Definitions


1. A Hausdorff space $X$ with a countable basis is called an $n$-dimensional manifold if every $x \in X$ has a neighborhood $U$ which is homeomorphic to an open set in $\mathbb{R}^n$. It is called an $n$-dimensional manifold with boundary if every $x \in X$ has a neighborhood $U$ that is homeomorphic to an open set in $(\mathbb{R}^n)^+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$.

2. A surface is a 2-dimensional manifold, a surface with boundary is a 2-dimensional manifold with boundary.

3. Examples of surfaces:
   (a) The sphere $S^2$, the projective plane $P^2$, the torus $T$, the Klein bottle $K$ are surfaces.
   (b) The (closed) Möbius strip $M$ is a surface with boundary.

4. Triangulation of a surface is defined in KC, p. 54.

5. The Euler characteristic of a triangulation is $V - E + F$. KC p. 56.

6. A map of a surface is defined in KC p. 60, also p. 68.

7. If $M_1$ and $M_2$ are surfaces, their connected sum $M_1 \# M_2$ is defined in B.

8. If $X$ is a connected, locally path connected topological space, paths, homotopy classes of paths, composition of paths, and the fundamental group $\pi_1(X, x)$ are defined in H pp 25 – 27, M pp. 56 – 62.

9. If $f : X \rightarrow Y$ is continuous, the induced homomorphism $f_* : \pi_1(X, x) \rightarrow \pi_1(Y, f(x))$ is defined in H p. 34 or M p 63.

10. If $f, g : X \rightarrow Y$, $A \subset X$ and $f|A = g|A$, we say that $f$ and $g$ are homotopic relative to $A$ if there is a continuous map $F : X \times I \rightarrow Y$ such that $F(x, 0) = f(X)$, $F(x, 1) = g(x)$ and $F(x, s) = f(x) = g(x)$ for all $x \in A$ and $s \in I$. M p. 64.

11. If $A \subset X$ is a closed subset and $i : A \rightarrow X$ is the inclusion, a continuous map $r : X \rightarrow A$ is called a retraction if $ri = id_A$ and is called a deformation retraction if, in addition, $ir$ is homotopic to $id_X$ relative to $A$. (M p 66).

12. A continuous map $p : \tilde{X} \rightarrow X$ of connected, locally path connected spaces is called a covering map iff each $x \in X$ has a neighborhood $U$ so that for each component $V$ of $p^{-1}(U)$, the map $p|V : V \rightarrow U$ is a homeomorphism.

Theorems

1. Every compact surface has a triangulation. (Accepted without proof.)
2. Know explicit triangulations for $S^2$, $P^2$, $T$, $K$.

3. If $T_1$ and $T_2$ are two triangulations of a compact surface $X$, then $\chi(T_1) = \chi(T_2)$. Thus can define $\chi(X)$ as $\chi$(any triangulation). (Proof sketched last semester, theorem accepted as true this semester).

4. If $X$ and $Y$ are homeomorphic compact surfaces, then $\chi(X) = \chi(Y)$.

5. The Euler characteristic can be computed from any map.

6. For every triangulation of a surface, $2E = 3F$.

7. $\chi(S^2) = 2$, $\chi(P^2) = 1$, $\chi(T) = 0$, $\chi(K) = 0$.

8. Every compact, connected surface has a map with only one face. In particular, $X$ is obtained as a quotient of a polygon by identifying pairs of edges. KC pp. 69–70.

9. Every compact, connected surface is homeomorphic to either $S^2$ or to a connected sum of tori and projective planes. Every compact, connected, orientable surface is homeomorphic to either $S^2$ or to a connected sum of tori. (B).

10. $K$ is homeomorphic to $P^2 \# P^2$ (proved in class by gluing two Möbius bands).

11. Compact, connected, orientable surfaces are determined up to homeomorphism by their Euler characteristic. ($\chi(S^2) = 2$ and if $X_g$ is the connected sum of $g$ tori, $\chi(X_g) = 2 - 2g$.)

12. Know how to recognize a surface from a diagram showing a polygon and boundary identifications (KC, B).

13. If $x, y \in X$ and $\alpha$ is a path from $x$ to $y$, then there is an isomorphism $\phi_\alpha : \pi_1(X, x) \to \pi_1(X, y)$ defined by $\phi_\alpha(\gamma) = \alpha^{-1} \cdot \gamma \cdot \alpha$.

14. If $f, g : X \to Y$ are homotopic relative to $x_0$, then $f_* = g_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$.

15. $\pi_1(S^1) = \mathbb{Z}$. (H p. 29, M pp. 68–74).

16. There is no continuous retraction $r : D^2 \to S^1$, every continuous map $f : D^2 \to D^2$ has a fixed point. H pp. 31–32, M pp. 74–76. Here $D^2$ is the closed unit disk in the plane.

17. If $p : \tilde{X} \to X$ is a covering space, $x_0 \in X, \tilde{x}_0 \in p^{-1}(x_0)$ and $\alpha : I \to X$ is a continuous path with $\alpha(0) = x_0$, then there is a unique continuous $\tilde{\alpha} : I \to \tilde{X}$ such that $p \tilde{\alpha} = \alpha$ and $\tilde{\alpha}(0) = \tilde{x}_0$. If $Y$ is a compact, connected, locally path connected metric space, $F : Y \times I \to X$ is continuous, $f(y) = F(y, 0)$ and $\tilde{f} : Y \to \tilde{X}$ is a continuous lift of $f$: $p\tilde{f} = f$, then there is a unique continuous map $\tilde{F} : Y \times I \to \tilde{X}$ such that $p\tilde{F} = F$ and $\tilde{F}(y, 0) = \tilde{f}(y)$ for all $y \in Y$.

18. If $p : \tilde{X} \to X$ is a covering space, then $p_* : \pi_1(\tilde{X}, x_0) \to \pi_1(X, p(x_0))$ is injective.