(1) (2pts) Let X be a compact Hausdorff space, let $A, B \subset X$ be disjoint closed sets. Prove that there exist disjoint open sets $U, V \subset X$ with $A \subset U$ and $B \subset V$.

(2) (4pts) Let (X, d) be a metric space and let $A \subset X$, let $x \in X$. Define the *distance between x and A*, denoted $d(x, A)$, by

$$d(x, A) = \inf\{d(x, y) \mid y \in A\}$$

(a) Prove that $f(x) = d(x, A)$ is a continuous function of x.

Suggestion Prove that f is Lipschitz.

(b) Prove that $d(x, A) = 0$ if and only if $x \in \overline{A}$.

(c) Observe that it follows from (a) and (b) that if $A \subset X$ is closed, there exists a continuous function $f : X \to [0, \infty)$ such that $f(x) = 0$ if and only if $x \in A$.

Prove that if $A, B \subset X$ are disjoint closed subsets, there exists a function $g : X \to [0, 1]$ such that $g(x) = 0 \iff x \in A$ and $g(x) = 1 \iff x \in B$.

Conclude, from knowing that g exists, that if $A, B \subset X$ are disjoint closed sets, there exist disjoint open sets $U, V \subset X$ with $A \subset U$ an $B \subset V$.

Suggestion To find g, use some fraction with numerator and denominator simple expressions in $d(x, A)$ and $d(x, B)$.

(3) (2 pts) Recall that a topological space is said to be *locally connected* if and only if it has a basis consisting of connected open sets.

(a) Prove that if X is locally connected, then its connected components are open.

(b) Prove that if X is locally connected and Y is the space of its connected components, with the quotient (= identification) topology, then Y is discrete.

(4) (2 pts) A topological space X is said to be *totally disconnected* if and only if, for all $x \in X$, the connected component of x is simply $\{x\}$.

(a) Prove that, if for all $x, y \in X$ there exists a continuous function $\phi : X \to \{0, 1\}$ (where $\{0, 1\}$ has the discrete topology) such that $\phi(x) \neq \phi(y)$, then X is totally disconnected.

(b) Prove that $\{0, 1\}^{\mathbb{N}}$, with the product topology, is totally disconnected. (Recall that product topology makes sense since $\{0, 1\}^{\mathbb{N}} = \prod_1^\infty \{0, 1\}$)