Introduction to Algebraic and Geometric Topology Week 6

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$\boxed{\text{Basis}}$ for a Topology

- $(X, \mathcal{T})$ topological space.

$\mathcal{B} \subset \mathcal{T}$

- Definition

$\boxed{\mathcal{B}} \subset \cancel{2^X}$ is called a *basis for $\mathcal{T}$* $\iff$

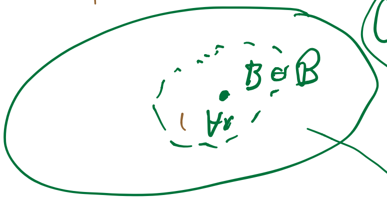
every element of $\mathcal{T}$ is a union of elements of $\mathcal{B}$.

$\mathcal{B}$
Basis $\to \mathcal{B}$ "generates" $\mathcal{T}$

$\mathcal{B} \subset \mathcal{T}_X$

- Equivalent statement:
- $\mathcal{B}$ is a basis $\iff$ $\forall U \in \mathcal{T}$ open,

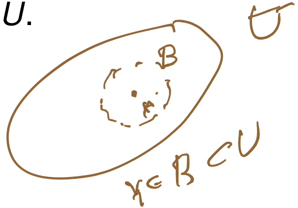
  $\forall x \in U$, $\exists B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$.



$U \in \mathcal{T}$

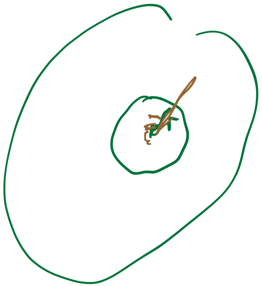
Yes

$\iff$



$x \in \underset{\in \mathcal{B}}{B} \subset U$

0 r



Metric spaces (X, d)

$U \subset X$ open $\iff$ $\forall x \in U \ \exists r > 0$ s.t. $B(x, r) \subset U$

$\exists k \in \mathbb{N}$ s.t. $B(x, 1/k) \subset U$

$$\left[\forall x\in U\ \exists r>0 \text{ s.t. } B(x,r)\subset U\right] \quad (1)$$

$$\left[\forall x\in U\ \exists k\in\mathbb{N} \text{ s.t. } B(x,1/k)\subset U\right] \quad (2)$$

$\{U \text{ satisfy } (1)\} = \mathcal{U}_1$

$\{ \ldots\ (2) = \mathcal{U}_2$

$\mathcal{U}_1 = \mathcal{U}_2$

(1) $\mathcal{U}_2 \subset \mathcal{U}_1$ ✓

~~$\exists r>0\ B(x,r)\subset U$~~

$\exists k$ s.t. $B(x,1/k)\subset U$

$\Rightarrow \exists r>0$

$r = 1/k$.

$\mathcal{U}_1 \subset \mathcal{U}_2$

$\exists r>0$ s.t. $B(x,r)\subset U$

$\overset{?}{\Rightarrow} \exists k\in\mathbb{N}$ s.t. $B(x,1/k)\subset U$.

$\forall r,\ 0<r,$

$\exists k\in\mathbb{N}$

s.t. $1/k<r$

Examples

- (X, d) metric space,

$$\mathcal{B} = \{B(x, r) : x \in X, \ r > 0\}$$

is a basis for topology of (X, d)

another ex:

$$\mathcal{B}' = \{B(x, 1/k) : x \in X, \ \forall k \in \mathbb{N}\}$$

- (X, d) metric space, $E \subset X$ *dense subset*
- ~~Recall~~ : E dense $\iff \overline{E} = X$.

$\forall x \in X$

$\exists \{x_n\} \subset E$

$\lim \{x_n\} = x$

- Def

$$\mathcal{B}' = \{B(x, \frac{1}{k}) : x \in \cancel{X}\ E, k \in \mathbb{N}\}.$$

Ex. $\mathbb{Q} \subset \mathbb{R}$ dense.

$\mathbb{Q}^n \subset \mathbb{R}^n$ "

- $(\mathbb{R}^n, d_{(2)})$ (or any equivalent d)
-

$$\boxed{\mathcal{B} = \{B(x, \frac{1}{k}) : x \in \mathbb{Q}^n,\ k \in \mathbb{N}\}.}$$

- In the last example $\mathcal{B}$ is *countable*
- $(X, \mathcal{T})$ is called *second countable* $\iff$ $\mathcal{T}$ has a countable basis.

If (X, d) has countable dense subset E

$\Downarrow$

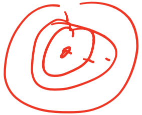
X has a countable basis

Ex: $\mathbb{Q} \subset \mathbb{R}$

$\mathbb{Q}^n \subset \mathbb{R}^n$

- Why “second”?
- Is there a “first countable”?
- Yes; a similar condition about any point $x \in X$

weaker



- $(X, \mathcal{T})$ topological space and $\mathcal{B}$ basis for $\mathcal{T}$.
- $f : (X', \mathcal{T}') \to (X, \mathcal{T})$ is continuous $\iff$ $f^{-1}(B)$ open $\forall B \in \mathcal{B}$.

$U = \bigcup_\alpha B_\alpha$

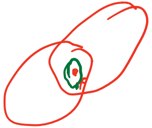
$f^{-1}(\bigcup B_\alpha) = \bigcup_\alpha f^{-1}(B_\alpha)$

- If $A \subset X$, then
  - $x \in A^o \iff \exists B \in \mathcal{B}$ with $x \in B \subset A$.
  - $x \in \overline{A} \iff B \cap A \neq \emptyset \;\; \forall B \in \mathcal{B}$ with $x \in B$.

Defining Topologies from a Basis

- X non-empty set, $\mathcal{B} \subset 2^X$ satisfying:
  1. $\forall x \in X \; \exists B \in \mathcal{B}$ such that $x \in B$.
  2. $\forall B_1, B_2 \in \mathcal{B}$ and $\forall x \in B_1 \cap B_2 \; \exists B \in \mathcal{B}$ such that $x \in B$ and $B \subset B_1 \cap B_2$.

$\bigcup_{\mathcal{B}} \mathcal{B} = X$



- Let

$$\mathcal{T} = \{U \subset X | \forall x \in U \; \exists B \in \mathcal{B} \text{ with } x \in B \subset U\} \cup \{\emptyset\}$$

$X \in \mathcal{T}$

$\emptyset \in \mathcal{T}$

Closed under $\bigcup$

- Then $\mathcal{T}$ is a topology on X and $\mathcal{B}$ is a basis for $\mathcal{T}$.

$\forall B_1, B_2 \in \mathcal{B}$, $B_1 \cap B_2 =$ union of elements of $\mathcal{B}$

Closed under finite $\cap$.

- Equivalent definition of $\mathcal{T}$:
- $\mathcal{T}$ is the collection of the unions of all subcollections of $\mathcal{B}$ (including the empty subcollection).

$X \in \mathcal{T}$, $\emptyset \in \mathcal{T}$

closed under arb. unions,

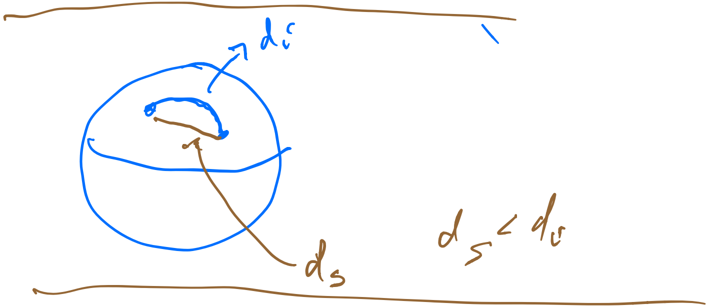
finite inters

Clear: union of unions = union

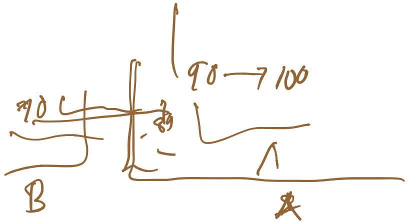
Need $B_1 \cap B_2$ = union of elements of $\mathcal{B}$

$$x \neq y \Rightarrow f(x) \neq f(y)$$

$$x = y \Leftarrow f(x) = f(y)$$

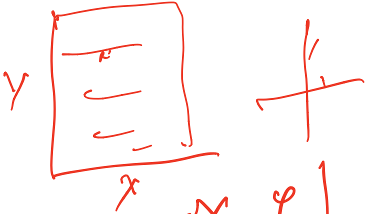


$$d_s < d_i$$


90 → 100
90
B
A

X, Y sets

$$X \times Y = \{(x,y) : x \in X,\ y \in Y\}$$

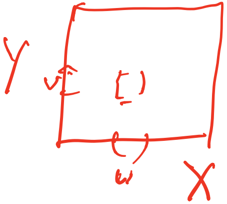


$(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ top spaces.

how to define top on $X \times Y$

Example: Product Topology

- $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ topological spaces.
- $X \times Y$ their Cartesian product.
- Let
$$\mathcal{B}_{X\times Y} = \{U \times V \mid U \in \mathcal{T}_X \text{ and } V \in \mathcal{T}_Y\}$$
- Then $\mathcal{B}_{X\times Y}$ satisfies conditions (1) and (2) above.
- Resulting $\mathcal{T}_{X\times Y}$ is a topology on $X \times Y$, called the *Product Topology*.



$U \in \mathcal{T}_X$

$V \in \mathcal{T}_Y$

$\{ U \times V : U \} \quad \mathcal{T}_{X \times Y}$

$X \times Y \in \mathcal{T}, \ \phi \in$

Ex: $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

$(A \times B) \cup (C \times D)$

need not be $(E \times F)$

- Useful fact: if $A_1, A_2 \subset X$ and $B_1, B_2 \subset Y$, then

$$(A_1 \times B_1) \cap (A_2 \times B_2) = (A_1 \cap A_2) \times (B_1 \cap B_2)$$

$X \times Y = \{(x,y) : x \in X, y \in Y\}$

$\cup$ $\cup$
A B

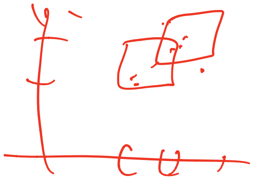
$A \times B \subset X \times Y$

$\mathcal{B} = \{U \times V : U \in \mathcal{J}_X, V \in \mathcal{J}_Y\}$ this a basis for a top.

$X \times Y \in \mathcal{B}$

$(x,y) \in (U_1 \times V_1) \cap (U_2 \times V_2)$

- $\mathcal{B}_{X \times Y}$ closed under finite intersections, but not under unions.



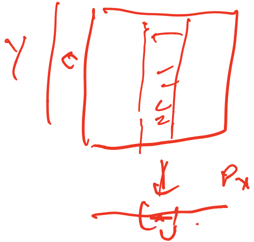
- Look at $\mathbb{R} \times \mathbb{R}$

$\{U \times V\}$ form,

- Projections $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$:

$$p_X(x,y) = x, \quad p_Y(x,y) = y.$$

- $\mathcal{T}_{X\times Y}$ is the smallest topology that makes both projections p_X and p_Y continuous.



P_X, P_Y cont $(X \times Y, \mathcal{T}_{prod}) \xrightarrow{P_X} X$

$\xrightarrow{P_Y} Y$

$P_X^{-1}(U)$ open $\forall\, U$ open in X

$P_Y^{-1}(V)$ open $\forall\, V$ open in Y

$\{(x,y) : \underbrace{P_X(x,y)}_{x} \in U\}$

$= \{(x,y) : x \in U\}$

$= \boxed{U \times Y} \in \mathcal{B}$

$P_Y^{-1}(V) = X \times V \in \mathcal{B}$

$\Rightarrow P_X, P_Y$ cont

Charac:

Suppose $\mathcal{T}'$ a top on $X \times Y$

st. P_X, P_Y both cont.

how does it relate to

$\mathcal{T}_{X \times Y}$ just defn: basis

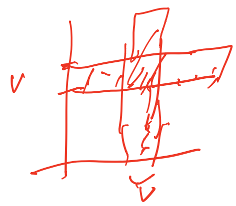
$\{U \times V : U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}$

$\mathcal{T}'$ has to contain

$\{U \times Y : U$ open in $X\} \subseteq \mathcal{T}'$

for P_X to be cont

$\{X \times V : V \text{ open in } Y\} \subset \mathcal{T}'$



$\Rightarrow (X \times V) \cap (U \times Y)$ opd

$\parallel$

$U \times V$

$\Rightarrow \mathcal{T}_{X\times Y} \subset \mathcal{T}'$

$\mathcal{T}' \supset \mathcal{T}_{X\times Y}$

$\mathcal{T}_{X\times Y}$ smallest top on $X \times Y$ that makes p_X, p_Y cont

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$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[rr, bend left, "p_X \circ f"] \arrow[rr, bend right] & X \times Y \arrow[r, "p_X"] \arrow[dr, "p_Y"'] & X \\
 & & Y
\end{tikzcd}$$

$p_Y \circ f$ — comp cor

Charac: f cont $\Rightarrow p_X \circ f,\ p_Y \circ f$ cont

$\Leftarrow$

$\Longleftrightarrow$

$p_X \circ f$ cont $\Rightarrow$ ~~f cont~~

$(p_X \circ f)^{-1}(U)$

$= f^{-1} \circ \underbrace{p_X^{-1}(U)}_{U \times Y}$

$f^{-1}(U \times Y)$ open $\forall_U$ op in X

$f^{-1}(X \times V)$ open

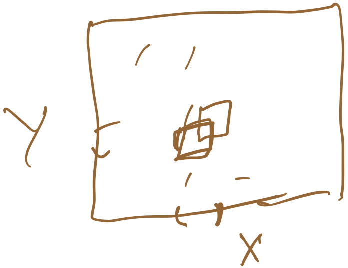
$\Rightarrow f^{-1}(\underbrace{(U \times Y) \cap (X \times V)}_{U \times V})$ open

$\Rightarrow f^{-1}(U \times V)$ open ~~$\forall U$~~

$\forall\, U \times V \in \mathcal{B}$

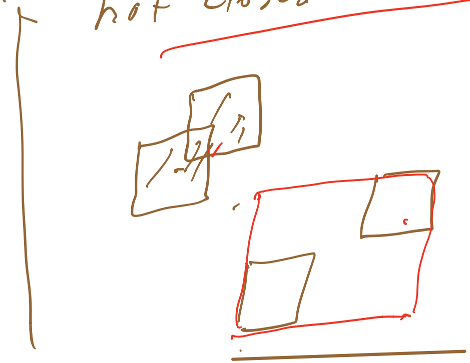
f cont

- $f : Z \to X \times Y$ continuous (w respect to $\mathcal{T}_{X\times Y}$) $\iff$ both compositions $p_X \circ f$ and $p_Y \circ f$ are continuous.



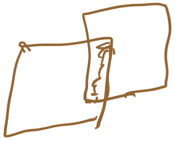
"rectangles"

not closed under uno



$(x,y) \in (U_1 \times V_1) \cap (U_2 \times V_2)$

$= (U_1 \cap U_2) \times (V_1 \cap V_2)$



$\subset$

$(x,y) \in U_1 \times V_1$	$x \in U_1$, $y \in V_1$
$\wedge$ $U_2 \times V_2$	$x \in U_2$, $y \in V_2$

$x \in U_1 \cap U_2$ $\qquad$ $y \in V_1 \cap V_2$

(3) closed under finite intersections

Infinite Products

ex: $\mathbb{Z}, \mathbb{N}, \mathbb{R}, \ldots$

- A an index set.
- $\{X_\alpha\}_{\alpha \in A}$ a collection of *non-empty* sets indexed by A.
- $\coprod_{\alpha \in A} X_\alpha$ their disjoint union.
- The *product* of the X_α is defined as

$$\prod_{\alpha \in A} X_\alpha = \{f : A \to \coprod X_\alpha \mid \forall \alpha \in A, f(\alpha) \in X_\alpha\}$$

$\prod_{\alpha \in A} X_\alpha$

$X_1, X_2, \ldots$

$X_1 \times X_2 \times \ldots$

Examples

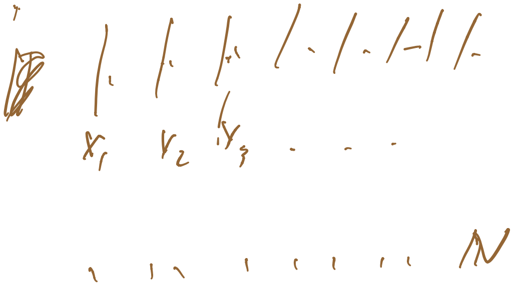
- $A = \{1, 2\}$ then

$$\prod_{\alpha \in \{1,2\}} X_\alpha = \{f : \{1, 2\} \to X_1 \coprod X_2 \mid f(1) \in X_1, f(2) \in X_2\}$$

Letting $x_1 = f(1)$ and $x_2 = f(2)$, this is the same as

$$\{(x_1, x_2) \mid x_1 \in X_1, x_2 \in X_2\}$$

which is the usual defintiion of $X_1 \times X_2$.



$\coprod X_i$

$f : A \longrightarrow \coprod X_\alpha$

$\boxed{\alpha \longmapsto X_\alpha}$

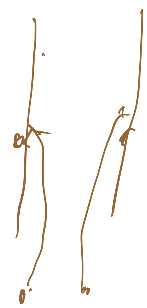
$f(\alpha) \in X_\alpha$

ex: $X_1 \times X_2$ $\quad \{1,2\}$

$\{1,2\} \to X_1 \sqcup X_2$

$f(1) \in X_1$

$f(2) \in X_2$



$\Leftrightarrow (f(1), f(2)) \in X_1 \times X_2$

("old def of $X_1 \times X_2$)

$f \in \prod_{\alpha \in A} X_\alpha$

$\Leftrightarrow$ choose $f(\alpha) \in X_\alpha$

"Axiom of Choice"

given a nonempty collection $\{X_\alpha\}$, each $X_\alpha \neq \emptyset$,

can choose an element $f(\alpha) \in X_\alpha$

$\Leftrightarrow \boxed{\prod_\alpha X_\alpha \neq \emptyset}$

- Similarly, if $A = \{1, 2, \ldots, n\}$, a fiinite set, then $\prod_{\alpha \in A} X_\alpha$ gives the usual definition

$$X_1 \times X_2 \times \cdots \times X_n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in X_i\}$$

Topology in Product Space

- Suppose A arbitrary and each X_α has a toplogy $\mathcal{T}_\alpha$.
- Let $\mathcal{B}_{\prod X_\alpha}$ be defined as follows.:
  - For each finite subset $F \subset A$ let $\mathcal{U}_F$ be a collection

$$U_F = \{U_\alpha\}_{\alpha \in F} \text{ where } U_\alpha \in \mathcal{T}_\alpha$$

  - Then let

$$B(F, \mathcal{U}_F) = \{f \in \prod X_\alpha \mid \; f(\alpha) \in U_\alpha \text{ for all } \alpha \in F\}$$

- Define $\mathcal{B}_{\prod X_\alpha}$ to be the collection of all $B(F, \mathcal{U}_F)$.
- Check: $\mathcal{B}_{\prod X_\alpha}$ is a basis.

- The resulting topology is called the *product topology*.

- Essence: Each $B(\mathcal{U}_F)$ restricts only finitely many coordinates.

- For A finite get same basis as before.

- Suppose $A = \mathbb{N}$ and all $X_i = X$.

  Then $B(F, U_F)$ is the set of all sequences $\{x_i\}$ such that $x_i \in U_i$ for $i \in F$.