Exam 1

Math 2210-001: Calculus III – Fall 2009

Problem	Score
1	15
2	10
3	25
4	25
5	25
Total	

1. Find the equation for the plane containing the points $3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, $3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$, and $\mathbf{i} + 4\mathbf{j}$.

   First, we need two vectors in the plane:

$$\begin{aligned} \mathbf{v}_1 &= 3\mathbf{i} - \mathbf{j} + 2\mathbf{k} - (3\mathbf{i} + 4\mathbf{j} - \mathbf{k}) \\ &= -5\mathbf{j} + 3\mathbf{k} \\ \mathbf{v}_2 &= \mathbf{i} + 4\mathbf{j} - (3\mathbf{i} + 4\mathbf{j} - \mathbf{k}) \\ &= -2\mathbf{i} + \mathbf{k} \end{aligned}$$

   Then take their cross product:

$$\begin{aligned} \mathbf{v}_1 \times \mathbf{v}_2 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -5 & 3 \\ -2 & 0 & 1 \end{vmatrix} \\ &= -5\mathbf{i} - 6\mathbf{j} + 10\mathbf{k} \end{aligned}$$

   So the equation is:

$$-5x - 6y + 10z = D$$

   To find D, plug in $(1, 4, 0)$:

$$\begin{aligned} D &= -5(1) - 6(4) + 10(0) \\ &= -5 - 24 \\ &= -29 \end{aligned}$$

   So the answer is

$$-5x - 6y + 10z = -29$$

2. **(Hard!)** Prove that

$$\|\mathbf{a} \times \mathbf{b}\|^2 = \|\mathbf{a}\|^2\|\mathbf{b}\|^2 - (\mathbf{a} \cdot \mathbf{b})^2$$

   Proof. Suppose θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

$$\begin{aligned} (\|\mathbf{a}\|\|\mathbf{b}\|\sin(\theta))^2 &= \|\mathbf{a}\|^2\|\mathbf{b}\|^2 - (\|\mathbf{a}\|\|\mathbf{b}\|\cos(\theta))^2 \\ \|\mathbf{a}\|^2\|\mathbf{b}\|^2\sin^2(\theta) &= \|\mathbf{a}\|^2\|\mathbf{b}\|^2 - \|\mathbf{a}\|^2\|\mathbf{b}\|^2\cos^2(\theta) \\ \sin^2(\theta) &= 1 - \cos^2(\theta) \\ \cos^2(\theta) + \sin^2(\theta) &= 1 \end{aligned}$$

$\square$

3. Albert has entered a catapult in the annual Pumkin' Chunkin' World Championship near Bridgeville, Delaware. The pumkin' it hurtles follows the path given by:

$$\mathbf{r}(t) = 20\sqrt{2}t\mathbf{i} + 20\sqrt{2}t\mathbf{j} + (40t - 5t^2)\mathbf{k}$$

(a) Find the velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ of the pumpkin.

$$\begin{aligned}
\mathbf{r}(t) &= 20\sqrt{2}t\mathbf{i} + 20\sqrt{2}t\mathbf{j} + (40t - 5t^2)\mathbf{k} \\
\mathbf{v}(t) &= \mathbf{r}'(t) \\
&= 20\sqrt{2}\mathbf{i} + 20\sqrt{2}\mathbf{j} + (40 - 10t)\mathbf{k} \\
\mathbf{a}(t) &= \mathbf{r}''(t) \\
&= -10\mathbf{k}
\end{aligned}$$

(b) Suppose Unicron eats the earth at time $t = 2$ and then disappears out of existence. As a result, gravity ceases to function and the pumpkin begins to travel in a straight line. Find the parametric equations for the tangent line the pumpkin travels along.

$$\begin{aligned}
\mathbf{L}(t) &= \mathbf{v}(2)t + \mathbf{r}(2) \\
&= \left(20\sqrt{2}\mathbf{i} + 20\sqrt{2}\mathbf{j} + (40 - 10(2))\mathbf{k}\right)t + \left(20\sqrt{2}(2)\mathbf{i} + 20\sqrt{2}(2)\mathbf{j} + \left(40(2) - 5(2)^2\right)\mathbf{k}\right) \\
&= \left(20\sqrt{2}\mathbf{i} + 20\sqrt{2}\mathbf{j} + 20\mathbf{k}\right)t + \left(40\sqrt{2}\mathbf{i} + 40\sqrt{2}\mathbf{j} + 60\mathbf{k}\right)
\end{aligned}$$

So then

$$\begin{aligned}
x &= 20\sqrt{2}t + 40\sqrt{2} \\
y &= 20\sqrt{2}t + 40\sqrt{2} \\
z &= 20t + 60
\end{aligned}$$

(c) Find the symmetric equations for the tangent line.

Solve the above equations for t, then set them all equal to get

$$\frac{x - 40\sqrt{2}}{20\sqrt{2}} = \frac{y - 40\sqrt{2}}{20\sqrt{2}} = \frac{z - 60}{20}$$

4. Given the curve

$$\mathbf{r}(t) = \cos(t^2)\mathbf{i} + \sin(t^2)\mathbf{j} \qquad 0 \le t \le \sqrt{2\pi}$$

(a) Find the curvature $\kappa(t)$

$$\begin{aligned}
\mathbf{r}(t) &= \cos(t^2)\mathbf{i} + \sin(t^2)\mathbf{j} \\
\mathbf{r}'(t) &= -2t\sin(t^2)\mathbf{i} + 2t\cos(t^2)\mathbf{j} \\
\mathbf{r}''(t) &= (-2\sin(t^2) - 4t^2\cos(t^2))\mathbf{i} + (2\cos(t^2) - 4t^2\sin(t^2))\mathbf{j}
\end{aligned}$$

$$\begin{aligned}
\kappa(t) &= \frac{|x'y'' - x''y'|}{\left((x')^2 + (y')^2\right)^{\frac{3}{2}}} \\
&= \frac{\left|-2t\sin(t^2)\left(2\cos(t^2) - 4t^2\sin(t^2)\right) - 2t\cos(t^2)\left(-2\sin(t^2) - 4t^2\cos(t^2)\right)\right|}{((-2t\sin(t^2))^2 + (2t\cos(t^2))^2)^{\frac{3}{2}}} \\
&= \frac{\left|-4t\sin(t^2)\cos(t^2) + 8t^3\sin^2(t^2) + 4t\cos(t^2)\sin(t^2) + 8t^3\cos^2(t^2)\right|}{\left(4t^2\sin^2(t^2) + 4t^2\cos^2(t^2)\right)^{\frac{3}{2}}} \\
&= \frac{\left|8t^3(\sin^2(t^2) + \cos^2(t^2))\right|}{\left(4t^2(\sin^2(t^2) + \cos^2(t^2))\right)^{\frac{3}{2}}} \\
&= \frac{\left|8t^3\right|}{(4t^2)^{\frac{3}{2}}} \\
&= \frac{8t^3}{(4t^2)^{\frac{3}{2}}} \qquad \text{since } t > 0 \\
&= \frac{8t^3}{8t^3} \\
&= 1
\end{aligned}$$

(b) Find the tangential and normal components of acceleration a_T and a_N.

$$\mathbf{T}(t) = \frac{\mathbf{r}'}{\|\mathbf{r}'\|}$$

$$\begin{aligned}
\|\mathbf{r}'(t)\| &= \sqrt{(-2t\sin(t^2))^2 + (2t\cos(t^2))^2} \\
&= \sqrt{4t^2\sin^2(t^2) + 4t^2\cos^2(t^2)} \\
&= \sqrt{4t^2(\sin^2(t^2) + \cos^2(t^2))} \\
&= \sqrt{4t^2} \\
&= 2t
\end{aligned}$$

$$\begin{aligned}
\mathbf{T}(t) &= \frac{1}{2t}\left(-2t\sin(t^2)\mathbf{i} + 2t\cos(t^2)\mathbf{j}\right) \\
&= -\sin(t^2)\mathbf{i} + \cos(t^2)\mathbf{j}
\end{aligned}$$

Both of the following computations can be done faster by extracting the information from the above computations. I'll do them out in detail.

$$
\begin{aligned}
a_T &= \mathbf{T} \cdot \mathbf{a} \\
&= (-\sin(t^2)\mathbf{i} + \cos(t^2)\mathbf{j}) \cdot ((-2\sin(t^2) - 4t^2\cos(t^2))\mathbf{i} + (2\cos(t^2) - 4t^2\sin(t^2))\mathbf{j}) \\
&= -\sin(t^2)(-2\sin(t^2) - 4t^2\cos(t^2)) + \cos(t^2)(2\cos(t^2) - 4t^2\sin(t^2)) \\
&= 2\sin^2(t^2) + 4t^2\sin(t^2)\cos(t^2) + 2\cos^2(t^2) - 4t^2\sin(t^2)\cos(t^2) \\
&= 2(\sin^2(t^2) + \cos^2(t^2)) \\
&= 2
\end{aligned}
$$

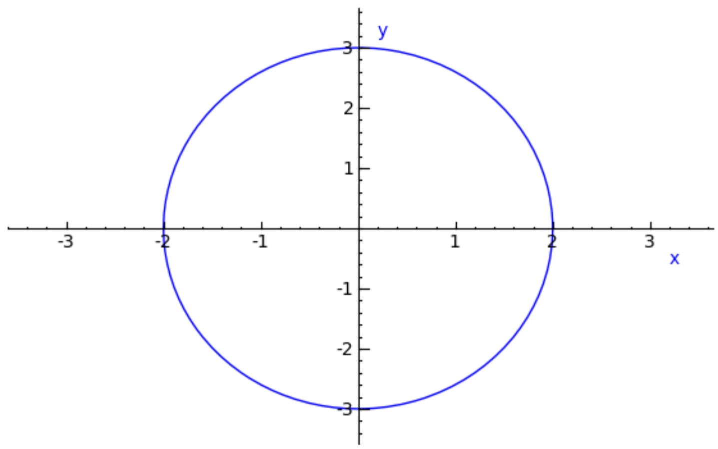
$$
\begin{aligned}
a_N &= \mathbf{T} \times \mathbf{a} \\
&= \left\| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin(t^2) & \cos(t^2) & 0 \\ -2\sin(t^2) - 4t^2\cos(t^2) & 2\cos(t^2) - 4t^2\sin(t^2) & 0 \end{vmatrix} \right\| \\
&= \left\| -\sin(t^2)(2\cos(t^2) - 4t^2\sin(t^2)) - \cos(t^2)(-2\sin(t^2) - 4t^2\cos(t^2)) \right\| \\
&= \left\| -2\sin(t^2)\cos(t^2) + 4t^2\sin^2(t^2)) + 2\cos(t^2)\sin(t^2) + 4t^2\cos^2(t^2) \right\| \\
&= \left\| 4t^2(\sin^2(t^2)) + \cos^2(t^2)) \right\| \\
&= \left\| 4t^2 \right\| \\
&= 4t^2
\end{aligned}
$$

5. Given the equation

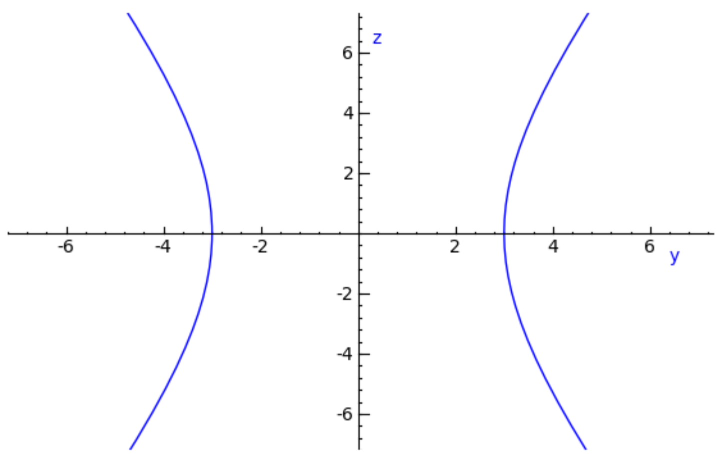
$$9x^2 + 4y^2 - z^2 = 36$$

(a) Quickly sketch its traces in the $xy-$, $yz-$, and $xz-$ planes.

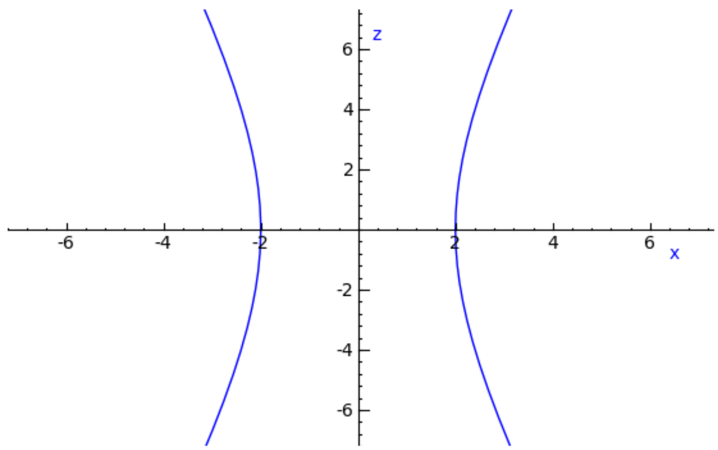
First plug in 0 for z and graph $9x^2 + 4y^2 = 36$:



Then plug in 0 for x and graph $4y^2 - z^2 = 36$:

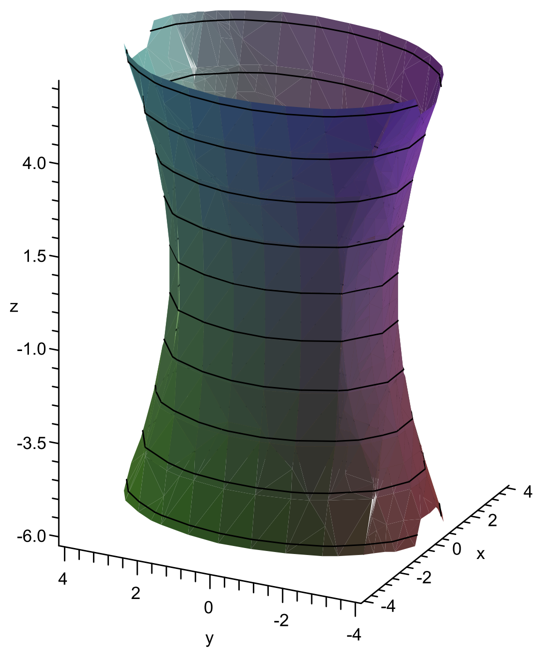


Finally plug in 0 for y and graph $9x^2 - z^2 = 36$:



(b) Graph the surface defined by the equation. Make sure your graph is neat and clear, with all axes labelled, as that will be part of your grade.

The final answer looks like this:



To get a nice graph by hand, you should be sure to compute a couple of cross-sections. You already have one – the section in the xy–plane. That is an ellipse with radius 3 in the y direction, and 2 in the x direction.

Here are 2 more cross sections:

When $z = \pm 6$, the equation becomes:

$$\begin{aligned} 9x^2 + 4y^2 - (\pm 6)^2 &= 36 \\ 9x^2 + 4y^2 - 36 &= 36 \\ 9x^2 + 4y^2 &= 72 \\ \frac{x^2}{8} + \frac{y^2}{18} &= 1 \\ \frac{x^2}{\left(2\sqrt{2}\right)^2} + \frac{y^2}{\left(3\sqrt{2}\right)^2} &= 1 \end{aligned}$$

So when $z = \pm 6$ the cross section is an ellipse with radius about 3 in the x direction and about 4.5 in the y direction. These are the cross sections at the top and bottom of the graph above.