1. (5 pts) It can be hard to prove that two metric spaces (X, d) and (X', d') are *not* isometric. This usually requires finding some isometry invariant that distinguishes them. Here is one possibility: for $x, y \in (X, d)$, define $E_d(x, y)$, the equality set of the triangle inequality, by

$$E_d(x, y) = \{z \in X \mid d(x, y) = d(x, z) + d(z, y)\}$$

   (a) Prove that if $f : (X, d) \to (X', d')$ is a *surjective* isometry (thus $f^{-1} : (X', d') \to (X, d)$) exists), then $f(E_d(x, y)) = E_{d'}(f(x), f(y))$ and $f : E_d(x, y) \to E_{d'}(f(x), f(y))$ is a surjective isometry, in particular, a homeomorphism.

   (b) Prove that $(\mathbb{R}^2, d_{(2)})$ and $(\mathbb{R}^2, d_{(1)})$ are *not* isometric. (You may use the fact, to be proved later, that an interval and a non-degenrate rectangle are not homeomorphic).

   (c) Prove that the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x_1, x_2) = (x_1 + x_2, x_1 - x_2)$ is an isometry $(\mathbb{R}^2, d_{(1)}) \to (\mathbb{R}^2, d_{(\infty)})$.

   (d) *Challenging extra credit problem*: Are $(\mathbb{R}^3, d_{(1)})$ and $(\mathbb{R}^3, d_{(\infty)})$ isometric?

   (e) Let $f : (\mathbb{R}^2, d_{(1)}) \to (\mathbb{R}^2, d_{(1)})$ be an isometry. Prove that f sends horizontal lines to horizontal or vertical lines.

   (f) Let $f : (\mathbb{R}^2, d_{(1)}) \to (\mathbb{R}^2, d_{(1)})$ be an isometry fixing the origin $(0, 0)$. Prove that there are only 8 possibilities for f, namely $f(x_1, x_2) = (\pm x_1, \pm x_2)$ or $(\pm x_2, \pm x_1)$.

2. (3 pts) Let $X = \{(x, 1) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$ (horizonal line in $\mathbb{R}^2$ distance one above the origin $(0, 0)$). Let d be the restriction to X of the French-railway distance d_{FR} (Paris at the origin), let d' be the restriction to X of the Euclidean distance $d_{(2)}$. Let $f : (X, d) \to (X, d')$ be the identity map.

   (a) Is f Lipschitz?

   (b) Is f bi-Lipschitz?

   (c) Is f a homeomorphism?

3. (2 pts) Let $\mathbb{R}^\infty$ denote the set of sequences of real numbers that are eventually 0, that is

$$\mathbb{R}^\infty = \{x = (x_1, x_2, \ldots) \mid x_i \in \mathbb{R} \text{ and } \forall x\ \exists N(x) \text{ such that } i > N \implies x_i = 0\}$$

   Since each $x \in \mathbb{R}^\infty$ is a finite sequence, the metrics $d_{(1)}, d_{(2)}, d_{(\infty)}$ are all defined on $\mathbb{R}^\infty$

   (a) Prove that $id : (\mathbb{R}^\infty, d_{(1)}) \to (\mathbb{R}^\infty, d_{(2)})$ is Lipschitz.

   (b) Prove that $id : (\mathbb{R}^\infty, d_{(2)}) \to (\mathbb{R}^\infty, d_{(1)})$ is *not* Lipschitz.

   (c) Relate this to the inequalities

$$d_{(2)}(x, y) \le d_{(1)}(x, y) \le \sqrt{n}\ d_{(2)}(x, y)$$

   from the last homework.