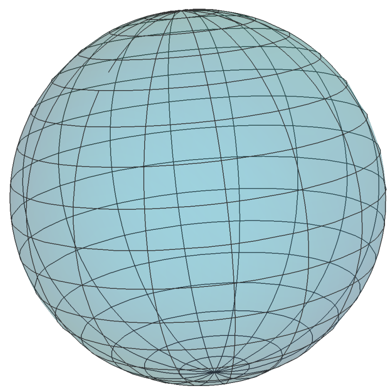
(1) (2 pts) *Geodesic curvature of parallels in* $S^2$: As usual $S^2$ is the unit sphere in $\mathbb{R}^3$. It can be parametrized by spherical coordinates $\phi, \theta$, where $0 \le \phi \le \pi$ is co-latitude and $0 \le \theta \le 2\pi$ is longitude:

$$(x, y, z) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi) \tag{1}$$

The parallels are the curves $\phi = c$, $0 \le c \le \pi$ a constant. The curves $\phi = 0, \pi$ degenerate



to the north and south pole respectively, and $\phi = \pi/2$ is the equator. *Compute the geodesic curvature* $|\frac{D\gamma'}{Ds}| = |\gamma''^T|$ *of the parallel* $\gamma = \gamma_\phi$ *for each value of* $\phi$

*Suggestion*:

(a) For fixed $\phi$, (**??**) gives a curve with parameter $\theta$, which is not arc-length unless $\phi = \pi/2$, the equator. Re-parametrize this curve by arc-length $s$, get a curve $\gamma(s)$ (depending on $\phi$)

(b) Figure our the tangential component $\gamma''^T$ and find its magnitude (answer: $\cot\phi$)

(2) (4 pts) *Gaussian curvature of the sphere of radius* $R$: Let $S^2(R)$ denote the sphere in $\mathbb{R}^3$ of radius $R$ and let $\mathbf{x}(\phi, \theta)$ be the parametrization of $S^2(R)$ obtained by multiplying (**??**) by $R$. Thus $\mathbf{x} : [0, \pi] \times [0, 2\pi] \to S^2(R)$ is surjective but maps $0 \times [0, 2\pi]$ to the north pole, etc.

(a) Not to forget topology, prove that $\mathbf{x}$ is an identification.

(b) Write the expression for $ds^2 = d\mathbf{x} \cdot d\mathbf{x}$ for this parametrization.

(c) Let $r$ be distance from the north pole. Express $\phi$ as a function of $r$ and change the expression for $ds^2$ to one that is a function of $r$ and $\theta$. It should be of the form

$dr^2 + g(r)^2 d\theta^2$ for a suitable function $g(r)$. Observe that the circumference of a circle centered at the north pole of radius $r$ is $\int_0^{2\pi} g(r) d\theta = 2\pi g(r)$.

(d) Show that the Taylor expansion of $g$ at $r = 0$ begins

$$g(r) = r - \frac{r^3}{6R^2} + \ldots$$

The quantity $1/R^2$ is called the "Gaussian curvature" of $S^2(R)$. It measures the deviation of the circumference of circles of radius $r$ from the value $2\pi r$ in Euclidean geometry.

(3) (4 pts) *Gaussian curvature of a cylinder*: Let $C$ be the cylinder $\{x^2 + y^2 = 1\} \subset \mathbb{R}^3$. "Parametrize " $C$ by the map $\mathbf{x} : \mathbb{R}^2 \to C$ defined by

$$\mathbf{x}(u, v) = (\cos u, \sin u, v).$$

(a) Even though $\mathbf{x}$ is not injective, show that $\mathbf{x}$ is locally injective, is an open map, and is an identification. So it can be used to compute $ds^2 = d\mathbf{x} \cdot d\mathbf{x}$.

(b) Find $ds^2 = d\mathbf{x} \cdot d\mathbf{x}$. Any surprises?

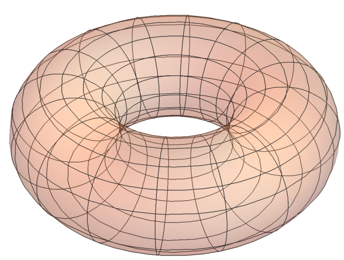
(c) Find the circumference of the geodesic circles of radius $r$, for small enough $r$, so that these curves will not intersect themselves. What's the Gaussian curvature?

(d) Find all geodesics in $C$ joining $(1, 0, 0)$ to $(1, 0, 2\pi)$. Show that there are infinitely many distinct ones. Draw a picture.

(4) *Extra credit problems on the torus*

Let $T$ be the torus $\mathbb{R}^2/(2\pi\mathbb{Z})^2 = \mathbb{R}^2/\sim$, where $(x, y) \sim (x', y')$ if and only if $x - x'$ and $y - y' \in 2\pi\mathbb{Z}$. Can measure lengths of curves in two ways:

- $ds_1^2 = dx^2 + dy^2$ on $\mathbb{R}^2$, is invariant under translations, is well defined on $T$.
- $ds_2^2 = d\mathbf{x} \cdot d\mathbf{x}$ where $\mathbf{x}(\phi, \theta) = ((2 + \cos\phi)\cos\theta, (2 + \cos\phi)\sin\theta, \sin\phi)$. Then $\mathbf{x} : \mathbb{R}^2 \to \mathbb{R}^3$ and $\mathbf{x}(\phi + 2\pi m, \theta + 2\pi n) = \mathbf{x}(\phi, \theta)$ for all $m, n \in \mathbb{Z}$, therefore $\mathbf{x}$ gives a map $\mathbf{x} : T \to \mathbb{R}^3$ with image the familiar torus of revolution in $\mathbb{R}^3$/



(a) For $(T, ds_1^2)$ find the Gaussian curvature and all the geodesics from $[(0, 0)]$ to $[(\frac{1}{2}, \frac{1}{2})]$.

(b) For $(T, ds_2^2)$ show that all circles $\theta = const$ are geodesics, and decide which of the circles $\phi = const$ are geodesics.

(c) Prove that there is no isometry between $(T, ds_1^2)$ and $(T, ds_2^2)$.