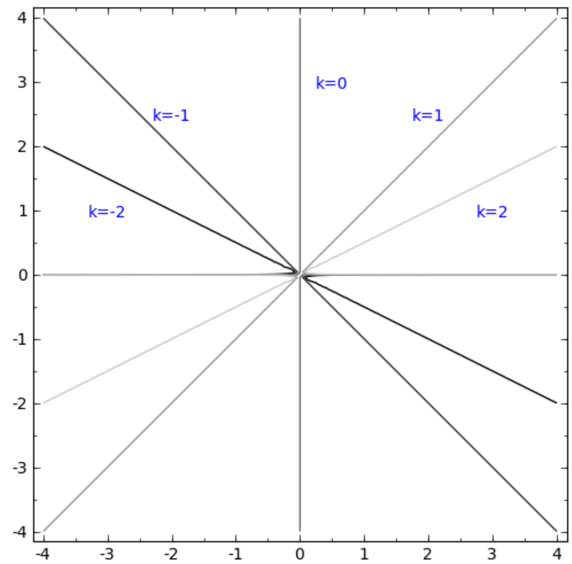
Exam 2 Practice Problems

Math 2210-001: Calculus III – Fall 2009

1. Sketch level curves of

$$f(x, y) = \frac{x}{y}$$

for $k = -2, -1, 0, 1, 2$.



2. Describe geometrically the domain of the function $f(x, y, z) = \sqrt{x^2 + y^2 - z^2}$

The domain will be where $x^2 + y^2 - z^2 > 0$; that is, where

$$x^2 + y^2 > z^2$$

Certainly we know that $x^2 + y^2 = z^2$ is a cone along the z-axis. But is it the inside of that cone, or the outside? Let's test a point inside the cone — say, the point $(0, 0, 1)$.

$$0^2 + 0^2 > 1^2$$

This is false, so the solution set is the **outside of a cone lying along the z-axis**.

3. Show that

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2}$$

does not exist. *(Hint: consider one path to the origin through the x-axis, and one along the line $y = x$).*

Following the hint, we split this problem into two limits.

- Approaching along the x-axis, $y = 0$ and the limit becomes:

$$\begin{aligned} \lim_{x\to 0} \frac{x \cdot 0}{x^2 + 0^2} &= \lim_{x\to 0} \frac{0}{x^2} \\ &= \lim_{x\to 0} 0 \\ &= 0 \end{aligned}$$

- Approaching along the line $x = y$, replace y with x and the limit becomes:

$$\begin{aligned} \lim_{x\to 0} \frac{x \cdot x}{x^2 + x^2} &= \lim_{x\to 0} \frac{x^2}{2x^2} \\ &= \lim_{x\to 0} 2 \\ &= 2 \end{aligned}$$

Since the limit is different when taken from different directions, the limit doesn't exist.

4. For each of the following functions:
   - Find $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$
   - Determine if $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$. If equality of mixed partials fails, explain why.

   (a) $f(x, y) = e^x \sin(x^2 y^3)$

$$\begin{aligned} f_x &= e^x \sin(x^2y^3) + y^3 2xe^x \cos(x^2y^3) \\ f_{xy} &= 3y^2x^2e^x \cos(x^2y^3) + 3y^2 2xe^x \cos(x^2y^3) - y^3 2x3y^2x^2e^x \sin(x^2y^3) \\ &= 3y^2x^2e^x \cos(x^2y^3) + 6xy^2e^x \cos(x^2y^3) - 6y^5x^3e^x \sin(x^2y^3) \\ f_y &= 3y^2x^2e^x \cos(x^2y^3) \\ f_{yx} &= 3y^2 2xe^x \cos(x^2y^3) + 3y^2x^2e^x \cos(x^2y^3) - 3y^2x^2e^x 2xy^3 \sin(x^2y^3) \\ & \quad \text{(that was just a huge application of the product rule)} \\ &= 6y^2xe^x \cos(x^2y^3) + 3y^2x^2e^x \cos(x^2y^3) - 6y^5x^3e^x \sin(x^2y^3) \end{aligned}$$

   And you can see that $f_{xy} = f_{yx}$.

   (b) $f(x, y) = \dfrac{\sin(x - y)}{x + y}$

   The first and second derivatives — after simplification — are:

$$\begin{aligned} f_x &= \frac{\cos(x-y)}{(x+y)} - \frac{\sin(x-y)}{(x+y)^2} \\ f_{xy} &= \frac{\sin(x-y)}{(x+y)} + 2\frac{\sin(x-y)}{(x+y)^3} \\ f_y &= -\frac{\cos(x-y)}{(x+y)} - \frac{\sin(x-y)}{(x+y)^2} \\ f_{yx} &= \frac{\sin(x-y)}{(x+y)} + 2\frac{\sin(x-y)}{(x+y)^3} \end{aligned}$$

   The computations are not so bad; just make sure you simplify the first derivatives as shown before trying to take the second derivatives.

   Notice that $f_{xy} = f_{yx}$, but we got kind of lucky. They are only equal because the derivatives (and in fact the function) are not defined at $(0, 0)$. The function is not continuous at $(0, 0)$, though — try taking limits along the x- and y- axes — so there is no reason to expect the mixed partials to match.

5. You and your navigator Jeremy are driving a rally car over the surface defined by $z = x^4 - x^2y^2 + 4$. You hit a bump at the point $(1, 1, 4)$, lose traction, and go airborne along a tangent line. Find the equation for the tangent line, assuming you were driving:

(a) Along the intersection with the plane $y = 1$

Since we are travelling in the plane $y = 1$, the slope of the tangent line is given by the partial derivative in the x direction. That is:

$$\frac{\partial}{\partial x}\left(x^4 - x^2y^2 + 4\right) = 4x^3 - 2xy^2$$

Plugging in $x = 1$, $y = 1$, we get

$$4(1)^2 - 2(1)(1) = 4 - 2 = 2$$

So as we increase x by one, z goes up by two. That means our direction line is $\mathbf{i} + 2\mathbf{k}$. Our starting point is $(1, 1, 4)$. Now just plug into the parametric equation for a line.

$$\mathbf{L} = (\mathbf{i} + 2\mathbf{k})t + (\mathbf{i} + \mathbf{j} + 4\mathbf{k})$$

(b) Along the intersection with the plane $x = y$. Be clever about this! You can do this part without any algebra.

Along the line $x = y$, the function is constant (notice that it becomes simply $z = 4$). So we can specify the tangent line $\mathbf{L}$ by the parametric equation

$$\mathbf{L} = (\mathbf{i} + \mathbf{j})t + 4\mathbf{k}$$

6. **(Hard.)** The city council is considering the following proposal for a new subway system:

- The main station will be underneath Temple Square. For the sake of argument, we will think of the temple as being the origin of the xy-plane.
- Every ray starting at the origin will have a subway line running along it, and there will be no other subway lines. This means, of course, that there will be an infinite number of subway lines, but the city council is expecting some serious stiumulus money for this project.
- You can get on a subway train at any point along its line.

Now, suppose $f(x, y)$ is the distance you have to travel along the subway to get from the point (x, y) to LCB, which is at about $(13.5, -2)$. Does this function have any discontinuities, and if so, where?

This is a bit weird, but bear with me.

If you are at LCB, you don't have to travel anywhere. The distance is 0.

But suppose you're a few feet from LCB. I'm requiring you to travel **along the subway** to get to LCB, so you have to travel all the way to temple square, which is about two miles or so, and then all the way back to LCB, which is another two miles or so. The total distance is about four miles.

By that logic, there is a discontinuity at $(13.5, -2)$, the location of LCB.

7. If $f(x, y) = \cos(x^2 + y^2) - xy$ and $\mathbf{p} = 2\mathbf{i} + 2\mathbf{j}$:

(a) Find $\nabla f(\mathbf{p})$.

$$\begin{aligned} \nabla f &= f_x\mathbf{i} + f_y\mathbf{j} \\ &= \left(-2x\sin\left(x^2 + y^2\right) - y\right)\mathbf{i} + \left(-2y\sin\left(x^2 + y^2\right) - x\right)\mathbf{j} \end{aligned}$$

(b) Find the equation of the tangent plane at $\mathbf{p}$.

There's a nice formula for this, but you can do it however you like. Say our point $\mathbf{p}$ is (x_0, y_0).

$$\begin{aligned}
z &= f(x_0, y_0) + \nabla f(x_0, y_0) \cdot \langle x - x_0, y - y_0 \rangle \\
&= f(2,2) + \nabla f(2,2) \cdot \langle x - 2, y - 2 \rangle \\
&= \cos(2^2 + 2^2) - (2)(2) \\
&\quad + \left(\left(-2(2)\sin\left(2^2+2^2\right) - (2)\right)\mathbf{i} + \left(-2(2)\sin\left(2^2+2^2\right) - 2\right)\mathbf{j}\right) \cdot ((x-2)\mathbf{i} + (y-2)\mathbf{j}) \\
&= \cos(8) - 4 + ((-4\sin(8) - 2)\,\mathbf{i} + (-4\sin(8) - 2)\,\mathbf{j}) \cdot ((x-2)\mathbf{i} + (y-2)\mathbf{j}) \\
&= \cos(8) - 4 + (-4\sin(8) - 2)(x-2) + (-4\sin(8) - 2)(y-2) \\
&= 16\sin(8) + \cos(8) + 4 - 4\sin(8)x - 4\sin(8)y - 2x - 2y
\end{aligned}$$

Collecting everything into standard form gives

$$(4\sin(8) - 2)x + (4\sin(8) - 2)y + z = 16\sin(8) + \cos(8) + 4$$

You can plug this into your calculator if you like numbers a whole lot.

8. Suppose your apartment is heated by a tiny sun in your living room. The temperature in your apartment (in degrees Celsius) is given by:

$$T(x,y,z) = \frac{5000}{10 + x^2 + y^2 + z^2}$$

with the origin at the center of the sun.

(a) Describe the level sets of this function geometrically. Then explain why the gradient must point either toward or away from the sun.

The level sets are concentric spheres, centered at the origin. Lower level sets are farther out. Since the gradient has to be perpendicular to the level sets, it's pointing either directly out or directly in.

(b) Compute the gradient, and use it to show that the direction of greatest temperature increase is always in the direction of the sun.

We can calculate the gradient quite quickly here, since the function is symmetric in x, y, and z.

$$T_x = -2x\frac{5000}{(10 + x^2 + y^2 + z^2)^2} T_y = -2y\frac{5000}{(10 + x^2 + y^2 + z^2)^2} T_z = -2z\frac{5000}{(10 + x^2 + y^2 + z^2)^2}$$

So then (after simplification) the gradient is

$$\nabla T = -2\frac{5000}{(10 + x^2 + y^2 + z^2)^2}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

The $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ term always points outward; the $-2\frac{5000}{(10+x^2+y^2+z^2)^2}$ coefficient is always negative, so it reverses the direction. The resulting vector must always point inward, toward the sun.

(c) You are standing at the point $(4, 0, -3)$ when you start walking in the direction of the vector $\mathbf{i} - \mathbf{j}$ to get some lemonade. Use the directional derivative to find the speed with which you feel the temperature changing. (Don't worry about units.)

The formula for the directional derivative is

$$D_{\mathbf{u}}T(x,y,z) = \mathbf{u} \cdot \nabla T(x,y,z)$$

Of course, this requires our direction vector to be a unit vector, which it isn't. Let's rectify that!

$$\begin{aligned}
\mathbf{u} &= \frac{\mathbf{i}-\mathbf{j}}{\|\mathbf{i}-\mathbf{j}\|} \\
&= \frac{\mathbf{i}-\mathbf{j}}{\sqrt{1^2+(-1)^2}} \\
&= \frac{\mathbf{i}-\mathbf{j}}{\sqrt{2}} \\
&= \frac{\sqrt{2}}{2}(\mathbf{i}-\mathbf{j})
\end{aligned}$$

Now just plug in.

$$\begin{aligned}
-2\frac{5000}{\left(10+(4)^2+(0)^2+(-3)^2\right)^2}(4\mathbf{i}+0\mathbf{j}-3\mathbf{k}) &= \frac{10000}{\left(10+16+9\right)^2}(4\mathbf{i}-3\mathbf{k}) \\
&= \frac{10000}{\left(35\right)^2}(4\mathbf{i}-3\mathbf{k}) \\
&= \frac{400}{49}(4\mathbf{i}-3\mathbf{k})
\end{aligned}$$

$$\begin{aligned}
D_{\mathbf{u}}T(x,y,z) &= \mathbf{u} \cdot \nabla T(x,y,z) \\
&= \frac{\sqrt{2}}{2}(\mathbf{i}-\mathbf{j}) \cdot \frac{400}{49}(4\mathbf{i}-3\mathbf{k}) \\
&= \frac{200\sqrt{2}}{49}(1(4)-1(0)-3(0)) \\
&= \frac{800\sqrt{2}}{49}
\end{aligned}$$

That represents quite a few degrees per foot walked.

9. **Use the chain rule** to find $\frac{\partial w}{\partial t}$ for each of the following expressions:

(a) $w = x^2y - \sin(xy)$; $x = 2t$, $y = e^t$

$$\begin{aligned}
\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} \\
&= \left(2xy - y\cos(xy)\right)(2) + \left(x^2 - x\cos(xy)\right)\left(e^t\right) \\
&= 8te^t - 2e^t\cos\left(2te^t\right) + 4t^2e^t - 2t\cos\left(2te^t\right)
\end{aligned}$$

(b) $w = \ln(x^2+y^2)$; $x = s\sin t$, $y = t\cos s$

$$\begin{aligned}
\frac{\partial w}{\partial t} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} \\
&= \frac{2x}{x^2+y^2} s\cos t + \frac{2y}{x^2+y^2}\cos s \\
&= \frac{2s\sin t}{s^2\sin^2 t + t^2\cos^2 s} s\cos t + \frac{2t\cos s}{s^2\sin^2 t + t^2\cos^2 s}\cos s \\
&= \frac{2s^2 \sin t\cos t + 2t\cos^2 s}{s^2\sin^2 t + t^2\cos^2 s}
\end{aligned}$$

10. The volume of a cone is given by the formula

$$V = \frac{1}{3}\pi r^2 h$$

where r is the radius of the base and h is the height of the cone. Suppose that the height and radius of the cylinder are changing, $h = 5$ and $r = 2$, $\frac{dh}{dt} = 3$, and $\frac{dr}{dt} = 7$. **Use the chain rule** to find $\frac{dV}{dt}$.

$$\begin{aligned}
\frac{dV}{dt} &= \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} \\
&= \frac{1}{3}\pi 2rh\frac{dr}{dt} + \frac{1}{3}\pi r^2\frac{dh}{dt} \\
&= \frac{1}{3}\pi(2)(2)(5)(7) + \frac{1}{3}\pi(2)^2(7) \\
&= \frac{1}{3}\pi(168) \\
&= \frac{168\pi}{3}
\end{aligned}$$

11. Consider the function

$$f(x, y) = \sqrt{x^2 + y^2}$$

Approximate $f(3.1, 4.1)$ using the following methods:

(a) A first-order Taylor series based at $(3, 4)$

(b) A second-order Taylor series based at $(3, 4)$

12. Find all of the critical points of the function

$$f(x, y) = \sin(x)\sin(y) \qquad -\pi < x, y < \pi$$

Determine if each of them is a local maximum, a local minimum, or a saddle point.

I'm going to list all of the derivatives first.

$$\begin{aligned}
f_x &= \cos x \sin y \\
f_y &= \sin x \cos y \\
f_{xx} &= -\sin x \sin y \\
f_{yy} &= -\sin x \sin y \\
f_{xy} = f_{yx} &= \cos x \cos y
\end{aligned}$$

The first step is to find the critical points.

$$\nabla f(x,y) = \cos x \sin y \mathbf{i} + \sin x \cos y \mathbf{j}$$

To find the critical points, then, we need to find when

$$\cos x \sin y = 0 \qquad\qquad \sin x \cos y = 0$$

In order:

$$\begin{array}{ccc} & \cos x \sin y = 0 & \\ \cos x = 0 & \text{or} & \sin y = 0 \\ x = \pm\dfrac{\pi}{2} & \text{or} & y = 0 \end{array}$$

When $x = \pm\frac{\pi}{2}$:

$$\begin{aligned} \sin\left(\pm\frac{\pi}{2}\right)\cos y &= 0 \\ \pm\cos y &= 0 \\ \cos y &= 0 \end{aligned}$$

If you're being really careful here, you'll notice that there are four points — $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$, with all possible signs on the two coordinates.

When $y = 0$:

$$\begin{aligned} \sin x \cos 0 &= 0 \\ \sin x &= 0 \\ x &= 0 \end{aligned}$$

That gives us one more point, which is $(0,0)$.

Now for the second partials test.

$$\begin{aligned} D(x,y) &= f_{xx}f_{yy} - f_{xy}f_{yx} \\ &= \sin^2 x \sin^2 y - \cos^2 x \cos^2 y \end{aligned}$$

At $(0,0)$, this becomes:

$$D(0,0) = \sin^2(0)\sin^2(0) - \cos^2(0)\cos^2(0) = -1$$

That means that $(0,0)$ is a saddle point.

When the point is $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$ or $\left(-\frac{\pi}{2}, -\frac{\pi}{2}\right)$, it becomes

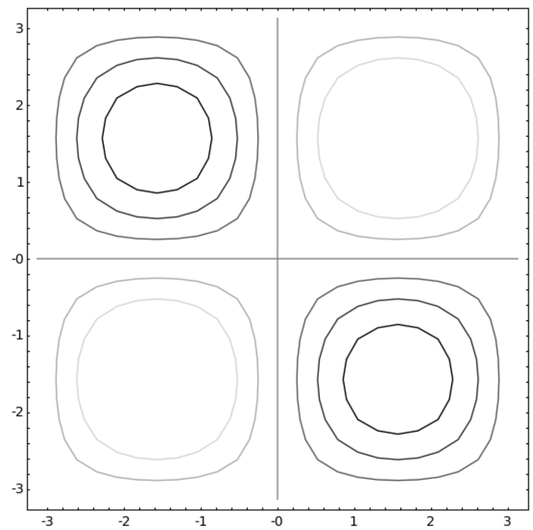
$$D(0,0) = 1 - 0 = 1$$

And the same for $\left(\frac{\pi}{2}, -\frac{\pi}{2}\right)$ or $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. So all four of these points are minmums or maximums.

Now check these last four points in f_{xx} to figure out of they are minima or maxima.

$$f_{xx}\left(\frac{\pi}{2},-\frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2}\right)\sin\left(-\frac{\pi}{2}\right) = 1 \qquad \text{(minimum)}$$
$$f_{xx}\left(-\frac{\pi}{2},\frac{\pi}{2}\right) = -\sin\left(-\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right) = 1 \qquad \text{(minimum)}$$
$$f_{xx}\left(\frac{\pi}{2},\frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right) = -1 \qquad \text{(maximum)}$$
$$f_{xx}\left(-\frac{\pi}{2},-\frac{\pi}{2}\right) = -\sin\left(-\frac{\pi}{2}\right)\sin\left(-\frac{\pi}{2}\right) = -1 \qquad \text{(maximum)}$$

Incidentally, the contour plot looks like this:



13. Find the maximum and minimum values of the function

$$f(x,y) = 4x^2 + 9y^2$$

on the closed set bounded by the triangle with vertices $(-1,-1)$, $(1,1)$, and $(1,-1)$. **Remember to check the boundaries.** Most of the work for the problem comes from having to find the maximum and minimum on the boundary. The interior is easy.