SOME EXERCISES IN CHARACTERISTIC CLASSES

1. GAUSSIAN CURVATURE AND GAUSS-BONNET THEOREM

Let $S \subset \mathbb{R}^3$ be a smooth surface with Riemannian metric g induced from $\mathbb{R}^3$. Its Levi-Civita connection ∇ can be defined by

$$\nabla_X Y = (d_X Y)^T = d_X Y - (d_X Y)^N$$

where $(d_X Y)^T$ and $(d_X Y)^N$ denote the tangential and normal components of the usual derivative $d_X Y$ of the vector function Y in direction of the vector X. Just check that ∇ as just defined preserves g and is torsion-free.

The normal component $(d_X Y)^N$ is $A^0(S)$=bilinear in X, Y and the scalar-valued bilinear form $< d_X Y, N >= - < Y, d_X N >$ on T_pS, called the *second fundamental form of S in $\mathbb{R}^3 S$*, is symmetric in X, Y, equivalently, the linear transformation $T_pM \to T_pM$ defined by $X \to -d_X N$, is self-adjoint. This is (up to sign) the differential of the Gauss spherical map $S \to S^2$ taking $p \in S$ to $N_p \in S^2$ and T_pS to $T_{N_p}S^2 \cong T_pS$ (canonically isomorphic by parallel translation) .

Various assertions above follow by differentiating inner product relations such as $< y, N >= 0$ or $< N, N >= 1$. For example, differentiating $< N, N >= 1$ we get $< d_X N, N >= 0$, so if $X \in T_pS$, we get $d_X N \perp N$, so indeed $d_X N \in T_pS$.

Definition 1. *The determinant of this linear transformation $T_pS \to T_pS$ is called the* Gaussian curvature *of S at p, denoted K_p.*

The easiest way to do calculations is as follows:

- Represent S as a parametrized surface $\Phi : U \to \mathbb{R}^3$ for some open set $U \subset \mathbb{R}^2$ and some smooth map $\Phi : U \to \mathbb{R}^3$ everywhere of maximal rank: Φ_u, Φ_v linearly independent at each $(u, v) \in U$. In terms of u, v the induced metric g on U has expression

$$g =< \Phi_u, \Phi_u > du^2 + 2 < \Phi_u, \Phi_v > dudv + < \Phi_v, \Phi_v > dv^2 \quad (1)$$

- Take a smooth orthonormal frame $\mathbf{e}_1(u, v), \mathbf{e}_2(u, v), \mathbf{n}(u, v)$ for $\Phi^*\mathbb{R}^3$: $\mathbf{e}_1(u, v), \mathbf{e}_2(u, v)$ an orthonormal basis for $T_{\Phi(u,v)}S$ obtained, say, by applying Gram-Schmidt to $\Phi_u(u, v), \Phi_v(u, v)$, and $\mathbf{n} = \mathbf{e}_1 \times \mathbf{e}_2$.

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- Since $d\mathbf{e}_1$ is perpendicular to both $\mathbf{e}_2$ and $\mathbf{n}$ (differentiate $< \mathbf{e}_1, \mathbf{e}_2 >= 0$ and D$< \mathbf{e}_1, \mathbf{n} >= 0$), using from the above paragraphs that the $\mathbf{e}_2$-component of $d\mathbf{e}_1 = \nabla \mathbf{e}_1$, and finally the definition of connection one-forms θ_i^j from the notes on connections, we get the first equation below. Similar reasoning for the second:

$$\begin{aligned} d\mathbf{e}_1 &= \theta_1^2 \mathbf{e}_2 + \lambda_1 \mathbf{n} \\ d\mathbf{e}_2 &= \theta_2^1 \mathbf{e}_1 + \lambda_2 \mathbf{n} \\ d\mathbf{n} &= -\lambda_1 \mathbf{e}_1 - \lambda_2 \mathbf{e}_2 \end{aligned} \tag{2}$$

for suitable one-forms $\theta_1^2, \theta_2^1 = -\theta_1^2, \lambda_1, \lambda_2 \in A^1(U)$.

The third equation is obtained from the first two by first using $< d\mathbf{e}_i, \mathbf{n} >= \lambda_i$, $i = 1, 2$, and then using $< d\mathbf{e}_i, \mathbf{n} >= - < \mathbf{e}_i, d\mathbf{n} >$. A consequence of the third equation is the following formula for the Gaussian curvature (as defined above, extrinsically):

$$K\ dA = \lambda_1 \wedge \lambda_2 \tag{3}$$

where dA is the (oriented) area element of the metric g on U:

$$dA = \sqrt{\det(g)}\, du\, dv \tag{4}$$

where g is as in (1).Since the determinant meaures the distortion in area, we see that (**??**) is equivalent to Definition 1. This is the *extrinsic* of curvature, meaning that it Uses the shape of the embedding of S in $\mathbb{R}^3$.

1.1. **The Exercises.**

(1) Compute $d^2\mathbf{e}_1$ by using (2), then set the resulting expression $= 0$ (since $d^2 = 0$). Show that this gives Gauss's *Theorema Egreguium*

$$K\ dA = -d\theta_1^2 \ \ (= d\theta_2^1), \tag{5}$$

where the left-hand side is defined as in Definition 1, or, equivalently, equation (3), is an *extrinsic* quantitity (defined in terms of the shape of the embedding in $\mathbb{R}^3$), while the right-hand side is an *intrinsic* quantity (depends just on the Levi-Civita connection of the induced metric, hence just on the induced metric.)

Remark Recall, from the notes on connections, that $d_\nabla^2 s = \Omega s$ for some $\Omega \in A^2(S, Sk(T))$, where $Sk(T) \subset End(T)$ denotes the bundle of skew-symmetric endomorphisms of T, and where we have changed the $K = K_\nabla$ of the notes to Ω because now we reserve K for the Gaussian curvature (a function on S). On a metric connection on a bundle of rank 2 , Ω is represented by a matrix of 2-forms

$$\Omega = \begin{pmatrix} 0 & \Omega_2^1 \\ \Omega_1^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & d\theta_2^1 \\ d\theta_1^2 & 0 \end{pmatrix} \text{ with } \Omega_2^1 = -\Omega_1^2.$$

From the intrinsic point of view, (5) says $K\ dA = \Omega_2^1 = Pf(\Omega)$.

(2) Let $\Phi(u, v) = (\sin u \cos v, \sin u, \sin v, \cos u)$, $0 \le u \le \pi$, $0 \le v \le 2\pi$ be the standard parametrization of the unit sphere centered at the origin by spherical coordinates (where the circles $u = 0, u = \pi$ collapse to the north pole, south pole respectively), Make the following choices for $\mathbf{e}_1, \mathbf{e}_2, \mathbf{n}$:

$$\begin{aligned} \mathbf{e}_1 &= \phi_u = (\cos u \cos v, \cos u \sin v, -\sin u), \\ \mathbf{e}_2 &= \Phi_v / \cos u = (-\sin v, \cos v, 0), \qquad (6) \\ \mathbf{n} &= \Phi. \end{aligned}$$

Work out explicitly all the equations (1) to (5) for these particular choices of $\Phi, \mathbf{e}_1, \mathbf{e}_2, \mathbf{n}$. In particular, what is the value of K?

(3) Suppose now that S is an oriented surface with a Riemannian metric g (not neccessarily induced from an embedding in $\mathbb{R}^3$) and associated Levi-Civita connection ∇. We have seen two ways of looking at the curvature d_∇^2;
   (a) $d_\nabla^2 s = \Omega s$ for some $\Omega \in A^2(S, Sk(T))$ as above and in the notes..
   (b) $d_\nabla^2 s = \iota_s R$ for some $R \in A^2(S, \Lambda^2 T) \subset \oplus_{p,q} S^p(S, \Lambda^q T)$ as in class. We have, for $x \in T$, a contraction operator $\iota_x : \Lambda^p T \to \Lambda^{p-1} T$ and in particular an isomorphism $\Lambda^2 T \xrightarrow{\cong} Sk(T)$ that takes $x \wedge y \in \Lambda^2 T$ to the skew-symmetric endomorphism $z \to \iota_z(x \wedge y) =< x, z > y- < y, z > x$ for $x, y, z \in T$. This isomorphism takes R to Ω.
   Let $\mathbf{u} \in A^0(S, \Lambda^2 T)$ be the unique section that is of unit length and is positive for the chosen orientation. Over any open set with positively oriented orthonormal frame $\mathbf{e}_1, \mathbf{e}_2$, $\mathbf{u} = \mathbf{e}_1 \wedge \mathbf{e}_2$. Note that the Pfaffian $Pf(R) \in A^2(S)$ is obtained from $R \in A^2(S, \Lambda^2 T)$ by

$$Pf(R) =< u, R >$$

   The Exercise:

(i) Let $\sigma \in A^0(S,T)$ be a vector field of constant length: $<\sigma,\sigma>=1$. Prove that

$$d_\nabla(\sigma \wedge d_\nabla \sigma) = R$$

consequently

(7)
$$d < u, \sigma \wedge d_\nabla \sigma >= Pf(R)$$

(ii) Now suppose $S = \bar{S} \setminus \{p_1, \ldots, p_n\}$ where $\bar{S}$ is a *closed* surface and $\{p_1, \ldots, p_n\}$ is the set of zeros of a vector field on $\bar{S}$ having only simple zeros. Apply Stokes's theorem and (7) to $\sigma = s/|s|$ to get

(8)
$$\int_{\bar{S}} Pf(R) = \sum_{i=1}^{n} \iota_{p_i}(s) = \chi(S)$$

(4) Apply the procedure of the last problem to the example of spherical coordinates (Exercise (2) above). Check that equations (7) and (8) for σ one of the fields $\mathbf{e}_1$ or $\mathbf{e}_2$.

2. Additional Exercises

(1) (Milnor-Stasheff 4B): Prove the following theorem of Stiefel: If $n+1 = m2^r$ with m odd, then $\mathbb{R}P^n$ does not have 2^r vector fields that are linearly independent at every point. In particular, show that $\mathbb{R}P^{4k+1}$ has a nowhere zero vector field but does not have 2 vector fields that are linearly independent at every point.

(2) (Milnor-Stasheff 5E): A vector bundle $E \to B$ is said to be of *finite type* if and only if B can be covered by finitely many open sets $U_1, \dots, U_k$ such that $E|_{U_i}$ is trivial for $i = 1, \dots, k$. Prove that the tautological line bundle $L \to \mathbb{R}P^\infty$ is not of finite type. This is the bundle denoted by γ^1 in Milnor-Stasheff. (Note that a bundle over a reasonable space of finite covering dimension (such as a finite-dimensional manifold) is necessarily of finite type. So to find an example not of finite type the base space B must be infinite-dimensional.)

(3) (Milnor-Stasheff 15B): Let $G_n(\mathbb{R}^\infty)$, respectively $\tilde{G}_n(\mathbb{R}^\infty)$ denote the Grassmannian of *unoriented*, respectively *oriented* n-dimensional subspaces of $\mathbb{R}^\infty$. Observe that $\tilde{G}_n(\mathbb{R}^\infty)$ is a two - sheeted covering space of $G_n(\mathbb{R}^\infty)$. Let Λ be an integral domain in which 2 is invertible. For instatnce, could take $\Lambda = \mathbb{Q}$. Recall that we have computed $H^*(\tilde{G}_n(\mathbb{R}^\infty), \Lambda)$.

Exercise: Prove that $H^*(G_n(\mathbb{R}^\infty), \Lambda)$ is the polynomial ring over Λ generated by the Pontrjagin classes $p_1(\gamma^n), \dots, p_{[n/2]}(\gamma^n)$ of the universal (= tautological) $\mathbb{R}^n$-bundle γ^n over $G_n(\mathbb{R}^\infty)$.

Suggestion Prove first the following general statement: For any double covering $\pi : \tilde{X} \to X$ with covering transformation $t : \tilde{X} \to \tilde{X}$ of order two, $\pi^* : H^*(X, \Lambda) \to H^*(\tilde{X}, \Lambda)$ is injective and its image is the fixed point set of the involution $t^* : H^*(\tilde{X}, \Lambda) \to H^*(\tilde{X}, \Lambda)$.

(4) Let $L \to M$ and $E \to M$ be a complex line bundle and a complex vector bundle of rank n (that is, fiber $\mathbb{C}^n$) respectively.

   (a) Compute the total Chern class $c(L \otimes E)$ in terms of the Chern classes of L and E. (*Suggestion*: Use the splitting principle).

   (b) Same for $c(Hom(L, E))$.

   (c) Suppose the base M has real dimension $2n$, and let L, E be as above. Give a necessary and sufficient condition, in terms of the Chern classes of L and E, for the existence of an *injective* bundle homomorphism $\phi : L \to E$.

(5) *A quick look at complex hypersurfaces in* $\mathbb{P}^{n+1}$:
Let $h \in H^2(\mathbb{P}^{n+1}, \mathbb{Z}) \cong \mathbb{Z}$ be the generator with the property that $h([\mathbb{P}^1]) = 1$, where $\mathbb{P}^1 \subset \mathbb{P}^{n+1}$ is the natural holomorphic embedding (actually linear). This is called the *positive generator* of H^2. The complex line bundles over $\mathbb{P}^{n+1}$ often denoted $\mathcal{O}(d)$, $d \in \mathbb{Z}$, notation chosen so that $c_1(\mathcal{O}(d)) = dh$. Thus for $d < 0$, $\mathcal{O}(d)$ has no holomorphic sections other than the identically zero section, and $\mathcal{O}(0)$ is the trivial bundle with its holomorphic sections constant functions on $\mathbb{P}^{n+1}$.

Now, if $d > o$, the space of holomorphic sections of $\mathcal{O}(d)$ is in bijective correspondence with the space $P(n+2, d)$ of homogeneous polynomials of degree d in $n+2$ variables. If $f \in P(n+2, d)$, the equation $f = 0$ defines a subset (analytic subvariety) of $\mathbb{P}^{n+1}$, the zero set of a holomorphic section of $\mathcal{O}(d)$. If the only common zero of the equations $f_{x_i} = 0$ (partial derivaatives with respect to all $n+2$ variables) is at the origin, then $f = 0$ defines a non-singular hypersuface, let's call it X_d. It is a complex manifold of complex dimension n. All non-singular f give diffeomorphic X_d, under diffeomorphisms that preserve the Chern classes.

Fix n and d, let $X = X_d$ and let $\iota_X : X \to \mathbb{P}^{n+1}$ be the embedding. The normal bundle of X in $\mathbb{P}^{n+1}$ is easily seen to be $\iota_X^*(\mathcal{O}(d))$ (from the fact that X is the zero set of a section of $\mathcal{O}(d)$ transverse to the zero section). So the standard decomposition (C^∞ but not holomorphic)

$$TX \oplus NX = \iota_X^* T\mathbb{P}^{n+1} \tag{9}$$

becomes

$$TX \oplus \iota_X^* \mathcal{O}(d) = \iota_X^* T\mathbb{P}^{n+1} \tag{10}$$

If we let $u = \iota_X^* h \in H^2(X, \mathbb{Z})$ (where $h \in H^2(\mathbb{P}^{n+1}, \mathbb{Z})$ is the positive generator as above), then (10) allows us to compute $c(X)$ as follows:

$$c(X) = \frac{(1+u)^{n+2}}{1+du} \tag{11}$$

(a) *Warm-up exercise* Use (11) to compute:

(i) $c_1(X)$ for any n and d in terms of u.

(ii) Argue that

$$u^n([X]) = d \tag{12}$$

by interpreting (by Poincaré duality) cup products as intersection products, and using that $u^n = \iota_X^* h^n$ and h^n is Poincaré dual to a line ($\mathbb{P}^1$) in $\mathbb{P}^{n+1}$.

(iii) Use the two previous problems to derive formulas for the Euler characteristic $\chi(X_d) = c_n(X_d)$ for $n = 1, 2, 3$.

(b) Now to Milnor-Stasheff 16- D. First we need to simplify the definition of the Chern class $s_k(E)$. For a line bundle, define $s_k(L) = c_1(L)^k \in H^{2k}(X)$. Extend to all bundles by requiring that it be additive and natural. This means: If $E = L_1 \oplus \cdots \oplus L_m$, $s_k(E) = c_1(L_1)^k + \cdots + c_1(L_m)^k$, and extend to all bundles by the splitting principle. (In terms of symmetic functions, the $s_k(t_1, \ldots, t_n)$ are the *power sums* $s_k(t_1, \ldots, t_n) = t_1^k + \cdots + t_n^k$ are, for fixed n and $k \leq n$, an alternative set of *algebra* generators for the algebra of symmetric functions in $t_1, \ldots, t_n$, related to the elementary symmetric functions by the famous Newton formulas.)

Observe that $s_k(E \oplus F) = s_k(E) + s_k(F)$.

Exercise Prove that $s_n(X_d) = d(n + 2 - d^n)$.

(6) (Milnor-Stasheff 16-E, F) This exercise gives more constructions of complex n-dimensional manifolds X with $s_n(X) \neq 0$ and of real n-dimensional manifolds Y with the analogous Stiefel-Whhitney class $s_n(Y) \neq 0$. Such a Y is not cobordant to a sum of products of lower dimensional manifolds

(a) Hypersurfaces in products of complex projective spaces:

The holomorphic line bundles in products $\mathbb{P}^m \times \mathbb{P}^n$ of two projective spaces are the bundles $\mathcal{O}(d_1, d_2) = \pi_1^*\mathcal{O}(d_1) \otimes \pi_2^*\mathcal{O}(d_2)$, where π_1, π_2 are the projections onto the two factors.

Assume $2 \leq m \leq n$ and let $X_{m,n} \subset \mathbb{P}^m \times \mathbb{P}^n$ be the zero set of a holomorphic section s of $\mathcal{O}(1, 1)$ which is transeverse to the zero section. Equivalently, $X_{m,n}$ is the zero set in $\mathbb{P}^m \times \mathbb{P}^n$ of a polynomial $f(x, y)$, for $(x, y) \in \mathbb{C}^{m+1} \times \mathbb{C}^{n+1}$ that is of bidegree $(1, 1)$: of degree one in x and in y, that is, $f(sx, ty) = stf(x, y)$ for all $s, t \in \mathbb{C}$, and so that the common solutions of $f_{x_i} = 0, f_{y_j} = 0$ are contained in $0 \times \mathbb{C}^{n+1} \cup \mathbb{C}^{m+1} \times 0$, for example

(13) $$f(x, y) = x_0 y_0 + x_1 y_1 + \cdots + x_m y_m$$

Then $X_{m,n}$ is a complex hypersurface in $\mathbb{P}^m \times \mathbb{P}^n$, therefore a complex manifold of complex dimension $m + n - 1$.

Exercise Prove that

(14) $$s_{m+n-1}(X_{m,n}) = -\frac{(m+n)!}{m!n!}$$

Suggestion Use the same reasoning in this situation as the one used in (9) and (10) to derive formulas for the characteristic classes of $X_{m,n}$. In particular, show that, since $T(\mathbb{P}^m \times \mathbb{P}^n) = \pi_1^* T\mathbb{P}^m \oplus \pi_2^* T\mathbb{P}^n$ of bundles of rank less than n, its s_{m+n-1}-class vanishes, so $s_{m+n-1}(X_{m,n}) = -s_{m_n-1}(\mathcal{O}(1,1))$. Then compute its Chern class from $\mathcal{O}(1,1) = \pi_1^*\mathcal{O}(1) \otimes \pi_2^*\mathcal{O}(1)$.

(b) Hypersurfaces in products of real projective spaces:

The procedure just studied for complex manifolds and Chern classes can be repeated for real manifolds and Stiefel-Whitney classes. For instance, the class $s_k(E)$ can be defined, using the splitting principle, as the class determined by $s_k(L) = w_1(L)^k$ when L is a real line bundle.

Let's just look at one example, which is particularly interesting in that it gives the smallest example of an oriented odd-dimensional manifold that is not a boundary. Let $Y \subset \mathbb{R}P^2 \times \mathbb{R}P^4$ be a non-singular hypersurfce of bi-degree $(1, 1)$: take $f(x, y) = 0$ with f as in (13) with $m = 2, n = 4$.

Exercise

(i) Prove that Y is orientable. (Suggestion: compute $w_1(Y)$)
(ii) Prove that $s_5(Y) \neq 0$. (Suggestion: derive and use the analogue of (14)).

(7) Show that the odd - dimensional complex projective spaces $\mathbb{C}P^{2m+1}$ bound by producing an explicit orientable manifold X of real dimension $4m + 3$ with boundary $\mathbb{C}P^{2m+1}$

Suggestion: $\mathbb{C}P^{2m+1}$ is the space of complex one-dimensional linear subspaces of $\mathbb{C}^{2m+2}$. Let $\mathbb{H}$ denote the quaternions, let $\mathbb{H}P^m$ denote the space of one-dimensional right quaternionic linear subspaces of $\mathbb{H}^{m+1}$. Identify $\mathbb{C}^{2m+2}$ with $\mathbb{H}^{m+1}$. Show that every complex line is contained in a unique right-quaternionic line, and that this leads to a fibration

$$S^2 \to \mathbb{C}P^{2m+1} \to \mathbb{H}P^m$$

Then find a real vector bundle E with fiber $\mathbb{R}^3$ whose sphere bundle is the above S^2-bundle.