Exam 2 Practice Problems

Math 2210-001: Calculus III – Fall 2009

1. Sketch level curves of
$$f(x,y) = \frac{x}{y}$$
for $k = -2, -1, 0, 1, 2$.

2. Describe geometrically the domain of the function
$$f(x,y,z) = \sqrt{x^2 + y^2 - z^2}$$

3. Show that
$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$
does not exist. *(Hint: consider one path to the origin through the x-axis, and one along the line $y = x$).*

4. For each of the following functions:
   - Find $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$
   - Determine if $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$. If equality of mixed partials fails, explain why.

   (a) $f(x,y) = e^x \sin(x^2 y^3)$

   (b) $f(x,y) = \dfrac{\sin(x-y)}{x+y}$

5. You and your navigator Jeremy are driving a rally car over the surface defined by $z = x^4 - x^2y^2 + 4$. You hit a bump at the point $(1,1,4)$, lose traction, and go airborne along a tangent line. Find the equation for the tangent line, assuming you were driving:

   (a) Along the intersection with the plane $y = 1$

   (b) Along the intersection with the plane $x = y$. Be clever about this! You can do this part without any algebra.

6. **(Hard.)** The city council is considering the following proposal for a new subway system:
   - The main station will be underneath Temple Square. For the sake of argument, we will think of the temple as being the origin of the xy-plane.
   - Every ray starting at the origin will have a subway line running along it, and there will be no other subway lines. This means, of course, that there will be an infinite number of subway lines, but the city council is expecting some serious stiumulus money for this project.
   - You can get on a subway train at any point along its line.

   Now, suppose $f(x,y)$ is the distance you have to travel along the subway to get from the point (x,y) to LCB, which is at about $(13.5, -2)$. Does this function have any discontinuities, and if so, where?

7. If $f(x,y) = \cos(x^2 + y^2) - xy$ and $\mathbf{p} = 2\mathbf{i} + 2\mathbf{j}$:

   (a) Find $\nabla f(\mathbf{p})$.

   (b) Find the equation of the tangent plane at $\mathbf{p}$.

8. Suppose your apartment is heated by a tiny sun in your living room. The temperature in your apartment (in degrees Celsius) is given by:

$$T(x, y, z) = \frac{5000}{10 + x^2 + y^2 + z^2}$$

with the origin at the center of the sun.

   (a) Describe the level sets of this function geometrically. Then explain why the gradient must point either toward or away from the sun.

   (b) Compute the gradient, and use it to show that the direction of greatest temperature increase is always in the direction of the sun.

   (c) You are standing at the point $(4, 0, -3)$ when you start walking in the direction of the vector $\mathbf{i} - \mathbf{j}$ to get some lemonade. Use the directional derivative to find the speed with which you feel the temperature changing. (Don't worry about units.)

9. **Use the chain rule** to find $\frac{\partial w}{\partial t}$ for each of the following expressions:

   (a) $w = x^2 y - \sin(xy)$; $x = 2t$, $y = e^t$

   (b) $w = \ln(x^2 + y^2)$; $x = s \sin t$, $y = t \cos x$

10. The volume of a cone is given by the formula

$$V = \frac{1}{3}\pi r^2 h$$

where r is the radius of the base and h is the height of the cone. Suppose that the height and radius of the cylinder are changing, $h = 5$ and $r = 2$, $\frac{dh}{dt} = 3$, and $\frac{dr}{dt} = 7$. **Use the chain rule** to find $\frac{dV}{dt}$.

11. Consider the function

$$f(x, y) = \sqrt{x^2 + y^2}$$

Approximate $f(3.1, 4.1)$ using the following methods:

   (a) A first-order Taylor series based at $(3, 4)$

   (b) A second-order Taylor series based at $(3, 4)$

12. Find all of the critical points of the function

$$f(x, y) = \sin(x)\sin(y) \quad -\pi < x, y < \pi$$

Determine if each of them is a local maximum, a local minimum, or a saddle point.

13. Find the maximum and minimum values of the function

$$f(x, y) = 4x^2 + 9y^2$$

on the closed set bounded by the triangle with vertices $(-1, -1)$, $(1, 1)$, and $(1, -1)$. **Remember to check the boundaries.** Most of the work for the problem comes from having to find the maximum and minimum on the boundary. The interior is easy.