(1) (3 pts) The purpose of this exercise is to prove the triangle inequality for the great-circle arc distance d on the unit sphere $S^2 = \{x \in \mathbb{R}^3 : |x| = 1\} \subset \mathbb{R}^3$. From elementary geometry, the length of the great circle arc from x to y is

$$d(x, y) = \cos^{-1}(x \cdot y) \tag{1}$$

where $x \cdot y$ is the usual dot product in $\mathbb{R}^3$. Thus, the triangle inequality for d is the same as the following inequality:

$$\cos^{-1}(x \cdot y) \leq \cos^{-1}(x \cdot z) + \cos^{-1}(z \cdot y) \quad \text{for all } x, y, z \in S^2 \tag{2}$$

(a) Show that (2) is equivalent to

$$\det\begin{pmatrix} x\cdot x & x\cdot y & x\cdot z \\ y\cdot x & y\cdot y & y\cdot z \\ z\cdot x & z\cdot y & z\cdot z \end{pmatrix} = \det\begin{pmatrix} 1 & x\cdot y & x\cdot z \\ y\cdot x & 1 & y\cdot z \\ z\cdot x & z\cdot y & 1 \end{pmatrix} \geq 0 \tag{3}$$

(Note that the two determinants agree in our case, because $x \cdot x = y \cdot y = z \cdot z = 1$, since $x, y, z \in S^2$.)

Suggestion: Apply the *decreasing* function $\cos$ to both sides of (2) to get

$$x \cdot y \geq \cos(\cos^{-1}(x \cdot z) + \cos^{-1}(z \cdot y))$$

Then use the addition fornula for the cosine $\cos(A+B) = \cos A \cos B - \sin A \sin B$ to the right-hand side, then the formula $\sin(\cos^{-1}(X)) = \sqrt{1 - X^2}$ to get rid of all trigonometric functions. Then get rid of all square roots. Check that the resulting expression you get matches with the second determinant in (3).

(b) Use the fact from linear algebra that for any three vectors x, y, z in $\mathbb{R}^n$, $n \geq 3$, the first determinant in (3) is ≥ 0 and $= 0$ if and only if $\{x, y, z\}$ is linearly dependent. Conclude that the triangle inequality holds for the spherical metric (1). Moreover, equality holds in (2) if and only if x, y, z lie on a great circle.

(2) (3 pts) Let $(\mathbb{R}^2, d_{FR})$ be $\mathbb{R}^2$ with the French railway metric

$$d_{FR}(x, y) = \begin{cases} |x - y| \text{ if } x \text{ and } y \text{ are in same ray from } 0 \\ |x| + |y| \text{ otherwise,} \end{cases}$$

Consider the following subspaces of $(\mathbb{R}^2, d_{FR})$ with the subspace metric d:

(a) $y = 1$
(b) $y = x$
(c) $x^2 + y^2 = 1$

In each case the subspace metric is homeomorphic to either $\mathbb{R}$ or the unit circle S^1 with the discrete metric

$$d_{disc}(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise,} \end{cases}$$

or to $\mathbb{R}$ with the usual euclidean metric $d_E(x, y) = |x - y|$. For each of the three metrics d in (a), (b), (c) answer the following questions:

(a) Identify to which of the three altenatives d' is it homeomorphic.

(b) Once you know the alternative d', then answer the questions: is d isometric to d'? Is d bi-Lispschitz to d'?

(3) (4 pts) Let (X, d) be a metric space, and let $f : [0, \infty) \to [0, \infty)$ be a strictly increasing function with $f(0) = 0$ and subadditive: $f(x + y) \leq (x) + f(y)$ for all $x, y \in [0, \infty)$. In a previous homework you proved that if we define $d'(x, y) = f(d(x, y))$, then (X, d') is also a metric space.

(a) Suppose, in addition, that f is *continuous*. Prove that d and d' give the same topology on X. (This means, $U \subset X$ is open in (X, d) if and only if it is open in (X, d')).
Remark: Since f is strictly increasing and continuous, then, by the intermediate value theorem, if follows easily that $f([0, \infty))$ is an interval $[0, a)$ for some a, $0 < a \leq \infty$ and $f^{-1} : [0, a) \to [0, \infty)$ exists and is continuous. You may use this in your proof.

(b) As in the previous homework, apply this to the function $f(x) = \frac{x}{1+x}$. Conclude that for any metric space (X, d), the metric space $(X, \frac{d}{1+d})$ has the same topology as (X, d).

(c) Conclude that the topology of any metric space can be defined by a *bounded* metric.

Extra Credit Problems:

(1) (5 pts)Prove the triangle inequality for the hyperbolic metric on the upper half $x_3 > 0$ of the hyperboloid $X = \{x_1^2 + x_2^2 - x_3^2 = -1\}$ in Minkowski space

$$d(x, y) = \cosh^{-1}(x \diamond y)$$

where $x, y \in X$ and $x \diamond y$ is the Minkowski inner product. See Lectures, Week 2 and Notes, 1.19 to 1.23 for more details.

(2) (2 pts) Problem (3) shows that (X, d) and $(X, \frac{d}{1+d})$ are homeomorphic. When can they be bi-Lipschitz equivalent? Give necessary and sufficient conditions.