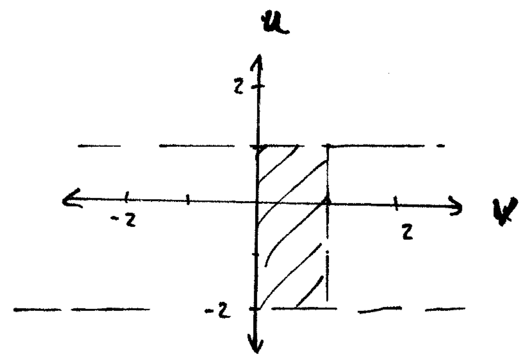
$$= \boxed{\frac{1}{2}\left[e - 1\right]}$$

15. Let's choose

$$u = 2x - y$$
$$v = x + 3y$$

So solving for x, y:

$$3u + v = 7x \qquad\qquad u - 2v = -7y$$
$$x = \frac{3u+v}{7} \qquad\qquad y = \frac{2v-u}{7}$$



$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 3/7 & 1/7 \\ 2/7 & -1/7 \end{vmatrix} = -3/49 - 2/49 = -5/49$$

Bounding curves:

$u = 1 \qquad v = 0$

$u = -2 \qquad v = 1$

Function:

$$49x^2y = 49\left(\frac{3u+v}{7}\right)^2\left(\frac{2v-u}{7}\right)$$
$$= 7^2 \cdot \frac{1}{7^3} \cdot (3u+v)^2(2v-u)$$
$$= \frac{1}{7}(9u^2+6uv+v^2)(2v-u)$$
$$= \frac{1}{7}(18u^2v + 12uv^2 + 2v^3 - 9u^3 - 6u^2v - uv^2)$$
$$= \frac{1}{7}(12u^2v + 11uv^2 + 2v^3 - 9u^3)$$

So we want:

$$\int_0^1 \int_{-2}^1 \frac{1}{2}(12u^2v + 11uv^2 + 2v^3 - 9u^3)\, du\, dv$$
$$= \frac{1}{2}\int_0^1 6v^3 - \frac{33}{2}v^2 + 36v + \frac{135}{4}\, dv$$
$$= \boxed{\frac{191}{8}}$$

Note: Again, these are just polynomials; I've skipped some steps.