Introduction to Algebraic and Geometric Topology Week 5

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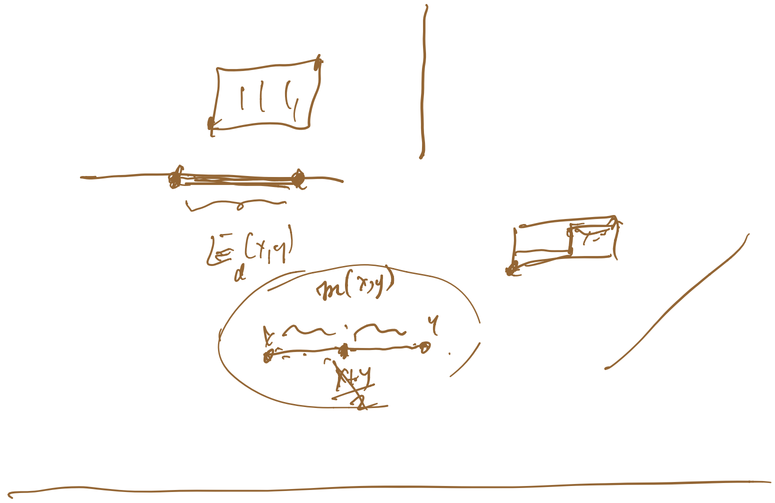


E_d(x,y)
m(x,y)

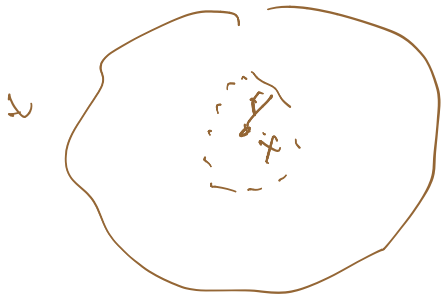
Topology of Metric Spaces

- (X, d) metric space.
- Recall the definition of *Open sets*:

Definition

$$U \subset X \text{ open set} \iff \forall x \in U \;\; \exists r > 0 \text{ so that}$$

$$B(x, r) \subset U.$$



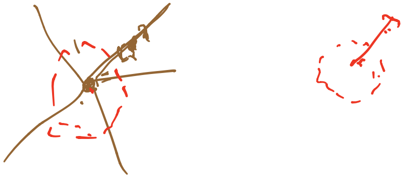
Examples

$r>0$

$B(x,r) = \{ y : d(x,y) < r \}$

$\bar{B}(x,r)$ $\le$

- $(\mathbb{R}^n, d_{(2)})$ usual open sets.
- (X, d) discrete metric space $\Longrightarrow$ all sets are open.
- Open sets in French railway metric.



- Examples on non-open sets:

$x \geq 0$ in $\mathbb{R}^2$

$B(x,r) \not\subset$ set

$\exists x \ \forall r > 0$

$d_{(2)}, \ d_{(1)}, \ d_{(\infty)}$

- Have seen that different metrics can give same open sets.
- Example: $(\mathbb{R}^n, d_{(1)})$ or $(\mathbb{R}^n, d_{(\infty)})$: same open sets.
- Will concentrate on the collection of open sets, rather than the metric,
- This collection will be called the *Topology*
- First look more closely at open sets.

- ## Theorem

  (X, d) metric space, $x \in X$ and $r > 0 \implies$

  $B(x, r)$ is an open set

  - Proof?

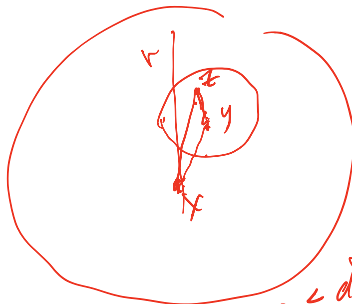
$\forall y \in B(x,r) \ \exists \ s > 0$

$B(y,s) \subset B(x,r)$

$s = r - d(x,y)$

$z \in B(y,s) : \ d(y,z) < s$

$d(x,z) \le d(x,y) + d(y,z)$



$s = r - d(x,y)$

$d(x,z) \le d(x,y) + d(y,z)$

- ## Theorem
  (X, d) metric space, $x \in X$ and $r \geq 0 \implies$

  $$\{y \in X | d(x, y) > r\} \textit{ is an open set .}$$

  - Proof?

Closed sets

- **Definition**

  $F \subset X$ *closed set* $\iff X \setminus F$ is open.

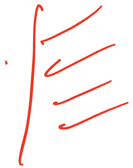
$\{y : d(x,y) > r\}$ is open

$X - \{\ \} = B(x,r)$

- Examples of closed sets:

$$\underbrace{B(x,r)}_{\text{closed.}} = X - \underbrace{\{y : d(x,y) > r\}}_{\text{open}}$$

$x \geq 0$ in $\mathbb{R}^2$



$\{x < 0\}$ open.

f:

$f: X, d \to Y, d'$

$\forall x \in X \quad \forall \varepsilon > 0$

$\exists \delta > 0$ s.t. $d(x,y) < \delta \Rightarrow d(f(x), f(y)) < \varepsilon$

$f(B(x,\delta)) \subset B(f(x), \varepsilon)$

Continuous maps

- (X, d) and (Y, d') metric spaces, $f : X \to Y$
- Theorem
  f is continuous

  $\iff$

  $\forall U \subset Y$,
  U *open in* $(Y, d') \implies f^{-1}(U)$ *open in* (X, d).
- Briefly:
  f continuous

  $\iff$

  the preimage of every open set is open.

- Review preimage f^{-1}

Sets:

$$\begin{array}{ccc} f: X & \to & Y \\ & & \cup \\ & & U \end{array}$$

$$f^{-1}(U) = \{x : f(x) \in U\}$$

f^{-1} operation on Sets

not pts.

- Prove Theorem

f cont ε-δ

U open

Suppose $f: X \to Y$

U open $\Rightarrow$ $f^{-1}(U)$ open.

$x \in X$, $f(x)$, $B(f(x), \varepsilon)$

open set

$x_1 \in f^{-1}(B(f(x), \varepsilon))$

$f(x) \in B(f(x), \varepsilon)$

$f^{-1}(B(f(x), \varepsilon))$ open

$\exists \ \delta > 0$ s.t. $B(x, \delta) \subset f^{-1}(B(f(x), \varepsilon))$

$\Rightarrow f(B(x, \delta)) \subset B(f(x), \varepsilon)$

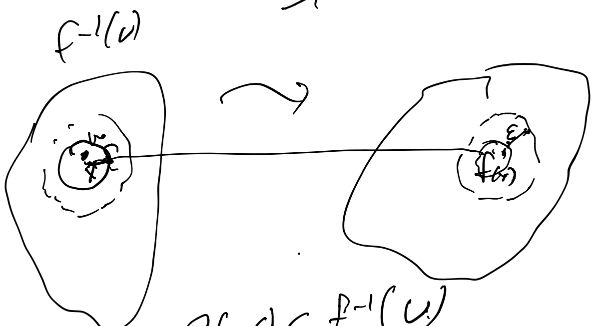
ε, δ def $\Rightarrow$ f^{-1}(open) is open

$f^{-1}(U)$ open.

$x \in f^{-1}(U) \Leftrightarrow f(x) \in U$

$\Rightarrow \exists\, \varepsilon > 0$ s.t. $B(f(x), \varepsilon) \subset U$

ε-δ def $\Rightarrow \exists\, \delta > 0$ s.t. $f(B(x,\delta)) \subset B(f(x),\varepsilon)$



$B(x,\delta) \subset f^{-1}(U)$

$\rightarrow B(f(x),\varepsilon) \subset U$.

Official def of Continuity

f^{-1}(open) is open.

- Similarly, f is continuous $\iff$

  the preimage of every closed set is closed:
- Proof: use $f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B)$.

$f: X \to Y$

$x \in f^{-1}(\cancel{Y} - A)$

$f(x) \in Y$, $f(x) \notin A$

$= X - f^{-1}(A)$

$x \notin f^{-1}(A)$

$f(x) \notin A$

$\boxed{C}$

- *Theorem* Composition of continuous maps is continuous.
- Knew this already, but now have shorter proof, since

$$\boxed{(f \circ g)^{-1}(U) = g^{-1}(f^{-1}(U))}$$

$(f \circ g)^{-1} = g^{-1}(f^{-1}\ )$

check

U open $\Rightarrow f^{-1}(U)$ open $\Rightarrow g^{-1}(f^{-1}(U))$ open

$= (f \circ g)^{-1}(U)$,

- $f : (X, d) \to (Y, d')$ continuous. Then

  f is a homeomorphism $\iff$

  f is bijective and $f(U) \subset Y$ is open for all $U \subset X$ open.

Cont: $\overset{Y}{U}$ open $\to$ $\boxed{f^{-1}}_{\text{set}}(\overset{X}{U})$ open

f^{-1} cont

if map f^{-1} exists then $(f^{-1})^{\text{set}}(A)$

$(f^{-1})^{\text{map}}(A)$

- $f : X \to Y$ bijective map (not assumed continuous). Then

  f is a homeomorphism $\iff$

  $\forall U \subset X$, U is open in $X \iff f(U)$ is open in Y.

The collection of open sets in (X, d)

- Let
  - A an index set (any cardinality)
  - $\{U_\alpha\}_{\alpha \in A}$ a collection of open sets in (X, d) indexed by A.
- Then

$$\bigcup_{\alpha \in A} U_\alpha \qquad \exists$$

is open.

pf $\quad x \in \bigcup_\alpha U_\alpha$

$\boxed{\exists \alpha}$ s.t. $x \in U_\alpha$

U_α open $\Rightarrow$

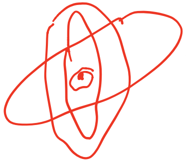
$\exists r > 0$

s.t. $B(x,r) \subset U_\alpha$.

$\Rightarrow B(x,r) \subset \bigcup_\alpha U_\alpha$

- Let $U_1, \ldots, U_k$ be a *finite* collection of open sets.
- Then
$$\bigcap_{i=1}^{k} U_i.$$
is open.

$$x \in \bigcap_{i=1}^{n} U_i.$$

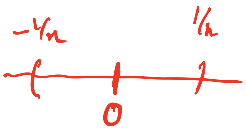


$$x \in U_i \quad \boxed{\forall i}$$

$\therefore$

$\forall i$ then $\Rightarrow \exists\, r_i$ s.t. $B(x, r_i) \subset U_i$

$U_n = (-1/n, 1/n)$



$r = \min\{r_1, \dots, r_n\}$

$B(x,r) \subset \bigcap U_r$

$\bigcap_{n=1}^{\infty} (-1/n, 1/n) = \{0\}$

$\exists\, r > 0$ s.t. $B(0,r) \subseteq \{0\}$

- Summary: The collection of open sets in (X, d) is closed under the operations of
  - Arbitrary union.
  - Finite intersection.

- Equivalent statement:

  The collection of closed sets in (X, d) is closed under the operations of

  - Arbitrary intersection.
  - Finite union.

F_α = closed $\alpha \in A$

$X \setminus \bigcap F_\alpha$

$= \bigcup (X - F_\alpha)$ open.

$\Rightarrow \bigcap F_\alpha$ closed

$F_1, \ldots, F_n$

$X - (F_1 \cup \cdots \cup F_n)$

$= \bigcap_{i=1}^{n} (X - F_i)$ open

$\Rightarrow \bigcup_{i=1}^{n} F_i$ closed.

$X = n$ elements

$\#(2^X) = 2^n$

$\{f : X \xrightarrow{f} \{0,1\}\}$

$\longleftrightarrow$ Subsets

$f \longrightarrow \{x \mid f(x) = 1\}$

$f : X \to \{0,1\} \longleftarrow S \subset X$

$f(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & x \notin S. \end{cases}$

"2" $= \{0, 1\}$

$\{0, 1, \ldots\}$

$Y^{X} = \{$all fns $(X \to Y)\}$

Topologies

- Turn these properties into a definition:

- **Definition**

$X \neq \emptyset$, set a set, $\boxed{\mathcal{T} \subset 2^X}$.

2^X = Set of Subsets of X.

$\mathcal{T}$ is called a *Topology on X* iff

- $\boxed{\emptyset \in \mathcal{T} \text{ and } X \in \mathcal{T}}$
- A any index set, $\{U_\alpha\}$ a collection of elements $U_\alpha \in \mathcal{T}$ indexed by A, then

$$\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}$$

- $U_1, \ldots, U_k$ any finite collection of elements $U_i \in \mathcal{T}$, then

$$\bigcap_{i=1}^{k} U_i \in \mathcal{T}$$

closed under arb union, fin interse

- Briefly, a topology $\mathcal{T}$ on X is
a collection of subsets of X ($\mathcal{T} \subset 2^X$)
which contains the empty set, the whole set, and is closed under the operations of arbitrary union and finite intersection.

Def $\mathcal{T}$ is a topology on X,
the elements of $\mathcal{T}$ are called
the ($\mathcal{T}$-) open sets

Examples of Topologies

- (X, d) any metric space, $\mathcal{T}_{(X,d)}$ the collection of its open sets.
- Two extreme examples of topologies::

Given X, what is the largest possible topology on X?

  - X any set, $\mathcal{T}_{disc} = 2^X$. Every set is open
  - This is the *discrete topology*, can be defined by the discrete metric.

Smallest

  - X any set, $\mathcal{T}_{ind} = \{\emptyset, X\}$, the *indiscrete topology*
  - Indiscrete topology not defined by any metric (if cardinality of X at least 2).

- Intermediate example:
  - X any infinite set. Define $\mathcal{T}_{CF} \subset 2^X$ by

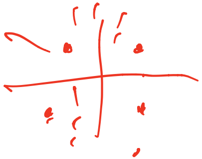
$$U \in \mathcal{T}_{CF} \text{ if and only if } \begin{cases} \boxed{U = \emptyset} \text{ or} \\ \boxed{X \setminus U \text{ is a finite set.}} \end{cases}$$

Complement of Finite

$X = \mathbb{Z}$

$X = \mathbb{R}$

$X = \mathbb{R}^2$



$F \subset X$ closed $\iff \begin{cases} F = X \\ \text{or finite} \end{cases}$

X, ϕ closed?

Closed under

arb intersec

finite unions

✓

Topological Spaces

- *Definitions*:
- A *Topological space* is a pair $(X, \mathcal{T})$, where
  - X is a set.
  - $\mathcal{T} \subset 2^X$ is a topology on X.
- If $(X, \mathcal{T})$ is a topological space, then
  - $U \subset X$ is an open set if and only if $U \in \mathcal{T}$.
  - $F \subset X$ is a closed set if and only if $X \setminus F$ is open.

- If $\mathcal{T}$ is a topology on X, its closed sets satisfy:
  - X and $\emptyset$ are closed sets.
  - If A is any index set and $\{F_\alpha\}_{\alpha \in A}$ is any collection of closed set indexed by A, then

    $$\bigcap_{\alpha \in A} F_\alpha$$

    is closed.
  - If $F_1, \ldots, F_k$ is a finite collection of closed sets,

    $$F_1 \cup \cdots \cup F_k$$

    is closed.

- A topology can be defined in terms of its closed sets.
- Example: X any set, define

$$F \subset X \text{ is closed} \iff \begin{cases} F = X & \text{or} \\ F \text{ is finite.} \end{cases}$$

- Then the topology

$$\mathcal{T} = \{X \setminus F \mid F \text{ is closed} \}$$

is the same as $\mathcal{T}_{CF}$ above.

(X, d) metric: distance

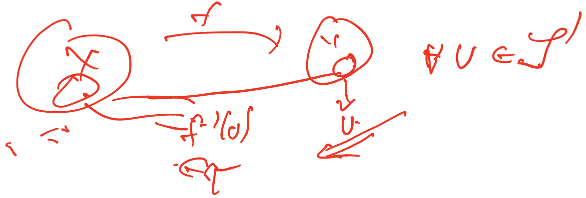
$(X, \mathcal{T})$ top: "neighborhoods"

$\leftrightarrow$ "close".

Continuous Maps

- ## Definition

  If $(X, \mathcal{T})$, $(X', \mathcal{T}')$ are topological spaces and $f : X \to X'$, then $f : (X, \mathcal{T}) \to (X', \mathcal{T}')$ is *continuous* iff for all $U \in \mathcal{T}'$, $f^{-1}(U) \in \mathcal{T}$.



- Equivalent Characterization: f continuous $\iff$ for all $\mathcal{T}'$-closed sets, $f^{-1}(F)$ is $\mathcal{T}$-closed.

Examples of Continuous Maps

X any set, $Y, \mathcal{T}$ top sp

$X \to Y$

- $(X, \mathcal{T})$ any topological space.
  - $f^{-1}(U) \in \mathcal{T}$ disc
  - $f : (X, \mathcal{T}_{disc}) \to (X, \mathcal{T}) \ni U$
  - $f : (X, \mathcal{T}) \to (X, \mathcal{T}_{ind})$

    $f^{-1}(\emptyset) = \emptyset$

    $f^{-1}(X) = X$
- Any $(X_1, \mathcal{T}_1), (X_2, \mathcal{T}_2)$, $f : X_1, \to X_2$ constant.

$f : X_1 \to X_2$ $\quad \boxed{f(x) = c \in X_2}$ $\forall x \in X_1$

$f^{-1}(U) = \begin{cases} \emptyset & \text{if } c \notin U \\ X & \end{cases}$

$x_1, c \in U$

$\mathbb{R}, \cancel{d_E}\ \tau_E$

$\downarrow f$

$\mathbb{R}, \mathcal{T}_{cF}$

- $X = \mathbb{R}$, two topologies: $\mathcal{T}_E$ Euclidean, $\mathcal{T}_{CF}$ as above.

  Compare usual continuous maps with continuous maps

  - $(\mathbb{R}, \mathcal{T}_E) \to (\mathbb{R}, \mathcal{T}_{CF})$

    f^{-1}(finite) closed.

    $\Leftrightarrow$ $f^{-1}(x)$ is closed $\forall x \in \mathbb{R}$,

  - $(\mathbb{R}, \mathcal{T}_{CF}) \to (\mathbb{R}, \mathcal{T}_E)$

    f^{-1}(closed) is closed

  - $(\mathbb{R}, \mathcal{T}_{CF}) \to (\mathbb{R}, \mathcal{T}_{CF})$

    f^{-1}(finite) $= \{ x_i$ finite

$\forall x \in \mathbb{R},\ \underline{f^{-1}(x)\ \text{finite}}$ or $\underline{X}$

$f = \text{const.}$

$f^{-1}(\text{closed}) = \begin{cases} X \\ \text{finite.} \end{cases}$

X

$\mathcal{T}_{CF} \to \mathcal{T}_{E}$

X

$CF \to CF$

Composition of Continuous Maps

- $(X, \mathcal{T}), (X', \mathcal{T}'), (X'', \mathcal{T}'')$ topological spaces.
- $f : (X, \mathcal{T}) \to (X', \mathcal{T}')$ and $g : (X', \mathcal{T}') \to (X'', \mathcal{T}'')$ continuous.
- Then $g \circ f : (X, \mathcal{T}) \to (X'', \mathcal{T}'')$ is continuous.

Neighborhoods

- $(X, \mathcal{T})$ topological space, $x \in X$.

- Definition

  A *Neighborhood of* x

  is an open set $U \subset X$ containing x.

  In other words, $x \in U \subset X$

$x \in X$

Limits

- $(X, \mathcal{T})$ topological space, $\{x_n\}$ sequence in X, $x \in X$.
- *Possible definition of* $\lim\{x_n\} = x$:
- $\forall$ neighbornoods U of x $\exists$ N such that

$$n > N \implies x_n \in U$$

$= \lim x_n$

- Problem: are limits unique?
- Example: In $(\mathbb{R}, \mathcal{T}_{CF})$, let $x_n = n$.

$\mathbb{R}$ $\quad$ $x_n = n$

$\lim \{x_n\}$

$\forall x \in \mathbb{R}$

$\lim \{x_n\} = x$

$\forall n > N$?

$N > \max\{N_1, \dots, N_s\}$

$n > N$

$n \to 0$ in CF top

$n \to 1$

$n \to 55$

Haus~~s~~dorff Spaces

Hausdorff

- Need definition:

- **Definition**
  $(X, \mathcal{T})$ is called a *Hausdorff Space* $\iff$

  $\forall x, y \in X,\ x \neq y,\ \exists$ nbds U of x, V of y s.t. $U \cap V = \emptyset$.

- If X is Hausdorff, limits are unique.

$\lim x_n = x$ & $x \neq y$

$\lim x_n = y$

Pf $x \neq y$

$\exists\ U, V$ $x \in U$, $y \in V$, $U \cap V = \emptyset$

$\lim x_n = x$ $\exists N$ $x_n \in U$ $\forall n \geq N$

$\lim x_n = y$ $\exists N'$ s.t. $x_n \in V$ $\forall n > N'$

$\{x_n\} \in U \cap V$ $\forall n > N, N'$

- Example:
- If (X, d) is a metric space, then it is Hausdorff.

$U \cap V = \emptyset$

Metric $\Rightarrow$ Hausdorff

open U, V

$x \in U = B\left(x, \frac{d(x,y)}{2}\right)$

$y \in V = B\left(y, \frac{d(x,y)}{2}\right)$

$U \cap V = \emptyset$

Interior, closure, boundary

- $(X, \mathcal{T})$ and $E \subset X$. Define:
- E^o, the *interior of E* by

$$E^0 = \bigcup \{U \subset X | U \text{ open and } U \subset E\}.$$

- $\overline{E}$, the *closure of E* by

$$\overline{E} = \bigcap \{F \subset X | F \text{ closed and } E \subset F\}.$$

- ∂E, the *boundary of E* by

$$\partial E = \overline{E} \setminus E^0.$$

largest open set $\subset E$

smallest closed set $\supset E$

$\mathcal{T}_{CF}$ on $\mathbb{R}$

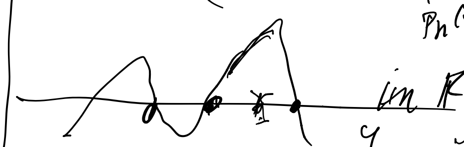
$\Longleftrightarrow$ Zariski Top

(Algebraic Geometry

$A \subset \mathbb{R}^n$ is closed

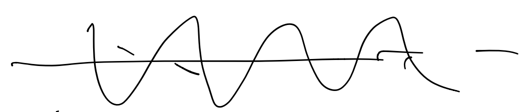
$\Longleftrightarrow \exists$ poly $p_1, \ldots, p_n$

$A = \{ x \in \mathbb{R}^n : p_1(x) = 0, \; p_2(x) = 0, \; \ldots, \; p_n(x) = 0 \}$



in $\mathbb{R}$

$\mathcal{T}_Z = \mathcal{T}_{CF}$



$\mathcal{T}_Z$ is not Hausdorff

- E^0 is open.

- E^0 is the largest open set contained in E.

- Possibly $E^0 = \emptyset$

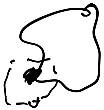
$E^\circ \subset E \subset \bar{E}$

- $\overline{E}$ is closed.

- $\overline{E}$ is the smallest closed set containing E.

- Possibly $\overline{E} = X$.

- $\boxed{E \text{ is open} \iff E = E^0}$
- $\boxed{E \text{ is closed} \iff E = \overline{E}.}$



- $x \in E^0 \iff \exists$ nbd U of x with $U \subset E$.

$X \setminus \overline{E}$ is open

- $x \in \overline{E} \iff \forall$ nbds U of x, $U \cap E \neq \emptyset$ $\cancel{x \in \overline{E}}$

- $x \in \partial E \iff \forall$ nbds U of x, $U \cap E \neq \emptyset$ and $U \cap E^c \neq \emptyset$

$\emptyset = \mathbb{Z}^{\circ} \subset \mathbb{Z} \subset \overline{\mathbb{Z}} = \mathbb{R}$

$\boxed{\mathbb{R}_{CF}}$

$\mathbb{Z} \subset \mathbb{R}$

$\overline{\mathbb{Z}} = \mathbb{R}$

$\mathbb{Z}^{\circ}$

E closed,

$E \supset \mathbb{Z}$

$E = \{ \; \mathbb{R}$

Open

$\mathbb{R} - \mathbb{Z} \subset$

$\emptyset \subset$

- $((E^c)^0)^c = ?$

- $(\overline{E^c})^c = ?$

$\mathbb{R}$

E / E'

$\bar{E} = E$

$E^0 = \phi$

Limit pt of E

$\forall$ nbs $\textcircled{x}$ of x,

$U \cap E \neq \phi$

Accumulation pt $\forall$ nbs U of x, $(U - \{x\})$

$\cap E \neq \phi$

Basis for a Topology

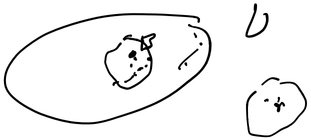
- $(X, \mathcal{T})$ topological space.

$\mathcal{B} \subset \mathcal{T} \subset 2^X$

- Definition
  $\mathcal{B} \subset 2^X$ is called a *basis for* $\mathcal{T}$ $\Longleftrightarrow$
  every element of $\mathcal{T}$ is a union of elements of $\mathcal{B}$.

- Equivalent statement:
- $\mathcal{B}$ is a basis $\iff$ $\forall U \in \mathcal{T}$ open,

  $\forall x \in U$, $\exists B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$.



Examples

- (X, d) metric space,

$$\mathcal{B} = \{B(x, r) : x \in X,\ r > 0\}$$

$$\{B(x, 1/n) : x \in X,\ n \in \mathbb{N}\}$$

$$\mathbb{R} \quad \{B(x, 1/n) : n \in \mathbb{N},\ x \in \mathbb{Q}\}$$

- (X, d) metric space, $E \subset X$ *dense subset*
- Recall : E dense $\iff \overline{E} = X$.
- $$\mathcal{B}' = \{B(x, \frac{1}{k}) : x \in X,\ k \in \mathbb{N}\}.$$

- $(\mathbb{R}^n, d_{(2)})$ (or any equivalent d)
- $$\mathcal{B} = \{B(x, \frac{1}{k}) : x \in \mathbb{Q}^n,\ k \in \mathbb{N}\}.$$

- In the last example $\mathcal{B}$ is *countable*

- $(X, \mathcal{T})$ is called *second countable* $\Longleftrightarrow$ $\mathcal{T}$ has a countable basis.