Introduction to Algebraic and Geometric Topology Week 9

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Fall 2017

$X \times Y$

Prod top : basis

$B(x_0, r) \times B(y_0, r')$

Metric topl d^∞

$B((x_0,y_0), r'')$

$B((x_0,y_0), r)$

$= B(x_0,r) \times B(y_0,r)$

$\mathcal{T}_{\mathcal{B}} = \mathcal{T}_{\mathcal{B}'}$

(x,y) :

$\max\{d(x_0,x), (y_0,y)\} < r$

$\Rightarrow d(x_0,x) < r, \; d(y_0,y) r$

$(x,y) \in B(x_0,r) \times B(y_0,r)$

$(x,y) \in B(x_0,r) \times (y_0, r)$

$d(x_0,x) < r \;\&\; d(y_0,y) < r$

$\Rightarrow \max\{d(x_0,x), d(y_0,y)\} < r$

$(x,y) \in B(x_0,r) \times B(y_0,r)$

$\mathcal{B}, \mathcal{B}'$

they define same topology

Suff cond :

$\forall x \in X \;\&\; B \in \mathcal{B}$, s.t. $x \in B$

$\exists B' \in \mathcal{B}'$ s.t. $x \in B' \subset B$

$\mathcal{T}_{\mathcal{B}} \subset \mathcal{T}_{\mathcal{B}'}$

and the way around

$\forall x \in X,\; B' \in \mathcal{B}'$

$x \in B'$,

$\exists B \in \mathcal{B}$ s.t. $x \in B \subset B'$

$\mathcal{T}_{\mathcal{B}'} \subset \mathcal{T}_{\mathcal{B}}$

$r_+ = \max\{r, r'\}$

$r_- = \min(r, r')$

$$\boxed{B(x_0,r_-) \times B(y_0,r_-) \subset B(x_0,r) \times B(y_0,r') \subset B(x_0,r_+) \times B(y_0,r_+)}$$

$\boxed{\text{Friday}}$

Recall: Connected Spaces

Definition

A topological space $(X, \mathcal{T})$ is *connected*

$\Longleftrightarrow$

if $Y \subset X$ is both open and closed, then either $Y = X$ or $Y = \emptyset$

Equivalent Formulations

Theorem

X is connected

$\iff$

Whenever $U, V \subset X$ are open sets with $X = U \cup V$ and $U \cap V = \emptyset$, then either

$$U = \emptyset \text{ or } V = \emptyset$$

better: X is disconnected

$\iff X = U \cup V$, $U \cap V = \emptyset$, $U \neq \emptyset$, $V \neq \emptyset$

open

open cl

$X = \bar{U} \cup \bar{V}$, $\bar{U} \cap \bar{V} = \emptyset$

$\Rightarrow U = \emptyset$ or $V = \emptyset$

Theorem

X is connected

$\Longleftrightarrow$

Whenever $E, F \subset X$ are closed sets with $X = E \cup F$ and $E \cap F = \emptyset$, then either

$$E = \emptyset \text{ or } F = \emptyset$$

Theorem

Let $\{0, 1\}$ have the discrete topology. Then X is connected

$\Longleftrightarrow$

Every continuous map $f : X \to \{0, 1\}$ is constant.

$X \xrightarrow{f} \{0,1\}$

$f^{-1}(0)$, $f^{-1}(1)$ open | closed

f cont $\Leftarrow$ $f^{-1}(0) \neq \emptyset$ $\cap f^{-1}(1) \neq \emptyset$ $X = f^{-1}(0) \cup f^{-1}(1)$, $f^{-1}(0) \cap f^{-1}(1) = \emptyset$

If $Y \subset X$, Y connected means: connected in the subspace topology.

$A \subset B \subset \bar{A}$

$\Downarrow$

B connected

$A \subset X$ connected subspace

$\Rightarrow \bar{A}$ is connected

$f : \bar{A} \to \{0,1\}$ $f|A$ constant

$f|_A : A \to \{0,1\}$ constant

$\Rightarrow f$ = const on $\bar{A}$

Theorem

If $f : X \to Y$ is continuous and X is connected, then $f(X)$ is connected.

$f : X \to Y = f(X)$ cont, surjective, X connected

$Y = U \cup V \Rightarrow Y$ connected

$\bar{U} \cap V = \emptyset$

$X = f^{-1}(U) \cup f^{-1}(V)$

$f^{-1}(U) \cap f^{-1}(V) = \emptyset$

$\Rightarrow$ one is empty $\Rightarrow$ one of $U, V = \emptyset$

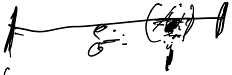
$X \xrightarrow{f} Y \xrightarrow{\varphi} \{0,1\}$

$\varphi \circ f$ const, f surj $\Rightarrow \varphi$ const

$[0,1] \cap \mathbb{Q}$

Theorem

The unit interval $[0,1] \subset \mathbb{R}$ *is connected.*



$[0,1]$

$\varphi : [0,1] \to \{0,1\}$

cont.

$\varphi(0) = 0$

$\varphi(1) \neq \varphi$??

$\{x \in [0,1] : x \in \mathbb{Q}\}$ not conn

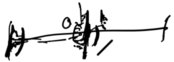
$\{x < \sqrt{1/2}\}$

$\{\sqrt{1/2} < x\}$ disconnect

Any $A \subset \mathbb{R}$ $A \neq \emptyset$, bdd, ~~has~~ inf, sup.

below $\Rightarrow$ inf A exist.

$a = \inf \varphi^{-1}(1)$



$a < 0$ impossible

$0 < a < 1$

$\varphi(x) = 0$ for $x < a$

$\forall \varepsilon > 0$, $\exists x$, $a < x < a + \varepsilon$

s.t. $\varphi(x) = 1$

Path-Connected Spaces

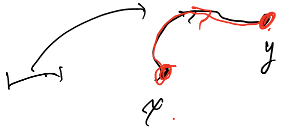
Definition

A topological space X is said to be *path connected*

$\iff$

For all $x, y \in X$ there exists a continuous map $\gamma : [0,1] \to X$ with

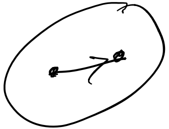
$$\underline{\gamma(0) = x} \quad \text{and} \quad \underline{\gamma(1) = y}.$$



Example: Convex subsets of $\mathbb{R}^n$

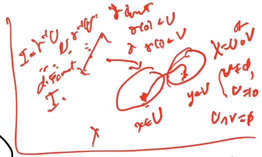
Corollary: $C \subset \mathbb{R}^n$ is a <u>convex</u> set

$\Rightarrow C$ is connected.



Theorem

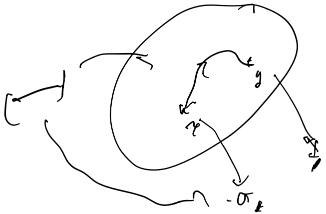
X path connected $\Longrightarrow$ X connected.



$X \xrightarrow{f} \{0,1\}$ non-const

$\cup [0,1] \cup \{0,1\}$

$\exists\, x, y \in X$, $f(x)=0$, $f(y)=1$



$\gamma : [0,1] \to X$ cont, $\gamma(0)=x$, $\gamma(1)=y$

$f\circ\gamma : [0,1] \to \{0,1\}$ non-const.

impossible.

locally P $\iff$ $\exists$ basis : P

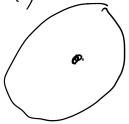
Definition

A topological space $(X, \mathcal{T})$ is said to be *locally path connected*

$\iff$

$\mathcal{T}$ has a basis $\mathcal{B}$ such that every $B \in \mathcal{B}$ is path connected.

$\forall x \in X$, $\forall U$ open in X, $\exists V$ path conncted s.t. $x \in V \subset U$.



$X \xrightarrow{f} Y$ f cont surj

X conn $\Rightarrow$ Y conn

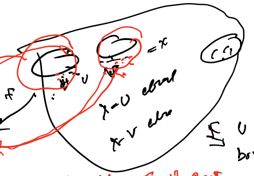
Connected:

X not conn $\iff$ X not conn

$\{0\}$ open $\{1\}$

$f^{-1}(U)$ $f^{-1}(V)$ disconnect X

$\{0,1\}$



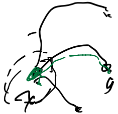
$\exists$ U, V both open & close

X disconnected $\iff \exists f : X \to \{0,1\}$ cont, not constant

Theorem

X connected and locally path connected
$\implies$ *X is path connected.*

Given $x \in X$, define $A = \{ y \in X : \exists\, \gamma : [0,1] \to X$



$\neq \emptyset$

s.t. $\gamma(0) = x,\ \gamma(1) = y \}$

Claim: A is open, $X - A$ is open.

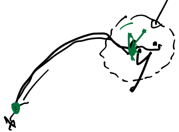
if true

$\Rightarrow A = \emptyset$ or $X - A = \emptyset$

$A \neq \emptyset$ $\Rightarrow$ $X - A = \emptyset$

$\Leftarrow$ X path connected.

A open: $y \in A$ $\exists$ V open, $y \in V \subset A$

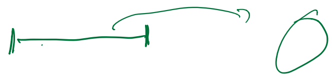


V = path conn nbd of y

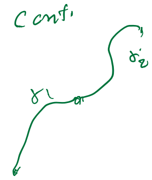
$\forall z \in V$ $\exists$ path

γ from y to z

Concatenate the two paths

Path: 

$\gamma : [0,1] \longrightarrow X$ cont.



γ_1, γ_2 paths,

$\gamma_1(1) = \gamma_2(0)$

define $\gamma_1 \cdot \gamma_2 : [0,1] \longrightarrow X$

formally $\gamma : [0,2] \longrightarrow X$

$\gamma_1 \cdot \gamma_2 [t] = \begin{cases} \gamma_1(t) & t \in [0,1] \\ \gamma_2(t-1) & t \in [1,2] \end{cases}$

better

$\gamma_1 \cdot \gamma_2 [0,1] \longrightarrow X$

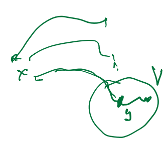
$\gamma_1 \cdot \gamma_2 = \begin{cases} \cancel{\gamma_1(1/2)} \quad \cancel{t \in} \\ \gamma_1(2t) & \text{if } t \in [0, \tfrac{1}{2}] \\ \gamma_2(2(t - \tfrac{1}{2})) & [\tfrac{1}{2}, 1] \end{cases}$

$\boxed{\gamma_1, \gamma_2 \text{ cont} \Rightarrow \gamma_1 \cdot \gamma_2 \text{ cont.}}$

$A = \{ y \in X : \exists\, \gamma : I \to X$

$\gamma(0) = x$

$\gamma(1) = y$



$y \in A$

$\exists$ convex $V \ni y$

$\forall z \in V$

$\exists$ path $\lambda : [0,1] \to V$

$\lambda(0) = y$

$\lambda(1) = z$

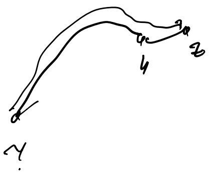
$\gamma \cdot \lambda$ path

from x to z

$\Rightarrow z \in A$

$y \in V \subset A$.

A open.



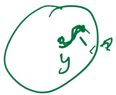
$\Rightarrow V \subset A$

$\Rightarrow A$ is conn.

$X - A$ is open

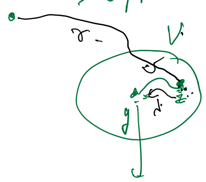
$y \in X - A \Rightarrow$ no path from x to y.

$\exists$ nbhd V of y path conn. $\forall z \in V$, $\exists$ path from y to z



Claim: $z \in X - A$.

Suppose $z \in A$



$\gamma.$

$\delta : [0,1] \to V \subset X$

$\delta(0) = y$

$\delta(1) = z$

$\delta^{-1} : [0,1] \to V$

$\delta^{-1}(t) = \delta(1-t)$

$\gamma \cdot \delta^{-1}$ path from x to z

$\longrightarrow \in$

$z \in X - A$

$$X = \underbrace{A}_{\text{open}} \cup \underbrace{X - A}_{\text{open}}$$

& X conn

& X loc path

$\Rightarrow$ either $A = \phi$

or $X - A = \phi$

X loc path conn

$\forall x \in X$, $\exists$ path conn nbhd U of x

$\Rightarrow A \neq \phi$

$\Rightarrow X - A = \phi$

$= A = X$

$\Leftrightarrow X$ is path conn.

Quotient Spaces

Recall General Principle:

Let X and Y be sets and let $f : X \to Y$.

ets

1. Given a topology $\mathcal{T}_Y$ on Y there is a smallest topology $\mathcal{T}_X$ on X that makes f continuous,

   namely $\mathcal{T}_X = \{f^{-1}(U) : U \in \mathcal{T}_Y\}$.

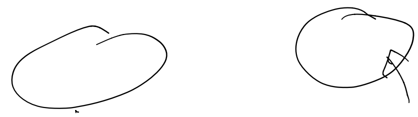
   (Ex: $f = i_A$, $A \subset X$
   subspace)

2. Given a topology $\mathcal{T}_X$ on X there is a largest topology $\mathcal{T}_Y$ that makes f continuous.,

   namely $\mathcal{T}_Y = \{U \subset Y : f^{-1}(U) \in \mathcal{T}_X\}$.

   (Main case: f is surjective.)

$X \longrightarrow Y$

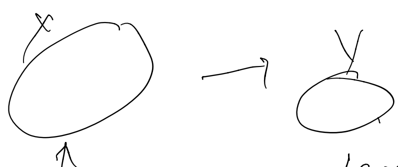


given top

Smallest top that makes f cont $= \{ f^{-1}(U) : U \text{ open in } Y \}$

Ex subspace top $X \overset{f}{\subset} Y$

$f^{-1}(U) = X \cap U$



given top

largest top makes f cont.

$\{ U \subset Y : f^{-1}(U) \text{ open in } X \}$

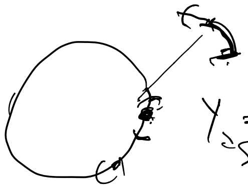
if f cont

U open $\Rightarrow f^{-1}(U)$ open

$\Leftarrow$

U open $\Longleftrightarrow f^{-1}(U)$ open

$X = [0, 2\pi)$



$Y = |z| = 1$

$\cong S^1$

U open in $\cancel{Y}$

$\Rightarrow f^{-1}(U)$ open in X

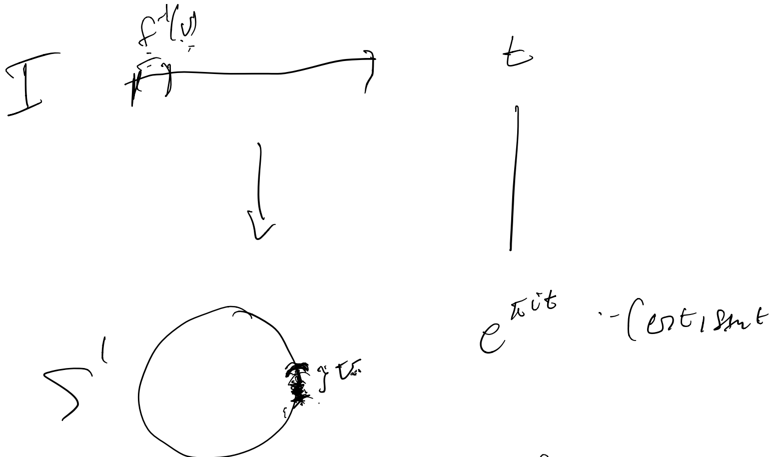
$f(t) = e^{it} \in \mathbb{C}$

$= (\cos t, \sin t) \in \mathbb{R}^2$

not conversely

$U = [0, \varepsilon)$ open set in X

$V = f[0, \varepsilon)$ not open in Y



Usual tops, f Cont

U open in $S^1 \to f^{-1}(U)$ open in I

Analogy gps homo

surjective $G \to$ quotient gps.

~~When talking~~ about quotient top. $X \xrightarrow{f} Y$ consider only f surjective

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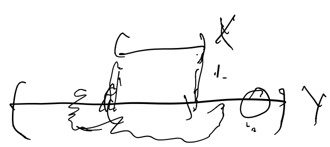


quotient top wouldn't any U with $U \cap f(X) = \emptyset$ open

any $U \supset f(X)$ open $\to f^{-1}(U) = \emptyset$

$f^{-1}(U) = X$

Examples

: $f^{-1}(V) = U$ open in X.

1. The subspace topology for $A \subset X$:

   Given a topology $\mathcal{T}_X$ on X, define

$$\mathcal{T}_A = \{A \cap U \mid U \in \mathcal{T}_X.\}$$

2. The *quotient topology* or *identification topology* for a surjective map

$$f : X \to Y$$

   Given a topology $\mathcal{T}_X$ on X, define

$$T_Y = \{U \subset Y \mid f^{-1}(U) \in \mathcal{T}_X\}$$

$$\begin{tikzcd} X \arrow[r, "\text{Cont, surj}"] & Y \\ f^{-1}(U) \arrow[u, phantom, "\cup"] \arrow[r] & U \arrow[u, phantom, "\cup"] \end{tikzcd}$$

Identification

- A continuous surjective map $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is called an *identification* if T_Y is the identification topology just defined:

$$\boxed{T_Y = \{U \subset Y \mid f^{-1}(U) \in \mathcal{T}_X\}}$$

- Equivalently: $f : X \to Y$ is an identification $\iff$

$$\boxed{U \text{ open in } Y \iff f^{-1}(U) \text{ is open in } X}$$

- Equivalently: $f : X \to Y$ is an identification $\iff$

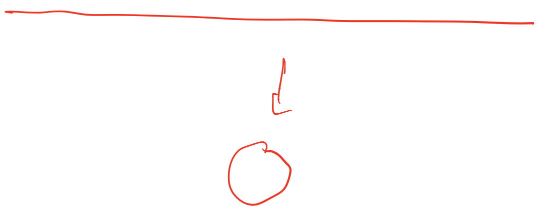
$$\boxed{F \text{ closed in } Y \iff f^{-1}(F) \text{ is closed in } X}$$

Examples

1. $f_1 : \mathbb{R} \to S^1$ defined by $f(t) = (\cos t, \sin t)$

2. $f_2 : [0, 2\pi) \to S^1$ same formula.

3. $f_3 : [0, 2\pi] \to S^1$ same formula.



$t \to e^{it}$

both

identifying?

$0 \qquad 2\pi$

S^1

$t \to e^{it}$

$\longrightarrow F \subset S^1$ closed $\Rightarrow f^{-1}(F)$ closed

$\Leftarrow$?

$f : X \to Y$

closed map. $F \subset X$ closed $\Rightarrow f(F)$ closed.

open map $U \subset X$ open $\Rightarrow f(U)$ open!

A open / closed $\Rightarrow f^{-1}(A)$ $\frac{\text{open}}{\text{closed}}$

cont

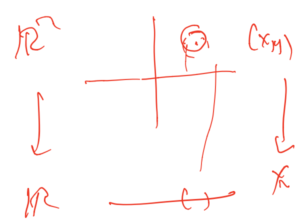
$$f^{-1}(X - A) = f^{-1}(X) - f^{-1}(A)$$

$$f(X - A) = f(X) - f(A) \quad ??$$

f = onto, $A \subset X$

$f(X-A) = \ldots$

$f(X) - A$ $\ldots$ ϕ

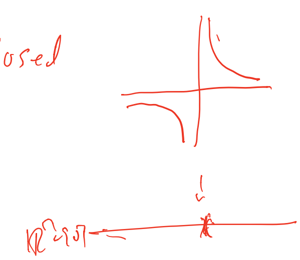


(x,y) open, not close

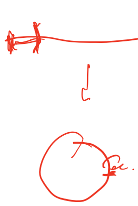
open

not closed

$\{xy = 1\}$



closed



not open

Recall: $f: X \to Y$ cont

bijection

& f closed

$(f^{-1})^{-1}(\text{closed}) = \text{closed}$

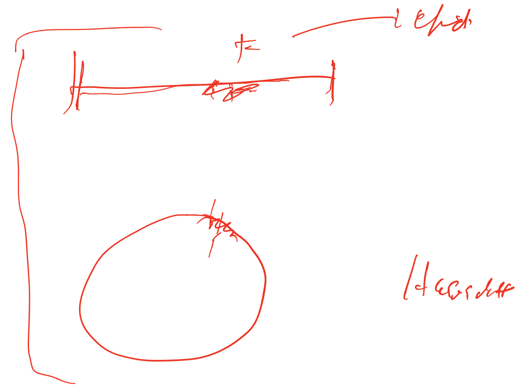
$\frac{1}{(1/x)}$

$f(\text{closed}) = \text{closed}$

$(x^{-1})^{-1} = x$

Cont closed bijec $\Rightarrow$ homeo

" open " $\Rightarrow$ "



$X \to Y$ cont, X cpt,

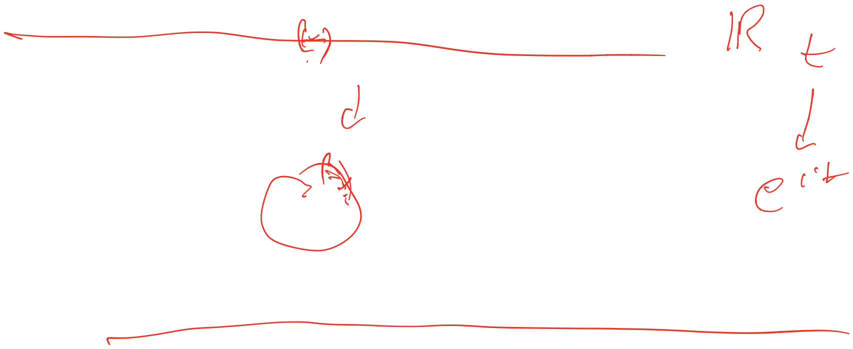
Y Hausdorff

$\Rightarrow$ closed map

Sufficient Conditions

1. $f : X \to Y$ continuous, surjective and *open* $\Longrightarrow$ identification.

2. $f : X \to Y$ continuous, surjective and *closed* $\Longrightarrow$ identification.



open map

~~Examples~~

$f: X \to Y$ cont, open

want: $U \subset Y$ open

$\Leftarrow f^{-1}(U) \subset X$ open.

f cont: U open $\Rightarrow f^{-1}(U)$ open.

need: $\underline{f^{-1}(U)}$ open $\Rightarrow$ $\underline{U}$ open f open, $f(f^{-1}(U))$ open

$f(\underbrace{f^{-1}(U)}_{\ni x}) \in U$

$f(U) \subset U$ $\subset$

f $\supset$

f empty, y_0 $y_0 \notin A$ $A \neq \emptyset$.

$f(f^{-1}(A)) = \emptyset \subset A$

If f surjective

$U \subset f(f^{-1}(U))$

$y \in U \Rightarrow \exists x \in X$ s.t. $f(x) = y$

$x \in f^{-1}(U)$

$f(x) = y$ $\quad y \in f(f^{-1}(U))$

f cont, open, surject $\Rightarrow$ identification

closed.

Checking Identifications

- Given $A \subset X$, when is $A = f^{-1}(B)$ for some $B \subset Y$?
- Useful facts for this purpose:
- Suppose $f : X \to Y$, $A \subset X, B \subset Y$. Then

1. $f(f^{-1}(B)) \subset B$
2. If f is surjective, $f(f^{-1}(B)) = B$
3. $A \subset f^{-1}(f(A))$.
4. If f is surjective, then

$$A = f^{-1}(B) \text{ for some } B \subset Y \iff A = f^{-1}(f(A))$$

and, in this case, $B = f(A)$.

Proofs

- First three assertions clear.
- Suppose f surjective and $A = f^{-1}(B)$ for some $B \subset Y$.
- Then $f(A) = f(f^{-1}(B)) = B$ by 2.
- So $A = f^{-1}(B) \implies B = f(A) \implies$ equality in 4:

$$A = f^{-1}(f(A))$$

Equivalence Relations

- $f : X \to Y$ surjective map of sets $\Longleftrightarrow$

  equivalence relation on X:

$$x_1 \sim x_2 \iff f(x_1) = f(x_2).$$

- $f : X \to Y$ surjective map of sets $\Longleftrightarrow$

  Partition of X into disjoint subsets

$$X = \coprod_{y \in Y} f^{-1}(y)$$

$x \sim x$

$x_1 \sim x_2 \iff x_2 \sim x_1$

$x_1 \sim x_2$

$x_2 \sim x_3$

$\Rightarrow x_1 \sim x_3$

Y = Set of equiv classes

$$x \to \{y \in Y : y \sim x\} \subset Y$$

Continuous Maps

- X, Y, Z topological spaces.
- $f : X \to Y$ identification,
- $g : Y \to Z$ a map.
- Then g is continuous $\iff$ $g \circ f$ is continuous

$$\begin{tikzcd} X \arrow[r, "g\circ f"] \arrow[d, "f"'] & Z \\ Y \arrow[ur, "g"'] & \end{tikzcd}$$

always

$g \circ f$ cont $\Rightarrow$ g cont

$(g \circ f)^{-1}(U)$ open $\Rightarrow$ $g^{-1}(U)$ open.

$\underbrace{f^{-1}(\underline{g^{-1}(U)})}$ $\underline{g^{-1}(U)}$

$f^{-1}(g^{-1}(U))$ open $\Leftrightarrow$ $g^{-1}(U)$ open.

- Equivalent Formulation:
- X, Y, Z topological spaces, $f : X \to Y$ an identification.
- $h : X \to Z$ a map that is constant on the fibers $f^{-1}(y)$ of f.
- Then the map g in the following diagram is defined:

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "f"'] & Z \\ Y \arrow[ur, "g"'] & \end{tikzcd}$$

- g is continuous $\iff$ h is continuous.

- Example: Periodic functions $h : \mathbb{R} \to \mathbb{R}$

$$
\begin{tikzcd}
\mathbb{R} \arrow[r, "h"] \arrow[d, "f"'] & \mathbb{R} \\
S^1 \arrow[ur, "g"'] &
\end{tikzcd}
$$

where $f(t) = (\cos t, \sin t)$.

Ex of equiv rel topo

X space

$x \sim y \iff \exists$ connected set $C \subset X$

$x, y \in C$.

Check: equiv rel

equiv classes: connected components of X.