1. (a) Circle the bullet-point next to each function that could be a general solution to a linear, homogeneous, third-order differential equation.

SOLUTION: A general solution for this type of equation would have to be a linear combination of three, linearly independent functions.

- $y = c_1 e^x + c_2 e^{-x}$ — NO (there are only two functions)

- $y = c_1 \sin^2 x + c_2 + c_3 \cos^2 x$ — NO (The three functions $\sin^2 x$, 1, and $\cos^2 x$ are not linearly independent, because $\sin^2 x + \cos^2 x = 1$ for all $x$)

- $y = c_1 + c_2 x + c_3 x^2$ — YES ($1$, $x$, and $x^2$ are three linearly independent functions)

- $y = c_1 \sin(\ln x) + c_2 \cos(\ln x) + c_3 x$ — YES (Again, three linearly independent functions)

- $y = 13c_1 \cos x - 7c_2 \cos x + c_3 \sin x$ — NO (Two of the functions are constant multiples of $\cos x$, so there are not three linearly independent functions)
(b) Circle the bullet-point next to each set of functions that could form a basis for the solution space of a linear, homogeneous differential equation with constant coefficients. Assume that the constant coefficients are real numbers.

SOLUTION: The order of the equation is not specified, so the set could contain any number of linearly independent functions. However, since we are told that the equation has constant coefficients, we know that the functions that make up the basis must follow certain patterns according to the types of roots we could have for the characteristic equation (real roots, complex conjugate pairs of roots, and repeated roots).

- \( \{e^{7x}, e^{8x}, 1\} \) — YES (corresponds to roots 7, 8, and 0 of the characteristic equation)
- \( \{\cos x, e^x\} \) — NO (if \( \cos x \) is in the basis, then \( \sin x \) must also be in the basis, corresponding the pair of roots \( \pm i \))
- \( \{1, 2, 3, 4\} \) — NO (these are not linearly independent functions)
- \( \{e^x \cos x, e^x \sin x, xe^x \cos x, xe^x \sin x\} \) — YES (corresponds to a repeated pair of roots \( 1 \pm i \))
- \( \{xe^x, x^2 e^x, xe^{-x}, x^2 e^{-x}\} \) — NO (if these functions are in the basis, then the functions \( e^x \) and \( e^{-x} \) must also be in the basis, corresponding to the roots 1 and \(-1\), each with multiplicity three)
2. Find a general solution $y(x)$ to the following differential equation:

$$ y'' - 2y' + y = x \sin x $$

SOLUTION: First we find the complementary solution $y_c$, which is the general solution to

$$ y'' - 2y' + y = 0 $$

Plugging in $y = e^{rx}$ gives us the characteristic equation $r^2 - 2r + 1 = 0$, or $(r - 1)^2 = 0$, which has a repeated root $r = 1$. Therefore our complementary solution is:

$$ y_c(x) = c_1 e^x + c_2 x e^x $$

Next we find a particular solution $y_p$ to the full equation. The method of undetermined coefficients is appropriate here. The right-hand side function $x \sin x$ has a finite number of linearly independent functions appearing in itself and its derivatives ($x \sin x$, $x \cos x$, $\sin x$, and $\cos x$). None of these are duplicated in the complementary solution, so plug in a solution of the form:

$$ y_p = Ax \sin x + B \sin x + Cx \cos x + Dx \cos x $$

Taking derivatives and rearranging terms we get

$$ y_p' = -Cx \cos x + (A - D) \sin x + Ax \cos x + (B + C) \cos x $$

$$ y_p'' = -Ax \sin x - (B + 2C) \sin x - Cx \cos x + (2A - D) \cos x $$

Then,

$$ y_p'' - 2y_p' + y_p = 2Cx \sin x + (2A - 2C + 2D) \sin x - (2B + 2C) x \cos x + (2A - 2B - 2C) \cos x $$

Matching terms we get $2C = 1$, $2(-A + D - C) = 0$, $-2(B + C) = 0$, and $2(A - B - C) = 0$, which leads to $A = 0$, $B = -\frac{1}{2}$, $C = \frac{1}{2}$, and $D = \frac{1}{2}$.

The final solution is $y = y_c + y_p$, or

$$ y = c_1 e^x + c_2 x e^x - \frac{1}{2} \sin x + \frac{1}{2} x \cos x + \frac{1}{2} \cos x $$
3. Consider the vectors \( \mathbf{u} = (4, 2, 2, 1) \), \( \mathbf{v} = (-1, 0, -1, 0) \), and \( \mathbf{w} = (1, 1, 1, 2) \).

(a) Are \( \mathbf{u} \), \( \mathbf{v} \), and \( \mathbf{w} \) linearly independent? Explain your reasoning.
SOLUTION: Yes, they are linearly independent. There are a few ways to demonstrate this fact. The most direct way is to use the definition of linear independence, which is that the equation

\[
c_1 \mathbf{u} + c_2 \mathbf{v} + c_3 \mathbf{w} = \mathbf{0}
\]

has only the trivial solution \( c_1 = c_2 = c_3 = 0 \).
In matrix form, the equation is

\[
\begin{bmatrix}
4 & -1 & 1 \\
2 & 0 & 1 \\
2 & -1 & 1 \\
1 & 0 & 2
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2 \\
c_3
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

Perform Gaussian elimination on the \( 4 \times 3 \) matrix:

\[
\begin{bmatrix}
4 & -1 & 1 \\
2 & 0 & 1 \\
2 & -1 & 1 \\
1 & 0 & 2
\end{bmatrix}
\rightarrow
\begin{bmatrix}
4 & -1 & 1 \\
0 & -1 & -1 \\
0 & 1 & -1 \\
0 & -1 & -7
\end{bmatrix}
\rightarrow
\begin{bmatrix}
4 & -1 & 1 \\
0 & -1 & -1 \\
0 & 0 & -2 \\
0 & 0 & 6
\end{bmatrix}
\rightarrow
\begin{bmatrix}
4 & -1 & 1 \\
0 & -1 & -1 \\
0 & 0 & -2 \\
0 & 0 & 0
\end{bmatrix}
\]

We end up with a matrix in row-echelon form, with no free variables, so there is only the unique solution \( c_1 = c_2 = c_3 = 0 \).

(b) Do \( \mathbf{u} \), \( \mathbf{v} \), and \( \mathbf{w} \) form a basis for \( \mathbb{R}^4 \)? Why or why not?
SOLUTION: No, a basis for \( \mathbb{R}^n \) must contain \( n \) linearly independent vectors in \( \mathbb{R}^n \). In this case \( n = 4 \) and we only have three vectors, so they do not form a basis even though they are linearly independent.
4. Consider the equation for motion of a pendulum,

\[ \theta'' + c\theta' + \frac{g}{L}\theta = f(t), \]

where \( \theta(t) \) is the angle the pendulum makes from vertical at time \( t \), and \( c, g, \) and \( L \) are positive parameters representing the damping constant, gravitational constant, and length of the pendulum, respectively. The function \( f(t) \) describes external forcing.

(a) In the case \( f(t) = 0 \), find conditions on the constant \( c \) that would result in the pendulum being underdamped, critically damped, and overdamped.

SOLUTION: When \( f(t) = 0 \) we have the homogeneous case, and the solution form \( e^{rt} \) gives us the characteristic equation

\[ r^2 + cr + \frac{g}{L} = 0, \]

and this equation has roots,

\[ r = -\frac{c}{2} \pm \frac{1}{2} \sqrt{c^2 - 4\frac{g}{L}}. \]

Overdamping occurs when the two roots above are real and negative (so that the pendulum decays to the equilibrium position without oscillating), and underdamping occurs when the roots are complex with negative real part, so that the pendulum oscillates in to the equilibrium. Critical damping occurs at the transition between overdamping and underdamping. So looking at the expression under the square root above, we derive:

\[ c > 2\sqrt{\frac{g}{L}} \rightarrow \text{overdamped} \]

\[ c < 2\sqrt{\frac{g}{L}} \rightarrow \text{underdamped} \]

\[ c = 2\sqrt{\frac{g}{L}} \rightarrow \text{critically damped} \]

(b) In the case \( c = 0 \) and \( f(t) = \cos \omega t \), find a condition on the constant \( \omega \) that would result in resonance.

SOLUTION: When \( c = 0 \), we have the complementary solution

\[ y_c = c_1 \cos \left( \sqrt{\frac{g}{L}} t \right) + c_2 \sin \left( \sqrt{\frac{g}{L}} t \right) . \]

Resonance occurs when \( f(t) \) duplicates part of the complementary solution (i.e., the external forcing exactly reinforces the natural motion). So we’d have resonance when \( \omega = \sqrt{\frac{g}{L}} \).
5. Consider the matrix
\[ A = \begin{bmatrix} -1 & 7 & 11 \\ -1 & 1 & 5 \\ -1 & 0 & 4 \end{bmatrix} \]

(a) Find a basis for the null space of \( A \).

**SOLUTION:** The null space of \( A \) is the solution space to the matrix equation \( Ax = 0 \). We use row operations to convert \( A \) to a matrix in echelon form:

\[ \begin{bmatrix} -1 & 7 & 11 \\ -1 & 1 & 5 \\ -1 & 0 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 7 & 11 \\ 0 & -6 & -6 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 7 & 11 \\ 0 & -6 & -6 \\ 0 & 0 & 0 \end{bmatrix} \]

So we are looking for solutions to the system

\[ \begin{bmatrix} -1 & 7 & 11 \\ 0 & -6 & -6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \]

\( x_3 \) is a free variable, so we let \( x_3 = t \). Then we use back substitution and get \( x_2 = -t \) and \( x_1 = 4t \). So there are infinite solutions of the form

\[ x = t \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} \]

Any non-zero value of \( t \) will give us a vector that is a basis for the null space of \( A \)

(b) Calculate the determinant of \( A \).

**SOLUTION:** The fact that \( Ax = 0 \) has infinite solutions tells us right away that the determinant of \( A \) will be zero (list of equivalent properties). We can also calculate this directly: using a cofactor expansion along the bottom row we get:

\[ \det A = -1 \begin{vmatrix} 7 & 11 \\ 1 & 5 \end{vmatrix} + 4 \begin{vmatrix} -1 & 7 \\ -1 & 1 \end{vmatrix} = -1(35-11)+4(-1+7) = -24+24 = 0 \]
6. Determine whether each of the following sets of vectors is a subspace of \( \mathbb{R}^2 \). Explain why or why not.

(a) The set of all vectors \((x_1, x_2)\) such that \(x_2 = 1\).

SOLUTION: This set is not closed under addition, nor is it closed under scalar multiplication, so it is not a subspace. For example \((1, 1)\) and \((2, 1)\) are both in the set, but \((1, 1) + (2, 1) = (3, 2)\) is not. Or, \(c(x_1, x_2)\) for \(c = 0\) gives the vector \((0, 0)\), which is not in the set. There are infinite other examples that would prove the lack of closure (any one would do).

(b) The set of all vectors \((x_1, x_2)\) such that \(\sin(x_1) \cos(x_2) = 0\).

SOLUTION:

\[
\sin(x_1) \cos(x_2) = 0 \text{ if either } \sin(x_1) = 0 \text{ or } \cos(x_2) = 0.
\]

We have \(\sin(x_1) = 0\) when \(x_1 = 0, \pm \pi, \pm 2\pi, \pm 3\pi, \ldots\).

We have \(\cos(x_2) = 0\) when \(x_2 = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \pm \frac{5\pi}{2}, \ldots\).

It is possible to find examples that show that this set is not closed under addition and not closed under scalar multiplication. For instance,

\( (0, \frac{\pi}{2}) \) is in the set, and \( (\frac{\pi}{2}, \frac{\pi}{2}) \) is in the set, but

\[
(0, \frac{\pi}{2}) + (\frac{\pi}{2}, \frac{\pi}{2}) = (\frac{\pi}{2}, \pi)
\]

is not in the set:

\[
\sin(\frac{\pi}{2}) \cos(\pi) = (1)(-1) = -1
\]
7. Find a general solution $y(x)$ to the following third-order differential equation,

$$y''' = y$$

SOLUTION: This is a homogeneous equation with constant coefficients $(y''' - y = 0)$. Plugging in $y = e^{rx}$ gives us the characteristic equation

$$r^3 - 1 = 0$$

We expect three roots, counting any multiplicities. One obvious root is $r = 1$, corresponding to a basis function $e^x$. This root is *not* repeated, as would be the case if the equation were $(r - 1)^3 = 0$.

So we still need to find the other two roots. Given that $r = 1$ is a root, we know that $(r - 1)$ will factor out of the $r^3 - 1$. Performing long division on $(r^3 - 1)/(r - 1)$ gives us

$$r^3 - 1 = (r - 1)(r^2 + r + 1) = 0$$

The quadratic factor $(r^2 + r + 1)$ has two complex conjugate roots, found by the quadratic formula:

$$r = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

which correspond to the basis functions $\{e^{-\frac{1}{2}x} \cos \left(\frac{\sqrt{3}}{2}x\right), e^{-\frac{1}{2}x} \sin \left(\frac{\sqrt{3}}{2}x\right)\}$

So our general solution is

$$y(x) = c_1 e^x + c_2 e^{-\frac{1}{2}x} \cos \left(\frac{\sqrt{3}}{2}x\right) + c_3 e^{-\frac{1}{2}x} \sin \left(\frac{\sqrt{3}}{2}x\right)$$