Graded midterms to be handed back in class on Tuesday, Mar. 25

Each numbered question is worth ten points, for a possible total of 60 points.

Class average midterm 2 score is approximately 39/60

There is also a letter grade on your graded midterm, which takes into account both of your midterm scores and your first seven homework scores.

To get the letter grade, I first calculated a number grade (from 0 to 100) using the fact that each midterm will be 20% of the final grade and each homework will be 2.5% of the final grade:

$$
\text{NumberGrade} = (100) \frac{H + M_1 + M_2}{.575},
$$

where

$$
H = \left( \frac{\text{TotalHWPoints}}{210} \right) (.175),
$$

$$
M_1 = \left( \frac{\text{Midterm1Points}}{60} \right) (.2) \quad \text{and} \quad M_2 = \left( \frac{\text{Midterm2Points}}{60} \right) (.2)
$$

I rounded the number grade to one decimal place and assigned a letter grade as follows: A (92.0–100); A- (88.0–91.9); B+ (83.0–87.9); B (77.0–82.9); B- (74.0–76.9); C+ (70.0–73.9); C (60.0–69.9); C- (55.0–59.9); D+ (50.0–54.9); D (40.0–49.9); D- (35.0–39.9)

The mean number grade was approximately 74, and the median number grade was approximately 75.
1. Consider the set $W$ of polynomials of the form $a_0 + a_1 x + a_2 x^2 + a_3 x^3$ such that the product of coefficients $a_0 a_1 a_2 a_3 = 0$. Determine whether $W$ is a subspace of the space $P$ of all polynomials. Explain your reasoning.

**SOLUTION:**

In order to be a subspace of $P$, $W$ must be a subset of $P$ that is closed under addition and closed under scalar multiplication. In this case, $W$ is closed under scalar multiplication, but $W$ is not closed under addition, so it is not a subspace.

For example, $p_1 = 1 + x$ is a member of $W$, and $p_2 = x^2 + x^3$ is a member of $W$, but $p_1 + p_2 = 1 + x + x^2 + x^3$ is not a member of $W$. 
2. Consider the following system of equations:

\[
\begin{align*}
  x - 2y + 3z &= a \\
  -2x + 3y + z &= b \\
  -2x + 6y + z &= c \\
  -3x + 7y + 5z &= d
\end{align*}
\]

Find a set of constants \(a, b, c,\) and \(d\) such that the system has a unique solution, or show that such constants do not exist.

**SOLUTION:**

For a given \(a, b, c,\) and \(d,\) this system has four equations and three unknowns, so we might expect that there will be no solutions \((x, y, z).\) However if exactly one equation is redundant (i.e., a linear combination of the other three) then there may be a unique solution. We can determine if this is a possibility through Gaussian elimination of the augmented matrix:

\[
\begin{bmatrix}
  1 & -2 & 3 & a \\
  -2 & 3 & 1 & b \\
  -2 & 6 & 1 & c \\
  -3 & 7 & 5 & d
\end{bmatrix}
\rightarrow
\begin{bmatrix}
  1 & -2 & 3 & a \\
  0 & -1 & 7 & 2a + b \\
  0 & 2 & 7 & 2a + c \\
  0 & 1 & 14 & 3a + d
\end{bmatrix}
\rightarrow
\begin{bmatrix}
  1 & -2 & 3 & a \\
  0 & -1 & 7 & 2a + b \\
  0 & 0 & 21 & 6a + 2b + c \\
  0 & 0 & 21 & 5a + b + d
\end{bmatrix}
\]

If the last two rows are the same, then one equation is redundant and we’re left with a \(3 \times 3\) system in echelon form. Therefore there will be a unique solution whenever \(6a + 2b + c = 5a + b + d,\) or more simply \(a + b + c = d.\)
3. Consider the matrix

\[
A = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 \\
0 & 1 & 1 & 0
\end{bmatrix}
\]

(a) Find a basis for the row space of \( A \).

\text{SOLUTION:}

The row space is the space spanned by the row vectors of \( A \). If we use row operations to convert \( A \) to a matrix in echelon form \( E \), then the non-zero row vectors of \( E \) will form a basis.

\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 \\
0 & 1 & 1 & 0
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 \\
0 & 1 & 1 & 1 \\
0 & 1 & 1 & 0
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\( E \) = \[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

The first three rows of \( E \) are three vectors that form a basis for the row space of \( A \) (note this is not the only possible answer).

(b) Find a basis for the null space of \( A \).

\text{SOLUTION:}

The null space of \( A \) is the solution space to the matrix equation \( Ax = 0 \). Using our work from part a), we can reduce the matrix equation to

\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\]

\( x_3 \) is a free variable, so we let \( x_3 = t \). Then we use back substitution and get \( x_4 = 0, x_2 = -t, \) and \( x_1 = 0 \). So there are infinite
solutions of the form

$$x = t \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

Any non-zero value of $t$ will give us a vector that is a basis for the null space of $A$
4. Consider the matrix

\[ A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \]

(a) Find \( A^{-1} \), or show that it does not exist

**SOLUTION:**

We perform Gaussian elimination on the 3 × 3 matrix \([AI]\):

\[
\begin{bmatrix}
1 & 0 & 1 & 1 & 0 & 0 \\
0 & 2 & 0 & 0 & 1 & 0 \\
0 & 0 & 3 & 0 & 0 & 1
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\
0 & 0 & 1 & 0 & 0 & \frac{1}{3}
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & 0 & 1 & 0 & -\frac{1}{3} \\
0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\
0 & 0 & 1 & 0 & 0 & \frac{1}{3}
\end{bmatrix}
\]

So,

\[ A^{-1} = \begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \]

(b) Find \( \det(A) \)

**SOLUTION:**

Since \( A \) is an upper-diagonal matrix, the determinant is the product of the diagonal elements:

\[ \det(A) = 1 \cdot 2 \cdot 3 = 6 \]

(c) Find \( \text{rank}(A) \)

**SOLUTION:**

Because \( A^{-1} \) exists (part a), or because \( \det(A) \neq 0 \) (part b), we know from the list of equivalent properties that \( \text{rank}(A) = n \), where \( n \) is the number of rows or columns of \( A \). Therefore:

\[ \text{rank}(A) = 3 \]
5. Consider the vectors \( \mathbf{u} = (1, 2, 3) \), \( \mathbf{v} = (4, 5, 6) \), and \( \mathbf{w} = (2, 1, 0) \).

(a) Write \( \mathbf{w} \) as a linear combination of \( \mathbf{u} \) and \( \mathbf{v} \), or show that it is not possible to do so.

**SOLUTION:**

We try to find \( c_1 \) and \( c_2 \) such that

\[
\begin{bmatrix}
  1 & 4 \\
  2 & 5 \\
  3 & 6
\end{bmatrix}
\begin{bmatrix}
  c_1 \\
  c_2
\end{bmatrix} =
\begin{bmatrix}
  2 \\
  1 \\
  0
\end{bmatrix}
\]

We can use Gaussian elimination on the augmented matrix:

\[
\begin{bmatrix}
  1 & 4 & 2 \\
  2 & 5 & 1 \\
  3 & 6 & 0
\end{bmatrix} \rightarrow
\begin{bmatrix}
  1 & 4 & 2 \\
  0 & -3 & -3 \\
  0 & -6 & -6
\end{bmatrix} \rightarrow
\begin{bmatrix}
  1 & 4 & 2 \\
  0 & -3 & -3 \\
  0 & 0 & 0
\end{bmatrix}
\]

This leaves us with the equations \(-3c_2 = -3\) and \(c_1 + 4c_2 = 2\), which have a unique solution: \( c_1 = -2 \) and \( c_2 = 1 \). This result means that:

\[ \mathbf{w} = -2\mathbf{u} + \mathbf{v} \]

(b) Are \( \mathbf{u} \), \( \mathbf{v} \), and \( \mathbf{w} \) linearly independent? Why or why not?

**SOLUTION:**

No, they are linearly dependent because one can be written as a linear combination of the other two, as shown in part a. In other words, there is a set of constants \( c_1 \), \( c_2 \), and \( c_3 \) not all zero such that

\[ c_1 \mathbf{u} + c_2 \mathbf{v} + c_3 \mathbf{w} = 0 \]

(c) Do \( \mathbf{u} \), \( \mathbf{v} \), and \( \mathbf{w} \) form a basis for \( \mathbb{R}^3 \)? Why or why not?

**SOLUTION:**

No. A basis for \( \mathbb{R}^3 \) must contain three linearly independent vectors, which \( \mathbf{u} \), \( \mathbf{v} \), and \( \mathbf{w} \) are not as explained in part b.
6. For each bullet-point statement, write whether the statement is always true (AT), sometimes true (ST), or never true (NT).

(a) Given: a $3 \times 3$ matrix $A$, where $\det(A) = 1$:

- $A$ is invertible
  SOLUTION: Always True (list of equivalent properties)
- $A$ is row equivalent to a $3 \times 3$ matrix of all zeros
  SOLUTION: Never True (row equivalent to identity matrix but not matrix of zeros)
- The null space of $A$ has dimension 3
  SOLUTION: Never True (null space has dimension 0 when $\det(A) \neq 0$)

(b) Given: a square matrix $A$ and three distinct vectors $x_1$, $x_2$, and $b$, where $Ax_1 = b$ and $Ax_2 = b$:

- The vectors $x_1$ and $x_2$ are the only two solutions to $Ax = b$
  SOLUTION: Never True (only possibilities are no solutions, one solution, or infinite solutions — never exactly two)
- The vectors $x_1$ and $x_2$ are linearly dependent
  SOLUTION: Sometimes True (if all solutions lie on a line through the origin, they will always be dependent, but otherwise they could be independent)
- The matrix $A$ does not have an inverse
  SOLUTION: Always True (List of equivalent properties — if $A$ was invertible, there would only be one solution to $Ax = b$)
- The row space of $A$ and the null space of $A$ are orthogonal complements
  SOLUTION: Always True (This is true for any matrix as described in Sect. 4.6 of the text.)
(c) Given: a 4 × 3 matrix $A$ and a 3 × 4 matrix $B$:

- $AB = BA$
  SOLUTION: Never True ($AB$ is 4 × 4 and $BA$ is 3 × 3, so they can’t be equal)
- The matrix product $AB$ results in a 3 × 3 matrix
  SOLUTION: Never True ($AB$ is 4 × 4)
- If $AB = I$, then $(AB)^{-1} = AB$
  SOLUTION: Always True (If $AB = I$, then the second statement can be written $I^{-1} = I$, which is true by definition)