A CRASH COURSE ON CONNECTIONS

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1. Introduction

These notes are meant to give a quick view of the theory of connections and curvature for vector bundles. Basically we follow the outline of §2 of [1], see also Appendix C of [3]. The main goal is to explain that the characteristic classes of a Riemannian manifold have local expressions in terms of the metric.

These notes are very preliminary. Watch for updates by looking at the date. Comments, corrections, requests for explanations, are all welcome.

2. Definition of Connections and Curvature

Everything will be C^∞. Let $E \to M$ be a smooth bundle over a smooth manifold. We write $A^0(E)$ for the space of smooth sections of E and $A^k(E)$ for the space of smooth sections of the vector bundle $\Lambda^k(T^*M) \otimes E$. In other words, $A^k(E)$ Is the space of smooth k-forms with coefficients in E. Alternatively, $A^k(E) = A^k(M) \otimes_{A^0(M)} A^0(E)$.

Definition 1. *A connection on E is an $\mathbb{R}$-linear map $\nabla : A^0(E) \to A^1(E)$ with the property that*

$$\nabla(fs) = df \otimes s + f\nabla s \tag{1}$$

holds for all $f \in A^0(M)$ and for all $s \in A^0(E)$. If $p \in M$ and $X \in T_pM$, and $i_X : T_p^*M \to \mathbb{R}$ denotes evaluation at X, then we write

$$\nabla_X s = (i_X \otimes id)\nabla s, \tag{2}$$

called the covariant derivative of s at p in the direction of X. *The defining equation (1) becomes*

$$\nabla(fs) = (Xf)s + f\nabla_X s \tag{3}$$

In other words, ∇s puts together the covariant derivatives $\nabla_X s$ in the direction of tangent vectors $X \in T_pM$ into a single element of $T^*M \otimes E$. This element is evaluated on each $X \in T_pM$ by means of (2).

The equation (1) means, in particular, that ∇ is a first-order differential operator $A^0(E) \to A^1(E)$. In other words, $(\nabla s)(p) = 0$ whenever s is a section that vanishes to second order at $p \in M$. A section vanishes to second order at p if and only if it can be written as $s = \sum f_i s_i$ where $f_i(p) = 0$ and $s_i(p) = 0$. This is a

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coordinate-free way of saying that in any local coordinate system, s and all its first order partial derivatives vanish at p. Then using (1) we see that

$$(\nabla s)(p) = \sum (df_i(p) \otimes s_i(p) + f_i(p)\nabla s(p)) = 0.$$

In particular, this implies that $(\nabla s)(p)$ depends just on the value of s on a neighborhood of p, in other words, it is a *local* operator. But it says much more: Define an equivalence relation on $A^0(E)$ by saying $s_1 \sim s_2$ if $s_1 - s_2$ vanishes to second order at p. Then $(\nabla s_1)(p) = (\nabla s_2)(p)$.

2.1. **Digression into jets.** It can be checked that the collection of these equivalence classes, as we vary p, forms a vector bundle over M, called the *bundle of one-jets of sections of* E, denoted $J^1(E)$. There is a well defined evaluation map $ev_p : J^1(E) \to E$ induced by $ev_p : A^0(E) \to E_p$. The latter has kernel the space of sections that vanish at p, thus the former has kernel the space of one-jets of sections that vanish at p. This last space is isomorphic to $T_p^*M \otimes E_p$. Thus the bundle $J^1(E)$ fits into an exact sequence

$$T^*M \otimes E \to J^1(E) \xrightarrow{ev} E \tag{4}$$

Let $j_p^1 : A^0(E) \to J^1 E$ denote the map that sends s to its equivalence class. A first order differential operator $D : A^0(E) \to A^0(F)$ (E and F vector bundles over M) is equivalent to a *bundle map* or *zeroth-order differential operator* $\tilde{D} : J^1 E \to F$ by the rule $Ds = \tilde{D} j^1 s$. Under this equivalence a connection is a bundle map

$$\tilde{\nabla} : J^1(E) \to T^*M \otimes E \text{ such that } \tilde{\nabla}|_{T^*M\otimes E} = id, \tag{5}$$

in other words, a splitting of (4).

2.2. **Space of connections.** The point of connections is that in a general vector bundle $E \to M$ there is no canonical way to differentiate sections. A connection gives a way to differentiate sections in all directions. First we should prove existence:

(1) In the trivial bundle $M \times \mathbb{R}^n$, or better, in the *trivialized bundle*, since we are choosing a particular trivialization, $A^0(E)$ is the same as $(A^0(M))^n$, vector functions $M \to \mathbb{R}^n$, and we can define $\nabla s = ds$, componentwise exterior derivative. More precisely, a section $s = (f_1, \ldots, f_n)$ where $f_i : M \to \mathbb{R}$ and we define ∇s by

$$\nabla s = ds = (df_1, \ldots df_n) \tag{6}$$

(2) The difference of two connections is a section of $T^*M \otimes End(E)$: If ∇_1 and ∇_2 are connections, then $(\nabla_1 - \nabla_2)(fs) = f(\nabla_1 - \nabla_2)(s)$, thus $\nabla_1 - \nabla_2$ is linear over $A^0(M)$, in other words, is a bundle map $E \to T^*M \otimes E$, in other words, a section of $T^*M \otimes End(E) = T^*M \otimes E \otimes E^*$, where E^* is the dual bundle of E.

The same computation shows that if ∇ is a connection and $A \in A^1(End(E))$, then $\nabla + A$ is a connection. So, if a connection exists, the space of connections is an affine space for the infinite-dimensional vector space $A^1(M, End(E))$.

(3) Given $E \to M$ and an open cover $\{U_\alpha\}$ of M with trivializations(bundle isomorphisms) $\phi_\alpha : E|_{U_\alpha} \to U_\alpha \times \mathbb{R}^n$, define a connection ∇^0_α on $E|_{U_\alpha}$ by

$$\nabla^0_\alpha s = (id \otimes \phi_\alpha^{-1}) d(\phi_\alpha s) \tag{7}$$

where $d(\phi_\alpha s)$ is as in (6).

(4) Let $\{\lambda_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$. Then define $\nabla s = \sum_\alpha \nabla^0_\alpha \lambda_\alpha s$

In summary, connections exist, they form an affine space for the vector space $A^1(End(E))$.

2.3. **Connection one-forms.** Let $E \to M$ be a vector bundle and choose an open cover $\{U_\alpha\}$ with local trivializations $\phi_\alpha : E|_{U_\alpha} \to U_\alpha \times \mathbb{R}^n$ as above. Let $\{e_i\}$ denote the standard basis for $\mathbb{R}^n$ and let e_i also denote the constant section e_i of $U_\alpha \times \mathbb{R}^n$. Let $s_i = \phi_\alpha^{-1}(e_i)$. Then $\{s_1, \ldots, s_n\}$ is a collection of sections of $E|_{U_\alpha}$ that forms a basis of E_p for each $p \in U_\alpha$. This collection is called a *frame* of $E|_{U_\alpha}$, or, more briefly, a *local frame* for E.

Recall the connection ∇^0_α defined in (7) which corresponds to d under ϕ_α. In terms of the frame $\{s_i\}$, ∇^0_α is characterized by $\nabla^0_\alpha s_i = 0$ for $i = 1, \ldots, n$.

Definition 2. *Let ∇ be a connection on E, and let ∇_α denote its restriction to U_α (more precisely, its restriction to $A^0(U_\alpha, E|_{U_\alpha})$). Then $\nabla_\alpha = \nabla^0_\alpha + \theta_\alpha$ for a unique $\theta_\alpha \in A^1(U_\alpha, End(E))$, called the* connection one-form *of ∇ with respect to $\{s_i\}$.*

Let us drop the subscript α and assume that we are working on $E|_U$ for which a frame $\{s_i\}$ exists. Then we can write

$$\theta = \{\theta^j_i\}, i, j = 1, \ldots, n, \text{ where } \theta^j_i \in A^1(U) \tag{8}$$

In other words, $\{\theta^j_i\}$ is a matrix of scalar one-forms representing $\theta \in A^1(End(E))$.

Since $\nabla = \nabla^0 + \theta$ and $\nabla^0 s_i = 0$, the θ^j_i are characterized by

$$\nabla s_i = \sum_{j=1}^n \theta^j_i s_j \quad \text{for } i = 1, \ldots, n \tag{9}$$

Once we have these forms we can find the covariant derivative of any section on U. Namely, any section s on U can be uniquely written as

$$s = \sum_{i=1}^n \xi^i s_i \quad \text{for some } \xi_i \in A^0(U). \tag{10}$$

Then (1) and (9) give us

$$\nabla s = \sum_{i=1}^n (d\xi^i \otimes s_i + \xi^i \nabla s_i) = \sum_{i=1}^n (d\xi^i \otimes s_i + \xi^i \sum_{j=1}^n \theta^j_i s_j)$$

Relabeling the indices in the second term $\sum_{i,j} \xi^i \theta_i^j s_j$ we get the final formula

$$\nabla s = \sum_{i=1}^{n} (d\xi^i \otimes s_i + \sum_{j=1}^{n} \xi^j \theta_j^i s_i) \tag{11}$$

for the covariant derivative of any section over U.

We can go one step further in (8) and assume that $U \subset M$ is the domain of a coordinate chart $\vec{x} : U \to \mathbb{R}^m$ where $\vec{x} = (x^1, \ldots, x^m)$. Then we can write

$$\theta_i^j = \sum_{\mu=1}^{m} \Gamma_{i,\mu}^j dx^\mu, \text{ where } \Gamma_{i,\mu}^j \in A^0(U), \tag{12}$$

and (11) becomes

$$\nabla s = \sum_{i=1}^{n} (d\xi^i \otimes s_i + \sum_{j=1}^{n} \sum_{\mu=1}^{m} \Gamma_{j,\mu}^i \xi^j dx^\mu s_i) \tag{13}$$

The functions $\Gamma_{i,\mu}^j$ are usually called the *Christoffel symbols*.

2.4. **Connections on induced bundles.** Let ∇ be a connection on $E \to M$, let N be a smooth manifold and let $f : N \to M$ be a smooth map. Then we have the induced bundle (or pull-back) $f^*E \to N$ defined by the requirement that it be a fiber product:

$$\begin{array}{ccc} f^*E & \longrightarrow & E \\ \downarrow & & \downarrow \\ N & \xrightarrow{f} & M \end{array}$$

A section s of s^*E is a map $s : N \to E$ satisfying $s(x) \in E_{f(x)}$ for all $x \in N$. If ∇ is a connection on E we want to define an *induced connection* $f^*\nabla$ on f^*E. Since f can be quite arbitrary, it is difficult to give a short definition that works in all generality. It seems best to use local frames. Namely, let $\{U_\alpha\}$ be an open cover of M so that there is a frame $\{s_i^\alpha\}$ for each $E|_{U_\alpha}$. Then $\{f^{-1}U_\alpha\}$ is a cover of N over which $E|_{f^{-1}U_\alpha}$ has frames $\{f^* s_i^\alpha\}$.

Definition 3. *Using the notation just explained, if θ_α are the connection forms for ∇ with respect to the frame $\{s_i^\alpha\}$ over U_α, then the* induced connection $f^*\nabla$ *is defined to be the connection on f^*E with connection forms $f^*\theta_\alpha$ over $f^{-1}U_\alpha$ with respect to the frames $\{f^* s_i^\alpha\}$.*

Explicitly, if U is an open set on which we have a frame $\{s_i\}$ and connection form θ (one of the U_α, drop the α for simplicity), then on $f^{-1}(U)$ we have that, as in (9), $f^*\nabla$ is defined by

$$(f^*\nabla)(f^* s_i) = \sum_{j=1}^{n} (f^*\theta_i^j)(f^* s_j) \tag{14}$$

As in (10) we have that any section of f^*E over $f^{-1}(U)$ can be written as $s = \sum \xi_i f^* s_i$, and we get

$$\nabla s = \sum_{i=1}^{n} (d\xi^i \otimes f^* s_i + \sum_{j=1}^{n} \xi^j (f^* \theta^i_j)(f^* s_i). \tag{15}$$

In terms of Christoffel symbols

$$(f^*\nabla)s = \sum_{i=1}^{n} (d\xi^i \otimes (f^* s_i) + \sum_{j=1}^{n} \sum_{\mu=1}^{m} (f^* \Gamma^i_{j,\mu}) \xi^j df^\mu f^* s_i) \tag{16}$$

where $f = (f^\mu)$ in local coordinates.

2.5. **Parallel translation.** Let ∇ be a connection on $E \to M$ and let $\gamma : [a, b] \to M$ be a smooth path.

Definition 4. *A section s of $\gamma^* E$, also called a section of E along γ, is called parallel along γ if and only if $(\gamma^*\nabla)(s) = 0$, where $\gamma^*\nabla$ is the induced connection.*

If we write out (13) explicitly, and also write $d\xi^i = \frac{d\xi^i}{dt} dt$, we get the following system of equations for parallel sections $s = \sum \xi^i \gamma^* s_i$:

$$\frac{d\xi^i}{dt} + \sum_{j,\mu} \Gamma^i_{j,\mu}(\gamma(t))\, \xi^j(t)\, \frac{d\gamma^\mu}{dt} = 0. \tag{17}$$

This is a system of ODE's. Therefore the existence and uniqueness theorem for ODE's tells us that given any vector $v \in E_{\gamma(a)}$ there exists a unique parallel section s along γ with $s(a) = v$. The vector $s(1)$ is called the *parallel translate* of v along γ, denoted $P(\gamma)^{\gamma(b)}_{\gamma(a)} v$. This defines a linear transformation $P(\gamma)^{\gamma(b)}_{\gamma(a)} : E_{\gamma(a)} \to E_{\gamma(b)}$, called parallel transport. Uniqueness implies that for a concatenation of paths the parallel transports compose: $P(\gamma)^{\gamma(c)}_{\gamma(b)} P(\gamma)^{\gamma(b)}_{\gamma(a)} = P(\gamma)^{\gamma(c)}_{\gamma(a)} : E_{\gamma(a)} \to E_{\gamma(b)}$. In general, if we take two paths γ_1, γ_2 from p to q, $P(\gamma_1)^q_p \neq P(\gamma_2)^q_p$, no matter how close the two paths γ_1, γ_2 may be. Equivalently, if γ is a loop at p, $P(\gamma)^p_p \neq id$ no matter how small γ is.

A closely related question is the following: suppose $\phi : R \to M$ is a mapping of a rectangle $R = [a, b] \times [c, d] \to M$, does there exist a section s along R, that is, a section of $\phi^* E$ that is parallel: $(\phi^*\nabla)s = 0$? The answer is in general *no*. The best way to see this u=is to first intrioduce thd concept of *curvature*.

2.6. **Curvature.** Let $\nabla : A^0(E) \to A^1(E)$ be a connection on E. Extend ∇ to a differential operator $d_\nabla : A^k(E) \to A^{k+1}(E)$ as follows: recall that $A^k(E) = A^k(M) \otimes_{A^0(M)} A^0(E)$ which in turn is the subspace of $A^k(M) \otimes_{\mathbb{R}} A^0(E)$ spanned by all $\omega \otimes s$ such that $(f\omega \otimes s) = \omega \otimes (fs)$ for all $f \in A^0(M)$.

Definition 5. *The* covariant exterior derivative*, denoted d_∇, is the operator $d_\nabla : A^k(E) \to A^{k+1}(E)$ defined by*

$$d_\nabla(\omega \otimes s) = d\omega \otimes s + (-1)^k \omega \wedge \nabla s. \tag{18}$$

For this definition to make sense we have to check that $d_\nabla((f\omega)\otimes s) = d_\nabla(\omega\otimes(fs))$ for all $f\in A^0(M)$. Let's check:

(1) $d_\nabla((f\omega)\otimes s) = d(f\omega)\otimes s + (-1)^k(f\omega)\wedge\nabla s = (df\wedge\omega)\otimes s + fd\omega\otimes s + (-1)^k(f\omega)\wedge\nabla s$
(2) $d_\nabla(\omega\otimes(fs)) = d\omega\otimes(fs) + (-1)^k\omega\wedge(df\otimes s + f\nabla s) = d\omega\otimes(fs) + (-1)^k(\omega\wedge df)\otimes s + (-1)^k(\omega\wedge f\nabla s)$

We see that the term $(df\wedge\omega)\otimes s$ in the first equals $(-1)^k(\omega\wedge df)\otimes s$ in the second. Similarly $(fd\omega)\otimes s$ is the same element of the tensor product as $d\omega\otimes(fs)$, similarly the two remaining terms.

An alternative, but equivalent, form of the definition of d_∇ is to use an obvious modification of one of the standard formulas for d: if $\eta\in A^k(M,E)$,

$$\begin{aligned}(d_\nabla\eta)(X_1,\dots,X_{k+1}) &= \sum(-1)^{i+1}\nabla_{X_i}(\eta(X_1,\dots,\hat{X}_i,\dots,X_{k+1}))\\ &+ \sum(-1)^{i+j}\eta([X_i,X_j],X_1,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots)\end{aligned} \tag{19}$$

(perhaps up to a factor) see, for example, §2 of chapter I of [2]. A straightforward calculation using (18) or (19) gives that d_∇^2 is a zeroth-order operator, rather than second order, therefore

$$d_\nabla^2 s = Ks \text{ for all } s \text{ and for some } K\in A^2(M,End(E)) \tag{20}$$

Definition 6. *The* curvature $K_\nabla\in A^2(M,End(E))$ *of* ∇ *is the element* $K\in A^2(M,End(E))$ *uniquely defined by (20).*

Warning: Some authors choose a minus sign in (20): $d_\nabla^2 s = -Ks$.
Remarks:

(1) If E is trivial then $d_\nabla = d$, the ordinary exterior derivative (on vector-valued forms). It is well-known that $d^2 = 0$. Since d_∇ is a first order operator with the same leading term as d, it is clear that d_∇^2 must have order strictlu less than two. Then a computation shows its order is actually zero.
(2) On $A^k(M,E)$, $k>0$, (20) becomes $d_\nabla^2(\omega\otimes s) = (\omega\wedge K)\otimes s$.

Now differentiate (20) again to get

$$d_\nabla^3 s = (d_\nabla K)s + Kd_\nabla s = (d_\nabla K)s + d_\nabla^3 s$$

thus $(d_\nabla K)s = 0$ for all s, hence the *Bianchi identity*

$$d_\nabla K = 0. \tag{21}$$

Proposition 1. *In the frame* $\{s_i\}$ *over* U, $K_\nabla\in A^2(U,End(E))$ *is represented by the matrix of scalar 2-forms* K_i^j *given by*

$$K_i^j = d\theta_i^j - \sum_{k=1}^n \theta_i^k\wedge\theta_k^j,\ \ i,j=1,\dots,n \tag{22}$$

Proof. Recall from (9) that $\nabla s_i = \sum_j \theta_i^j s_j$. Hence from Definition 5

$$d_\nabla \nabla s_i = \sum_j d\theta_i^j s_j - \theta_i^j \wedge \nabla s_j = \sum_j (d\theta_i^j - \sum_k \theta_i^k \wedge \theta_k^j) s_j$$

where one goes from the second to the third term by the usual trick of expanding $\nabla s_j = \sum_k \theta_j^k s_k$ and then suitably renaming indices. □

Exercise Write (22) in terms of Christoffel symbols.

2.7. **Geometric interpretation of curvature.** Suppose that ∇ is a connection on $E \to M$. We have just seen that for any parametrized curve $\gamma : [a, b] \to M$ there exist parallel sections, in fact, given any $v \in E_{\gamma(a)}$ there exiss a parralles section $s(t)$ along γ with $s(0) = v$. The situation is quite different for parametirzed surfacesas $\phi : [a, b] \times [c, d] \to M$. In fact, suppose that there exists a parallel section s over a non-degenerate such surface ϕ : $\nabla s = 0 = d_\nabla s$ holds at all $(u, v) \in [a, b] \times [c, d]$.. We can differentiate again and get $d_\nabla^2 s = Ks = 0$., thus $K(X \wedge Y)(s) = 0$ for all X, Y tangent to ϕ.

Proposition 2. *Let $\phi : [a, b] \times [c, d] \to M$ be a non-degenerate surface (that is, $d\phi$ has generically rank 2). Then there exist parallel sections $s_1, \ldots, s_n$ of $\phi^* E$ which span the fiber at each point if and only if $\phi^* K \equiv 0$.*

Proof. Suppose that there exist sections $s_1, \ldots, s_n$ of $\phi^* E$ that form a basis at each point and are all parallel: $d_\nabla s_i = 0$ for $i = 1, \ldots, n$. Then $d_\nabla^2 s_i = K s_i = 0$ for $i = 1, \ldots, n$, therefore $K(X \wedge Y) = 0$ for all X, Y tangent to the surface $\phi([a, b] \times [c, d])$, proving one direction.

For the converse, if parallel secitons of $\phi^* E$ exist, they can be constructed as follows: Take a parallel section $s(u, 0)$ of $\phi^* E|_{[a,b]\times c}$. For each $u \in [a, b]$ let $s(u, v)$ be the parallel section over $u \times [c, d]$ with initial condition our earlier $s(u, 0)$. This section is parallel along the bottom $[a, b] \times 0$ of the rectangle $[a, b] \times [c, d]$ and also along all vertical segments $u \times [c, d]$. It is parallel over $[a, b] \times [c, d]$ if and only if it satisfles the further equation $\nabla_u s(u, v) = 0$ on each horizontal line $[a, b] \times v$.

From the formula (19) for the exterior derivative we get $\nabla_u \nabla_v - \nabla_v \nabla_u = K(\frac{\partial}{\partial u} \wedge \frac{\partial}{\partial v})$. Thus $K = 0$ gives

$$0 = \nabla_u \nabla_v s(u, v) = \nabla_v \nabla_u s(u, v).$$

Thus $\nabla_u s(u, v)$ is the unique solution of $\nabla_v(\nabla_u s(u, v)) = 0$ with initial condition $\nabla_u s(u, 0) = 0$, hence $\nabla_u s(u, v) \equiv 0$ as desired. □

Proposition 3. *There exists a basis of parallel sections of $\phi^* E$ if and only if parallel transport along the boundaries of all subrectangles with vertical and horizontal sidesof $[a, b] \times [c, d]$ is the identity. This is in turn equivalent to parallel transport along all small loops being the identity, and also equivalent to $K = 0$.*

Proof. Exercise. □

This gives a good geometric interpretation of K. In particular we get that $K = 0$ if and only if parallel transport along all small loops is the identity. In this case the bundle is called *flat*.

We summarize some consequences:

(1) A connection ∇ on E is called *flat* if and only if $K_\nabla = 0$.
(2) $K_\nabla = 0$ if and only if parallel transport along all small loops in M is the identity.
(3) $K_\nabla = 0$ if and only if parallel transport along loops depends only of the homotopy class of the loop.
(4) $K_\nabla = 0$ if and only if for each fixed $p \in M$ parallel transport along loops at p gives a homomorphism $\rho : \pi_1(M,p) \to GL(E_p)$.
(5) In this case E can be described as follows: let $\pi_1(M,p)$ act on the universal cover $\tilde{M}$ of M by covering transformations. Choose a linear isomorphism $E_p \cong \mathbb{R}^n$ and let $\rho : \pi_1(M,p) \to GL(n,\mathbb{R})$ be the resulting representation. Then $E = \tilde{M} \times_\rho \mathbb{R}^n$, meaning the quotient of $\tilde{M} \times \mathbb{R}^n$ by the equivalence relation $(x,v) \sim (\gamma x, \rho(\gamma)v)$.

3. Bundles with additional structure

So far we have talked about general vector bundles $E \to M$ with fiber $\mathbb{R}^n$, with transition functions in $GL(n,\mathbb{R})$ and connections ∇ whose parallel transport $P(\gamma)_p^q$ can be an arbitrary linear isomorphism $E_p \to E_q$. These are called *linear connections*.

We can ask for additional structure. A useful general setting is to choose an associated bundle which has a very special section. This section will give E an extra structure. We consider only connections that preserve this additional structure.

Concrete examples are:

(1) Complex vector bundles: Assume that E admits a bundle map $J : E \to E$ such that $J^2 = -id$. Then we can make the complex numbers $\mathbb{C}$ act on E by $(x+iy)v = xv + yJv$. In other words, $i \in \mathbb{C}$ acts by J, and the action extends to $\mathbb{C}$ by $\mathbb{R}$-linearity. thus all the fibers of E are complex vector spaces. We only consider connections ∇ that commute with J: $\nabla(Js) = J\nabla s$. Otherwise said, $\nabla J = 0$ or J is parallel. This is the same as asking that parallel transport is always complex linear.
(2) Riemannian vector bundles: there is a section g of the bundle S^2E^* of symmetric 2-tensors that is positive definite at each point. This makes evert fiber E_p into a Euclidean vector space E_p, g_p. We only consider connections for which parallel transport is an isometry with respecto to g. This means that $dg(X,Y) = g(\nabla X, Y) + g(X, \nabla Y)$ or that g is parallel: $\nabla g = 0$. This is the same as asking that parallel translation is always an isometry. Thus E is an *orthogonal* vector bundle.
(3) We can combine the above two examples: ask that there is a $J : E \to E$ with $J^2 = -id$, a section h of S^2E^* which is positive definite at each point. Moreover ask that they be compatible in the sense that $h(JX,JY) = h(X,Y)$ for all $X, Y \in A^0(E)$, thus J is an isometry and h_p is a *hermitian form* on the complex vector space E_p. Finally we consider only connections that preserve both structures. This is the same as asking that all

parallel transports are complex linear isometries. Thus E is a *hermitian vector bundle*.

(4) Another example would be the existence of a skew-symmetric form $\omega \in A^0(\Lambda^2 E^*)$ which is non-degenerate at each point in the sense that $\omega^n \neq 0$ at any point, where $2n = \dim E_p$, and consider only connections that preserve this structure. This leads to *symplectic vector bundles* which we will probably not have the time to study.

3.1. **The space of connections preserving the extra structure.** In all the above examples the difference of two connections preserving the extra structure is an element of $A^1(M, \underline{\mathfrak{g}})$ where $\underline{\mathfrak{g}}$ is a bundle of Lie-subalgebras of $End(E)$. Taking the above examples (except the last) in the same order

(1) Complex vector bundles: The difference of two connections is an element of $A^1(End_J(E))$, where $End_J(E)$ is the bundle of J-linear endomorphisms. Each fiber $End_J(E_p)$ is a Lie subalgebra of E_p isomorphic to $\mathfrak{gl}(n, \mathbb{C})$. The space of connections preserving J is an affine space for the vector space $A^1(M, End_J(E))$.

(2) Riemannian (or orthogonal) vector bundles: The difference of two connections is an element of $A^1(M, End_g(E))$ where $End_g(E)$ is the bundle of infinitesimal isometries of (E, g):

$$End_g(E_p) = \{A \in End(E_p) : g(AX, Y) + g(X, AY) = 0\} \tag{23}$$

holds for all $X, Y \in E_p$. Each fiber $End_g(E_p)$ is a Lie subalgebra of $End(E_p)$ isomorphic to $\mathfrak{o}(n)$. The space of connections preserving g is an affine space for the vector space $A^1(M, End_g(E))$.

(3) Similarly for hermitian vector bundle the space of connections is an affine space for the vector space $A^1(M, End_{J,h}(E))$ with fiber $End_{J,h}(E_p)$ a Lie subalgebra of $End(E_p)$ isomorphic to $\mathfrak{u}(n)$.

3.2. **Frames and connection forms.** In all these examples it is best to choose local frames $\{s_i\}$ to be compatible with the extra structure and to make it standard:

(1) Complex vector bundles: $\{s_1, \ldots, s_n\}$ form a $\mathbb{C}$-basis at each point of U. The conection forms θ_i^j are complex-linear.

(2) Orthogonal (or Riemannian or Euclidean) bundles: the local frames are orthonormal: $g(s_i, s_j) = \delta_{ij}$. Then the connection matrices θ_i^j are *skew-symmetric*

(3) Hermitian bundles: $\{s_i\}$ a complex orthonormal basis, the connection matrices skew-hermitian.

3.3. **The curvature for additional structure.** In all the above situations the curvature of ∇ is a two-form of endomorphisms with extra structure:

(1) Complex vector bundles: $K_\nabla \in A^2(M, End_J(E))$. Corresponding matrices K_i^j complex-valued.

(2) Riemannian vector bundles: $K_\nabla \in A^2(M, End_g(E))$. Corresponding matrices K_i^j skew-symmetric.

(3) Hermitian vector bundles: $K_\nabla \in A^2(M, End_{J,h}(E))$. Corresponding matrices K_i^j skew-hermitian.

3.4. **Invariant polynomials and characteristic forms.** In all the above situations, given a vector bundle with connection, consider the corresponding Lie algebra $\mathfrak{g}$ being one of $\mathfrak{gl}(n, \mathbb{R}), \mathfrak{gl}(n, \mathbb{C}), \mathfrak{o}(n), \mathfrak{u}(n)$. Let P polynomial function of degree k on $\mathfrak{g}$ invariant under the adjoint action of the corresponding group G, where $G = GL(n, \mathbb{R}), GL(n, \mathbb{C}), SO(n, \mathbb{R}), U(n)$. The formalism of [1] gives that $P(K) \in A^{2k}(M)$ is a well-defined closed form whose cohomology class is independent of the choice of the connection ∇ (in the class of connections compatible with the extra structure). It turns out that for $GL(n, \mathbb{R})$ these give more forms than there are characteristic classes. The reason is that some of these forms are always exact, for example, $tr(K)$ is always exact. The bijective correspondence between invariant polynomials and characteristic classes (namely, $H^*(BG, \mathbb{R})$) works for compact groups (as $SO(n)$ and $U(n)$) and for the complexification of a compact group (as $GL(n, \mathbb{C})$ which is the complexification of $U(n)$) but not for $GL(n, \mathbb{R})$ which does not fall into either of these categories.

3.5. **Local nature of the characteristic forms.** For concreteness we will concentrate in the case of Euclidean (also called orthogonal or Riemannian) vector bundles. So the structure group is $SO(n)$. (We prefer to consider only the orientable situation and connected groups.) Let $\mathcal{P}(\mathfrak{o}(n))$ denote the algebra of polynomials on $\mathfrak{o}(n)$ that are invariant under the adjoint action of $SO(n)$. We will say more about the structure of this algebra later. By a *characteristic form* we mean a form $P(K_\nabla) \in A^{2k}(M, \mathbb{R})$. Let $\mathcal{C}(E)$ denote the space of metric connections on E. We need the following observation:

Proposition 4. *For each $P \in \mathcal{P}(\mathfrak{o}(n))$ the map $c_P : \mathcal{C}(E) \to A^*(M)$ that assigns to ∇ the form $P(K_\nabla)$ is a first-order differential operator. Moreover c_P is natural: if $f : M \to N$ there is an induced map $f^* : \mathcal{C}(E) \to \mathcal{C}(f^*E)$ and $c_P(f^*\nabla) = f^*(c_P(\nabla))$.*

Proof. The space $\mathcal{C}$ can be locally parametrized by the space of local connection forms θ. The formula (22):

$$K_\nabla = d\theta - \theta \wedge \theta$$

is clearly a (first order, non-linear) differential operator. The map that assigns to K_∇ the form $P(K_\nabla)$ involves no derivatives, is a zeroth-order (non-linear operator).

□

3.6. **Riemannian manifolds.** Suppose now that (M, g) is an n-dimensional Riemannian manifold. Then TM is an orthogonal vector bundle. Moreover, it is well known that TM has a unique metric connection ($\nabla g = 0$) that is also torsion free ($\nabla_X Y - \nabla_Y X = [X, Y]$), the *Levi-Civita connection* $\nabla^{LC}(g)$. Moreover $\nabla^{LC}(g)$ is constructed from g by an explicit formula that involves only first derivatives of g, so is a differential operator in g.

Let $\mathcal{M}(M)$ denote the space of Riemannian metrics on M, let $LC : \mathcal{M}(M) \to \mathcal{C}(TM)$ be the map that assigns to a metric g its Levi-Civita connection $\nabla^{LC}(g)$ and let $c_P : \mathcal{C}(TM) \to A^*(M)$ the map of §3.5. Denote the composed map $\kappa_P : \mathcal{M}(M) \to A^*(M)$.

Proposition 5. *For each $P \in \mathcal{P}(\mathfrak{o}(n))$ the map $\kappa_P : \mathcal{M}(M) \to A^*(M)$ is a second-order differential operator. Moreover this map is natural under local diffeomorphisms: if $f : M \to N$ is a local diffeomorphism, $\kappa_P \circ f^* = f^* \circ \kappa_P$.*

Proof. Immediate from Proposition 4 and the construction of the Levi-Civita connection. □

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