Introduction to Algebraic and Geometric Topology Week 8

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Recall:

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We proved the following statments about compactness:

- A compact metric space is bounded.
- A compact subspace of a metric space is closed.
- A subspace of $\mathbb{R}^n$ is compact if and only if it is closed and bounded (Heine-Borel theorem)

from 3

X discrete

X closed $\Leftrightarrow$ X finite

$\{x\}_{x\in X}$

- A compact subspace of a Hausdorff space is closed.

$\mathbb{R}, \mathcal{T}_{CF}$

- Hausdorff is needed.

- A closed subspace of a compact topological space is compact.

- A continuous image of a compact space is compact.

- $f : X \to Y$ continuous, $\boxed{C} \subset X$ compact $\implies f(C)$ compact.

U

$\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$.

$\{f^{-1}U_\alpha\}$ open cover of C

$\exists$ $f^{-1}(U_{\alpha_1}), \ldots f^{-1}(U_{\alpha_n})$ cover C

$\Rightarrow U_{\alpha_1}, \ldots U_{\alpha_n}$ cover $f(C)$.

Closed Maps and Homeomorphisms

- The last few statements combine to give:

Theorem

- *X compact space, Y Hausdorff space.*
- $f : X \to Y$ *continuous.*
- *Then* $C \subset X$ *closed* $\Longrightarrow$ $f(C)$ *is closed.*

Cpct Hausdorff

$X \xrightarrow{f} Y$

$\cup$

C closed f(C)

C closed ⇒ f(C) closed

X cpct ⇒ C cpct ⇒ f(C) cpct ⇒ f(C) closed

C closed f cont Y Hausdorff

Definition

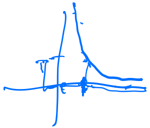
$f : X \to Y$ is a *closed map*

$\iff$

for all closed subsets $C \subset X$, $f(C)$ is closed in Y.

Recall: cont $\iff$ $f^{-1}(C)$ closed $\forall\, C \subset Y$

closed



$X = (0, \infty) \to \mathbb{R}$

$x \mapsto 1/x$

$f([1, \infty)) = \ldots$

$= (0, 1]$

$(0, \infty)$ closed in X

image not closed in $\mathbb{R}$

Rephrase:

Theorem

- *X compact space, Y Hausdorff space,*
- *$f : X \to Y$ continuous*
- *$\implies f$ is a closed map.*

Consequence:

Theorem

- X *compact space*, Y *Hausdorff space.*
- $f : X \to Y$ *continuous bijection.*
- $\implies f$ *is a homeomorphism.*

for enough: $\underbrace{(f^{-1})^{-1}(C)}_{f(C)}$ closed $\forall\, C$ closed

$X \underset{f}{\overset{f^{-1}}{\rightleftarrows}} Y$

C

$(f^{-1})^{-1}(C)$ closed

$f(C)$

$x : f^{-1}(x) \in C$

$\iff f(x) \in f(C)$

Cont.

Possible application:

Cantor Set

$\prod_{n=1}^{\infty}\{0,2\} = \{0,2\}^{\mathbb{N}} \longrightarrow \mathcal{C}$ = middle 3'rds Cantor set $\subset [0,1]$

$\{a_n\} \to \sum_{n=1}^{\infty} \frac{a_n}{3^n}$

cont bijection

is it a homeo?

If we knew $\{0,2\}^{\mathbb{N}}$ is compact

$\Longrightarrow$ Yes!

Another useful fact:

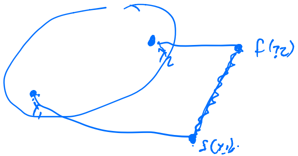
Theorem

X compact, $f : X \to \mathbb{R}$ continuous

$\Longrightarrow$

f has a maximum and a minimum,

ie, $\exists x_1, x_2 \in X$ *such that* $f(x_1) \leq f(x) \leq f(x_2)\ \forall x \in X$



Pf: X cpct, f cont $\Longrightarrow$

$f(X) \subset \mathbb{R}$ compact

$\Rightarrow f(X)$ closed & bdd

bdd $\Rightarrow \inf f(X), \sup f(X) \in \mathbb{R}$
exists

~~closed~~: $\inf f(X), \sup f(X) \in \overline{f(X)}$

$f(X)$ closed $\Rightarrow \overline{f(X)} = f(X)$

$\exists\, x_1, x_2$
$\inf f(X) = f(x_1)$
$\sup f(X) = f(x_2)$

"Existence thms"

Finite intersection property

- X compact $\iff$ whenever $\{F_\alpha\}_{\alpha\in A}$ is a collection of closed sets with the property that all finite intersections

$$F_{\alpha_1} \cap \cdots \cap F_{\alpha_n}$$

are non-empty, then

$$\bigcap_{\alpha\in A} F_\alpha \neq \emptyset$$

Bases and Compactness

Theorem

Let $\mathcal{B}$ be a basis for the topology of X. Then X is compact

$\Longleftrightarrow$

Every cover of X by elements of $\mathcal{B}$ has a finite sub-cover.

$\Downarrow$

Every open cover has a finite subcover

$\mathcal{U} = \{U_\alpha\}$

each $U_\alpha =$

$U_\alpha = \bigcup V_\beta$

Umgebung

U

F fermé

$$U_\alpha = \bigcup_{\beta \in B_\alpha} V_\beta$$

$\mathcal{B}$

$\{V_\beta\}_{\beta \in B_\alpha}$ cover of X by sets in $\mathcal{B}$.

α

1
2
3
4

Any cover by V_β's has a finite sub-cover.

each $V_{\beta_i} \subset$ some U_α

$V_{\beta_1}, \ldots, V_{\beta_n}$

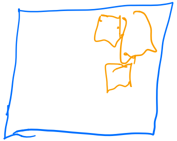
$\underbrace{U_{\alpha_1}, \ldots, U_{\alpha_n}}$ cover X.

Products

Theorem

X, Y compact $\Longrightarrow X \times Y$ is compact.

Enough to prove that any cover of $X \times Y$ by sets $\{U \times V \mid U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}$ has a finite sub-cover.



$\{U_\alpha \times V_\alpha\}$ $\quad \bigcup U_\alpha \times V_\alpha = X \times Y$

U_α cover X

V_α cover Y

$(x, y) \in X \times Y$

$\exists \alpha$ s.t. $(x, y) \in U_\alpha \times V_\alpha$

$\Rightarrow x \in U_\alpha$ $\quad \Rightarrow \{U_\alpha\}$ cover X
$y \in V_\alpha$ $\quad \{V_\alpha\}$ cover Y

What else is needed?

$\exists \alpha_1, \ldots, \alpha_n$ s.t. $U_{\alpha_1}, \ldots, U_{\alpha_n}$ cover X

$\alpha_1', \ldots, \alpha_m'$ s.t. $V_{\alpha_1'}, \ldots, V_{\alpha_m'}$ cover Y

$\Rightarrow \underbrace{U_{\alpha_i} \times V_{\alpha_i}}_{\text{a finite ex}}$ cover $X \times Y$

but is it a sub-cover?

$U_\alpha \times V_\alpha \leadsto U_{\alpha_1} \times V_{\alpha_1}, \ldots U_{\alpha_k} \times V_{\alpha_k}$

A way to find a sub-cover

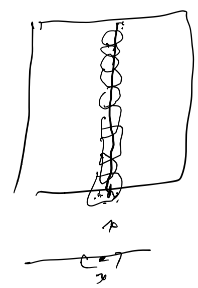
Start: $\{U_\alpha \times V_\alpha\}_{\alpha \in A}$

foreach $x \in X$, a finite cover $x \times Y$

find

$U_{\alpha_1(x)} \times V_{\alpha_1(x)}, \ldots,$

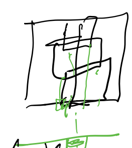
$U_{\alpha_{n(x)}(x)} \times V_{\alpha_{n(x)}(x)}$



$\alpha_1(x), \ldots, \alpha_{n(x)}(x)$

define

$U'(x) = \bigcap_{i=1}^{n(x)} U_{\alpha_i(x)}$



$\{U'(x)\}_{x \in X}$ open cover of X

$\Rightarrow U'(x_1), \ldots U'(x_k)$ finite sub.

$U_{\alpha_1(x_1)} \times V_{\alpha_1(x_1)}, \ldots U_{\alpha_{n(x_1)}(x_1)}, V_{\alpha_n(x_1)}$

$U_{\alpha_1(x_2)} \times V_{\alpha_1(x_2)} \ldots U_{\alpha_{n(x_2)}(x_2)} \times V_{\alpha_{n(x_2)}}$

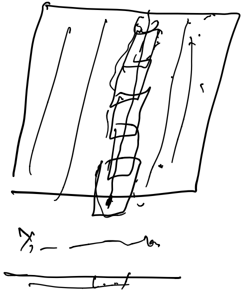
$\vdots$

finite, sub cover

$\{a_i\}^{\mathbb{N}} \to$ Cantor

$C[0,1]$

$(a_i) \to \sum a_i/3^i$



Compactness of Y

Compactness of X

$\prod_{i=1}^{\infty} \{0,1\}$

$= \{0,1\}^{\mathbb{N}}$

Compact

(Cantor set)

$\Rightarrow$ (Induction) $X_1, \dots, X_n$ compact $\Rightarrow X_1 \times \dots \times X_n$ cpct.

More difficult: Tychonoff's thm: $\{X_\alpha\}_{\alpha \in A}$ X_α cpct

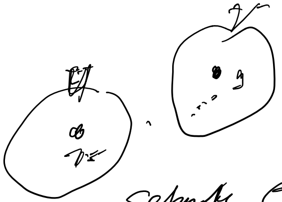
$\Rightarrow \prod_\alpha X_\alpha$ compact

Separation Properties

- A topological space X is said to be:
  - *Hausdorff* $\iff$ for all $x, y \in X$, $x \neq y$, there exist open sets $U, V \subset X$ such that

$$x \in U, \;\; y \in V, \;\; \text{and} \;\; U \cap V = \emptyset$$

nbds



$U \cap V = \phi$

separate $\iff$ disjoint nbds.

- *Regular* $\iff$ for all $x \in X$ and all closed sets $C \subset X$ with $x \notin C$, there exist open sets $U, V \subset X$ such that

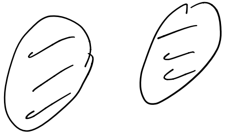
$$x \in U, C \subset V, \quad \text{and} \quad U \cap V = \emptyset$$

$x \notin C$

Separate pts & closed sets

- *Normal* $\iff$ for all closed sets $A, B \subset X$ with $A \cap B = \emptyset$ there exist open sets $U, V \subset X$ such that

$$A \subset U, \; B \subset V, \quad \text{and} \quad U \cap V = \emptyset$$



$A \cap B = \emptyset$

$\Rightarrow$ disjt nbds

Normal $\Rightarrow$ regular $\Rightarrow$ Hausdorff

Normal, Hausdorff

Theorem

A compact Hausdorff space is regular.

X Hausdorff

$C \subset X$ closed

$x \notin C$

find U, V

$x \in U$, $C \subset V$

$U \cap V = \emptyset$

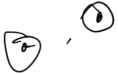
C closed

$x \notin C$

$\forall y \in C$

$\exists\ U_y, V_y$

x $\quad$ y



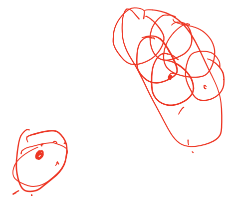
$U_y \cap V_y = \emptyset$

Use: $C \subset X$ closed, X compact $\Rightarrow C$ compact.

$(V_y)_{y \in C}$ an open cover of C

$\Rightarrow \exists\, y_1, \dots, y_n \in C$ s.t.

$C \subset V_{y_1} \cup \dots \cup V_{y_n}$



$V = V_{y_1} \cup \cdots \cup V_{y_n}$

$U = ?$

$V_{y_i} \cap U_{y_i} = \phi$
$\ni$
x

let $U = U_{y_1} \cap \cdots \cap U_{y_n}$

$U \cap V = \phi$

Theorem

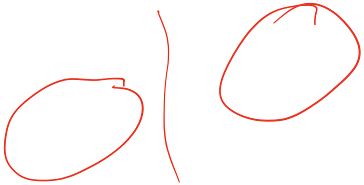
A compact Hausdorff space is normal.

Exercise

- (X, d) metric space. Have seen:
  - (X, d) is Hausdorff.

- Will see (in exercises)
  - (X, d) is regular.
  - (X, d) is normal.

distance fun



Connected Spaces

Definition

A topological space $(X, \mathcal{T})$ is *connected*

$\iff$

if $Y \subset X$ is both open and closed, then either $Y = X$ or $Y = \emptyset$

Equivalent Formulations

Theorem

X is connected

$$\Longleftrightarrow$$

Whenever $U, V \subset X$ are open sets with $X = U \cup V$ and $U \cap V = \emptyset$, then either

$$U = \emptyset \text{ or } V = \emptyset$$

$\Rightarrow U = \phi$
or
$V = \phi$

$A \subset X$ open & closed

A is open

$X - A$ is open.

disjt.

$X = A \cup (X - A)$

$\therefore$ open open

$\Leftarrow$ $X = C_1 \cup C_2$ C_1, C_2 open

$C_1 \cap C_2 = \phi$

$\Rightarrow C_1 = \phi$ or $C_2 = \phi$

Theorem

X is connected

$\iff$

Whenever $E, F \subset X$ are closed sets with $X = E \cup F$ and $E \cap F = \emptyset$, then either

$$E = \emptyset \text{ or } F = \emptyset$$

Theorem

Let $\{0,1\}$ have the discrete topology. Then X is connected

$\iff$

Every continuous map $f : X \to \{0,1\}$ is constant.

$f^{-1}\{0\}, f^{-1}\{1\}$ are open in X

disjoint,

$\cup = X$

f constant $\Rightarrow$ one is $\emptyset$.

If $Y \subset X$, Y connected means: connected in the subspace topology.

Theorem

If $f : X \to Y$ is continuous and X is connected, then $f(X)$ is connected.

$X \to f(x)$ cont

$f(X)$ subsp for Y

Pf

$X \xrightarrow{f} Y \to \{0,1\}$

surjective

$\varphi(f(x))$ cont

$\varphi(y)$ cont?

$y = f(x)$

$y_1 \neq y_2$

$\varphi(y_1) \neq \varphi(y_2)$ $\quad y_1 = f(x_1)$ $\quad y_2 = f(x_2)$

$X \xrightarrow{f} Y \xrightarrow{\varphi} \{0,1\}$

Sur

$\varphi(f(x_1)) \neq \varphi(f(x_2))$

X con $\Rightarrow$ $\varphi \circ f$ const.

$\varphi(f(x_1)) \neq \varphi(f(x_2))$

$\Rightarrow$ $\varphi \circ f$ const.

Known

Theorem

The unit interval $[0, 1] \subset \mathbb{R}$ *is connected.*

Completeness of $\mathbb{R}$

Path-Connected Spaces

Definition

A topological space X is said to be *path connected*

$\iff$

For all $x, y \in X$ there exists a continuous map
$\gamma : [0, 1] \to X$ with

$$\gamma(0) = x \;\text{ and }\; \gamma(1) = y.$$

Theorem

X path connected $\implies$ X connected.

not path $\Leftarrow$ X not conn

$\exists \varphi : X \to \{0,1\}$

~~[illegible]~~ continuous

$I \xrightarrow{\gamma} X$

$[a,b]$ $\; 0 \to x$, $\; 1 \to y$

$\exists x, y \in X$

$\varphi(x) = 0, \quad \varphi(y) = 1$

Definition

পথ মানে $I \to [0,1] \to X$

A topological space $(X, \mathcal{T})$ is said to be *locally path connected*

$\iff$

$\mathcal{T}$ has a basis $\mathcal{B}$ such that every $B \in \mathcal{B}$ is path connected.

Theorem

X connected and locally path connected
$\implies X$ is path connected.

For Homework,

need to know

Basis for Topology

Product Topology

Here's some
review!

Basis $\quad$ Top sp $(X, \mathcal{T})$

$\mathcal{B}$

~~Top sp~~

Topology on a set X:

$$\mathcal{T}_X \subset 2^X$$

s.t.
1) $\phi \in \mathcal{T}_X$, $X \in \mathcal{T}_X$
2) closed under arbitrary unions $\{U_\alpha\}_{\alpha \in A}$, each $U_\alpha \in \mathcal{T}$, $\Rightarrow \bigcup_\alpha U_\alpha \in \mathcal{T}_X$.
3) $U_1, \ldots, U_n \in \mathcal{T}_X$ $\Rightarrow U_1 \cap \cdots \cap U_n \in \mathcal{T}_X$

X

What are possible tops?

two extremes: ind $= \emptyset, \{\}$

$$\mathcal{T}_{disc} = 2^X \quad \text{met}$$

(X,d) metric space

metric top : open sets in (X,d)

U is open $\iff$ $\forall x \in U$ $\exists r_{(x)} > 0$

s.t. $B(x, r(x)) \subset U$.

$\iff$ U is a union of balls

$$U = \bigcup_{x \in U} B(x, r(x))$$

A topology is metrizable

$\iff$ $\exists$ metric d on X

s.t. $\mathcal{T}$ = topology of (X,d).

given $(X, \mathcal{T}_X)$

~~Kxdd~~ $\mathcal{B} \subset \mathcal{T}_X$ is called

a <u>basis</u> for $\mathcal{T}_X$

$\Longleftrightarrow$ every $U \in \mathcal{T}_X$ is a union of elements of $\mathcal{B}$

$\Longleftrightarrow$ $\forall U \in \mathcal{T}_X, \forall x \in U$

$\exists B \in \mathcal{B}$ s.t. $\boxed{x \in B \subset U}$

<u>Ex</u> Balls $B(x,r)$ form a basis for the top of (X, d).

Change pt of view:

~~Given set X~~

~~Defn~~ a collection $\mathcal{B} \subset 2^X$

define a topology on X
with $\mathcal{B}$ as a basis

define $\mathcal{T}_X$ = (all unions of elements of $\mathcal{B}$)
$\cup \{\emptyset\}$

$\mathbb{R}$ $\quad \mathcal{B} = \{(0,1), \{5\}, [1,\infty)\}$

$\mathcal{B}$ has to satisfy some
conditions for
collection of all unions to
be a topology.

Clear: Colection of all
unions is closed under
arb union ✓

$X, \emptyset$ $\quad \cancel{X} = \cup$ of elements of $\mathcal{B}$

$\cancel{\emptyset}$

$\mathcal{T}$ = (all union of elements of $\mathcal{B}$) $\cup \{\phi\}$

assume : $\boxed{X = \text{union of elements of } \mathcal{B}}$ request ①

2) need collection of unions to be closed under finite union.

at least ;

if $B_1, \dots, B_n \in \mathcal{B}$

$\Rightarrow B_1 \cap \dots \cap B_n$ = union of elements of $\mathcal{B}$

$\forall B_1, \dots, B_n \in \mathcal{B}$ and $\forall x \in \underline{B_1 \cap \dots \cap B_n}$

$\exists B \in \mathcal{B}$

s.t. $x \in B \subset B_1 \cap \dots \cap B_n$

Given a set X,

and $\mathcal{B} \subset 2^X$

Satisfying : 1) $\forall x \in X$, $\exists B \in \mathcal{B}$

s.t. $x \in B$

2) $\forall\ B_1, \dots, B_n \in \mathcal{B}$

and $\forall x \in B_1 \cap \dots \cap B_n$,

$\exists\, B \in \mathcal{B}$

with $x \in B \subset B_1 \cap \dots \cap B_n$

Then (collection of all unions of elements of $\mathcal{B}$) $\cup \{\emptyset\}$

is a topology $\mathcal{T}_X$ on X

and $\mathcal{B}$ is a basis for $\mathcal{T}_X$.

Prime Example

(X, d) a metric, $\mathcal{B} = \{ B(x, r) : x \in X, r > 0 \}$

defines a topology on X,

with $\mathcal{B}$ as basis.

other Basis for $\mathcal{T}_{(X,d)}$

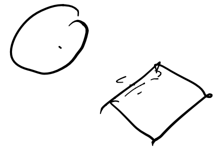
$$\{B(x, 1/n) : x \in X,\ n \in \mathbb{N}\}$$

$\mathcal{B}, \mathcal{B}'$ both satisfy cond for basis

When do $\mathcal{B}$ & $\mathcal{B}'$ define same topology

<u>Ex</u> $\mathbb{R}^2$, $\mathcal{B} = \{$Euclidean balls$\}$

$\mathcal{B}' = \{$Taxi-cab balls$\}$



each $B' \in \mathcal{B}'$ is a union of elts of $\mathcal{B}$

— — $B \in \mathcal{B}$ — — — — — — $\mathcal{B}'$

$\forall\ B' \in \mathcal{B}'$ and $\forall\ x \in B'$,

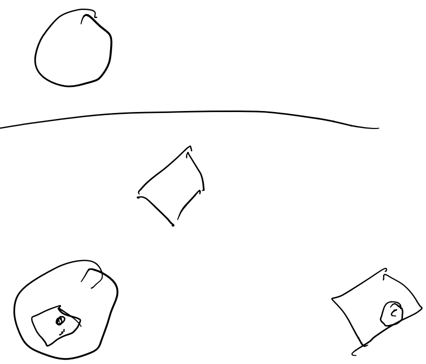
$\exists\ B \in \mathcal{B}$ s.th. $x \in B \subset B'$

$\forall\ B \in \mathcal{B}\ \forall\ x \in B$,

$\exists\ B' \in \mathcal{B}'$ such that $x \in B' \subset B$.

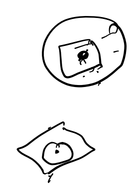
$\mathcal{B}$

$\mathcal{B}'$



$\mathcal{B}$ $\quad B(x,r)$

$B'(x,r)$



U open in $\mathcal{B}$:

$\forall\ x \in U\ \exists\ B(x,r) \subset U$

& $\exists\ r'$ s.th. $B'(x,r') \subset B(x,r)$

U open in $\mathcal{B}'$

U $\quad \forall x \in U \quad \exists r(x)$ st.

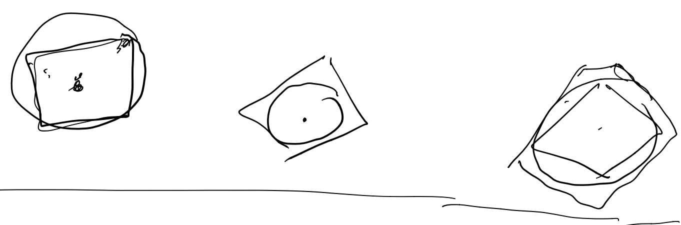
$$B(x, r(x)) \subset U$$

$$\exists r'(x) \text{ st } B'(x, r'(x)) \subset B(x, r(x))$$

$$B'(x, r'(x)) \subset U$$

$\forall x \in U$
$\exists B'(x, r) \subset U$

open in B'



$(X, d_X) \quad (Y, d_Y)$

metric spaces.

$X, \mathcal{T}_X, \; Y, \mathcal{T}_Y$ top spaces.

Product $X \times Y$ has basis $\{U \times V : U \in \mathcal{T}_X, \; V \in \mathcal{T}_Y\}$

topology on $X \times Y$

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)$$

$\Rightarrow$ Closed under finite intersections

$\Rightarrow$ Basis for a topology.

$\mathcal{B}_X$ basis for $\mathcal{T}_X$

$\mathcal{B}_Y$ — $\mathcal{T}_Y$

Then $\mathcal{B}_{\otimes}$ $\{ U \times V : U \in \mathcal{B}_X, V \in \mathcal{B}_Y \}$

is a basis for prod top.

Back $(X, d_X) \times (Y, d_Y)$

Product top on $X \times Y$

has basis $\{ B_X(x, r) \times B_Y(y, r') : x \in X, y \in Y, r, r' > 0 \}$

d^∞ on $X \times Y$

$B^\infty(x_0, y_0, r) = \{ (x, y) : \max d(x_0, x), d(y_0, y) < r \}$

$= B_X(x_0, r) \times B_Y(y_0, r)$