Introduction to Algebraic and Geometric Topology
Week 3

Domingo Toledo

University of Utah

Fall 2017

Lipschitz Maps

$x \neq y$

$\frac{d'(fx, fy)}{d(x,y)}$

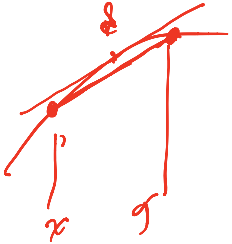
- Recall $f : (X, d) \to (X', d')$ is *Lipschitz* iff $\exists C > 0$ such that
$$\boxed{d'(f(x), f(y)) \leq C d(x, y)}$$
holds for all $x, y \in X$

bounded by C

- Lipschitz $\implies$ uniformly continuous $\implies$ continuous.
- None can be reversed.
- Keep examples in mind:
  - $f(x) = x^2$ continuous on $\mathbb{R}$, not uniformly continuous.
  - $f(x) = \sqrt{x}$ uniformly continuous on $[0, \infty)$, not Lipschitz.

Source of Lipschitz Maps

- Differentiable maps with bounded derivative are Lipschitz.
- Proof for $f : \mathbb{R} \to \mathbb{R}$:
  - Suppose $\exists C > 0$ such that $f'(x) \leq C$ for all $x \in \mathbb{R}$.
  - Mean Value Theorem: $\forall x, y \in \mathbb{R}\ \exists \xi$ between x and y s.t.
    $$f(x) - f(y) = f'(\xi)(x - y)$$
  - Then
    $$|f(x) - f(y)| = |f'(\xi)||x - y| \leq C|x - y|$$
  - Thus f is Lipschitz with Lipschitz constant C


x
y

Another Proof

- Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable and $\exists C > 0$ with $|f'(x)| \leq C \; \forall x \in \mathbb{R}$
- Fundamental Theorem of Calculus: $\forall x, y \in \mathbb{R}$

$$\boxed{f(y) - f(x)} = \int_x^y f'(t)dt$$

Σ

$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$

- Then (say, wlog, $x < y$)

$\leq C$

$$|f(y) - f(x)| = \left|\int_x^y f'(t)dt\right| \leq \int_x^y |f'(t)|dt \leq C|y - x|$$

- Again f is Lipschitz with Lipschitz constant $\boxed{C}$.
- This proof works in higher dimensions.

$f : \mathbb{R}^m \longrightarrow \mathbb{R}^n$

diff at x

$\exists$ lin map $d_x f : \mathbb{R}^m \to \mathbb{R}^n$

$x, h \in \mathbb{R}^m$ $\quad f(x+h) - f(x) = d_x f(h) + \underbrace{\varepsilon(x,h)}_{\to 0}$

$\frac{|\varepsilon(x,h)|}{|h|} \to 0$ faster than linear

$d_x f : \mathbb{R}^m \to \mathbb{R}^{(n)}$ $(f^1, \dots, f^n)$

given by a matrix

$$\begin{pmatrix} \frac{\partial f^1}{\partial x_1} & \frac{\partial f^1}{\partial x_m} \\ \vdots & \vdots & \cdots \\ \frac{\partial f^n}{\partial x_1} & \frac{\partial f^n}{\partial x_m} \end{pmatrix}$$

$m \times n$ matrix

Jacobian matrix

$f = f^1, \dots, f^n$

$f : \mathbb{R} \to \mathbb{R}^n$

$f(t) - f(s)$

$f^1(\xi_1), \dots f^n(\xi)$

$f^1(\xi_2) \quad f^n(\xi)$

$f^1(t) - f^1(s) = (f^1)'(\xi_1)(t-s)$

$f^2(t) - f^2(s) = f^1(\xi_2)(t-s)$

Higher Dimensions

- Let $f : \mathbb{R}^m \to \mathbb{R}^n$ is differentiable with bounded derivative df.
- Recall: for each $x \in \mathbb{R}^m$,

$$d_x f : \mathbb{R}^m \to \mathbb{R}^n$$

  is a linear transformation.
- $d_x f$ is the linear transformation that best approximates f near x

$|x| = 1$

$$\overset{\max}{\cancel{\sup}} \{ |Ax| : |x| = 1 \}$$

Norm of a Linear Transformation

- Let $A : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation.
- The *norm* of $|A|$ is defined as

$$|A| = \sup\{\frac{|Ax|}{|x|} \mid x \neq 0\}$$

- It has the property that

$$\boxed{|Ax| \leq |A||x| \text{ for all } x \in \mathbb{R}^m}$$

(and is the smallest number with this property)

|A|

$$\left(\text{largest eigenvalue} \left(A^t A \right) \right)^{1/2}$$

$$= |A|$$

Bounded derivative

- Let $f : \mathbb{R}^m \to \mathbb{R}^m$ be differentiable.
- Say that f has *bounded derivative* if $\exists C > 0$ so that

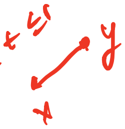
$$|d_x f| \leq C \text{ for all } x \in \mathbb{R}^m$$

- Suppose $x, y \in \mathbb{R}^m$.
- $\gamma(t) = (1 - t)x + ty, 0 \leq t \leq 1$ is the straight line segment from x to y.

$\mathbb{R}^m$

$\gamma(t) = (1-t)x + ty$

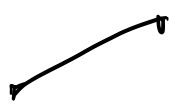
$0 \le t \le 1$



$\mathbb{R}^n$

$f(x)$

$f(y)$



$$\frac{d}{dt}\Big(f\big(\gamma(t)\big)\Big) \in \mathbb{R}^n$$

$$= \underbrace{\left(d_{\gamma(t)} f\right)}_{\text{Lin transf } \mathbb{R}^m \to \mathbb{R}^n}\left(\gamma'(t)\right)$$

vector in $\mathbb{R}^m$

- Fundamental Theorem of Calculus:

$$\boxed{f(y) - f(x) = \int_0^1 \frac{d}{dt}(f((1-t)x + ty)dt}$$

- Chain Rule

$$\frac{d}{dt}(f(\gamma(t))) = (d_{\gamma(t)} f)(\gamma'(t))$$

- For $\gamma(t) = (1-t)x + ty$,

$$\boxed{\gamma'(t) = y - x}$$

See more details below

- Putting all this together:

$$f(y)-f(x) = \int_0^1 (d_{\gamma(t)}f)(y-x)dt = \left(\int_0^1 (d_{\gamma(t)}f)dt\right)(y-x)$$

- Therefore

$$|f(y)-f(x)| = \left|\int_0^1 (d_{\gamma(t)}f)dt(y-x)\right| \leq \int_0^1 |d_{\gamma(t)}f|\, dt|y-x|$$

- which is $\leq \int_0^1 C\, dt|y-x| = C|y-x|$
- Thus

$$|f(y)-f(x)| \leq C|y-x|$$

$$f(y) - f(x) = \left(\int_0^1 (d_{\gamma(t)} f)\, dt\right)(y-x)$$

$$|f(y) - f(x)| = \left|\left(\int_0^1 (d_{\gamma(t)} f)\, dt\right)(y-x)\right|$$

$$\leq \left|\int_0^1 (d_{\gamma(t)} f)\, dt\right| \, |y-x|$$

$$\leq \left(\int_0^1 \underbrace{|d_{\gamma(t)} f|}_{\leq C}\, dt\right) |y-x|$$

$$= \underbrace{\left(\int_0^1 C\, dt\right)}_{C} |y-x|$$

$$= C|y-x|$$

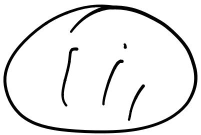
Conclusion

- Suppose
  - $f : \mathbb{R}^m \to \mathbb{R}^n$ is differentiable,
  - $\exists C > 0$ such that $|d_x f| \leq C$ for all $x \in \mathbb{R}^n$
- Then for all $x, y \in \mathbb{R}^n$,

$$|f(y) - f(x)| \leq C|y - x|$$

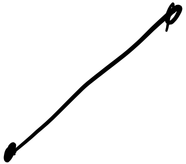
that is, f is Lipschitz with Lipschitz constant C.

Remark



- Note: f need not be defined on all of $\mathbb{R}^n$.
- Looking at the proof, all we needed was f defined on an open, convex subset of $\mathbb{R}^n$.

for diff



Close relation Lipschitz $\leftrightarrow$ differentiable

Done

- have seen that $f : \mathbb{R}^m \to \mathbb{R}^n$ differentiable, bounded derivative $\implies f$ Lipschitz.
- Also true: $f : \mathbb{R}^m \to \mathbb{R}^n$ Lipschitz $\implies f$ is differentiable "almost everywhere"
- More precisely
  - df exists a.e.
  - $|df|$ is bounded a.e.
- a. e. means outside set of measure zero.

"small"

$$a(x,y) = \begin{cases} \dfrac{f(y)-f(x)}{y-x} & x \neq y \\ f'(x) & \text{if } x = y \end{cases}$$

cont func in $\mathbb{R}\times\mathbb{R}$ (

- $f : \mathbb{R} \to \mathbb{R}$ continuously differentiable $\iff$

$$\boxed{f(y) - f(x) = a(x, y)(y - x)}$$

  for some continuous function $a(x, y)$
- In fact,

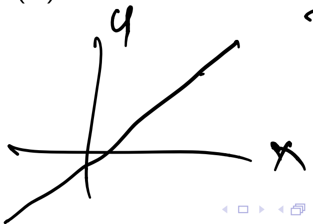
$$\boxed{a(x, y) = \frac{f(y) - f(x)}{y - x}}$$

$y \neq x$

extends

to $x = y$

- Observe $a(x, x) = f'(x)$



- In higher dimensions

$$\cancel{\frac{f(y)-f(x)}{y-x}}$$

doesn't make sense, but

$$\boxed{f(y)-f(x)=a(x,y)(y-x)}$$

makes sense if $a(x,y)$ is a matrix-valued function.

$a(x,y)$ - lin transf

$\mathbb{R}^n \to \mathbb{R}^m$

($m \times n$ matrix

- We have seen

$$\boxed{a(x,y)=\int_0^1 (d_{((1-t)x+ty)}f)\,dt}$$

works.

- observe that in higher dimensions $\boxed{a(x,y)}$ *not* unique
- Except when $x = y$:

$$\boxed{a(x,x) = d_x f}$$

- Lipschitz $\iff$

$$\frac{|f(y) - f(x)|}{|y - x|}$$

  is bounded.
- If f is differentiable, this is implied by $|a(x,y)|$ bounded.

More maps of metric spaces

- $f : (X, d) \to (X', d')$ is called *bi-Lipschitz* if there exist consants $C_1, C_2 > 0$ so that

$$C_1 d(x, y) \leq d'(f(x), f(y)) \leq C_2 d(x, y).$$

- If f is bi-Lipschitz, then
  - f is Lipschitz
  - f is injective: $f(x) = f(y) \implies d(x, y) = 0$
  - $f^{-1} : f(X) \to X$ is Lipschitz, with Lipschitz constant $\frac{1}{C_1}$.
  - If f is surjective, then $f^{-1} : X' \to X$ exists and is Lipschitz.

$$C_1\, d(x,y) \le d'(fx, fy)$$

$$fx = fy$$

$$\Rightarrow d'(fx, fy) = 0$$

$$\Rightarrow d(x,y) = 0$$

$f^{-1}u,\ f^{-1}v$ $\qquad u \quad v$

$$\boxed{d(x,y) \le \frac{d'(fx,fy)}{C_1}}$$

$\Rightarrow f$ injective

$\Rightarrow$ $f^{-1} : f(X) \longrightarrow X$

$u, v \in f(X)$

$u = fx$
$v = fy$

$f^{-1}u = x$
$f^{-1}v = y$

$d(f^{-1}u, f^{-1}v) \leq \frac{d'(u, v)}{c_1}$

- $f : (X, d) \to (X', d')$ is called an *isometry* iff

$$d'(f(x), f(y)) = d(x, y)$$

for all $x, y \in X$.

- Isometry = bi-Lipschitz with both $C_1, C_2 = 1$.
- f isometry $\implies$

$$f^{-1} : f(X) \to X$$

is also an isometry.

Equivalences between metric spaces

- Let $f : (X, d) \to (Y, d')$. Say:
  - f is a *homeomorphism* iff f is continuous, $f^{-1} : Y \to X$ exists, and f^{-1} is continuous.
  - If a homeomorphism f exists, we say that (X, d) and (Y, d') are *homeomorphic*.
  - f is a *bi-Lipschitz equivalence* iff f is surjective and bi-Lipschitz.
  - If a bi-Lipschitz equivalence exists we say that (X, d) and (Y, d') are *bi-Lipschitz equivalent*.
  - The spaces (X, d) and (Y, d') are *isometric* iff there exists a surjective isometry $f : (X, d) \to (Y, d')$.

Surjective: domain of f^{-1} is all of Y

$c_1, c_2 = 1$

f, f^{-1} both Lipschitz

- isometry $\Longrightarrow$ bi-Lipschitz $\Longrightarrow$ homeomorphism.
- None can be reversed.

invertible map

Examples

- $(\mathbb{R}^n, d_{(1)})$, $(\mathbb{R}^n, d_{(2)})$ and $(\mathbb{R}^n, d_{(\infty)})$ are all bi-Lipschitz equivalent.
- This is the content of the inequalities
  1. $d_{(2)}(x,y) \le d_{(1)}(x,y) \le \sqrt{n}\, d_{(2)}(x,y)$.
  2. $d_{(\infty)}(x,y) \le d_{(2)}(x,y) \le \sqrt{n}\, d_{(\infty)}(x,y)$.
  3. $d_{(\infty)}(x,y) \le d_{(1)}(x,y) \le n\, d_{(\infty)}(x,y)$.

$id: (\mathbb{R}^n, d_{(2)}) \to (\mathbb{R}^n, d_{(1)})$ is Lipschitz

$id: (\mathbb{R}^n, d_{(1)}) \to (\mathbb{R}^n, d_{(2)})$

1 $\quad$ n

Which make sense in $\mathbb{R}^\infty$?

$d_2(x, y)$

$\boxed{|x|_2 \le |x|_1}$ indep of n

depends on n $\to$ ~~$\boxed{|x|_1 \le \sqrt{n}\, |x|_2}$~~

$\mathbb{R}^\infty, d_{(1)} \xrightarrow{id} \mathbb{R}^\infty, d_{(2)}$

is Lipschitz

but $(\mathbb{R}^\infty, d_{(2)}) \xrightarrow{id} (\mathbb{R}^\infty, d_{(1)})$

unlikely to be Lipschitz

$|x|_1 \le C\, |x|_2$

$\left\{ \frac{|x|_1}{|x|_2} ; x \ne 0 \right\}$

bounded.

- Picture for $n = 2$

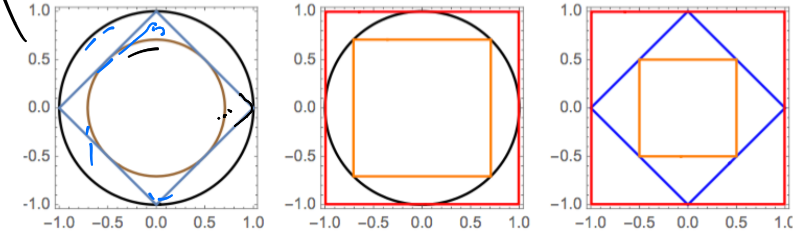


Figure: Bi-Lipschitz equivalence of the $1, 2, \infty$ metrics

- Are any two of these isometric?

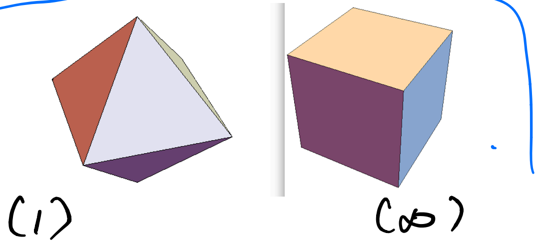
Euclidean taxicab

$(\mathbb{Z})$ & (1) not isometric

$\mathbb{R}^2, d_{(1)}$ isometric

$\mathbb{R}^2, d_{(\infty)}$

- Higher dimensions?
- For $n = 3$, look at the unit spheres of (1) and (∞) metrics



- Are these isometric?

$$(\underline{x_1, x_2, \ldots, 1}, \underline{0, 0, \ldots}, \ldots, \underline{0})$$

$$(y_1, \ldots, \overline{y_m, 0, \ldots}, \ldots)$$

$$x_1 + y_1, \ldots, x_n y_n, \ldots$$

Infinite dimensions

$\mathbb{R}^n$

- Look at the space $\boxed{\mathbb{R}^\infty}$ of sequences

$$x = (x_1, x_2, \dots)$$

of real numbers, which are eventually zero:

$$\forall x \exists N = N(x) \text{ such that } i > N \implies x_i = 0.$$

- It's important that $N = N(x)$. If it were independent of x, then we would be talking about $\mathbb{R}^N$.
- Equivalently, could think of $\mathbb{R}^\infty$ as the set of finite, but arbitrarily long, sequences of real numbers.

- The (1), (2) and ∞ metrics are still defined. It is easier to look at the norms. For a fixed $x \in \mathbb{R}^\infty$,
    - $|x|_{(1)} = \sum_{i=1}^{\infty} |x_i|$ is a finite sum.
    - $|x|_{(2)} = (\sum_{i=1}^{\infty} x_i^2)^{\frac{1}{2}}$ is a finite sum.
    - $|x|_{(\infty)} = \sup\{|x_i|\}$ is the sup of a finite set.
- Are these bi-Lipschitz equivalent?

Completeness

any X, d has a completion

$\mathbb{Q} \longrightarrow \mathbb{R}$

$\mathbb{R}^\infty, d_{C1} \longrightarrow$

not complete

Completions

- $\mathbb{R}^\infty$, with any of the three metrics, is *not* complete.
- The completions are spaces of infinite sequences

$$x = (x_1, x_2, \dots), \quad x_i \in \mathbb{R}$$

with appropriate convergence properties:

  - Completion of the (1) norm: the space ℓ^1 of infinite sequences x satisfying

$$\sum_{i=1}^{\infty} |x_i| < \infty$$

  - Completion of (2)-norm: the space ℓ^2 of all x with

$$\sum_{i=1}^{\infty} |x_i|^2 < \infty$$

  - For (∞)-norm, the space ℓ^∞

$$\sup\{|x|_i, i = 1, 2, \dots\} < \infty$$

l l_1

$$X_n = (1, \tfrac{1}{2}, \ldots, \tfrac{1}{n}, 0, \ldots 0 \text{———} 0)$$

(or the classic ∞-seq (a_i) of positive real nums $\sum_{i=1}^{\infty} a_i < \infty$

Choose $\{1/n^2\}$

$$X_n = (1, 1/2^2, \ldots 1/n^2, 0, 0 \ldots \ldots 0)$$

$$X_m = (1, 1/2^2, \ldots 1/m^2, 0 \longrightarrow \ldots)$$

$m < n$

$$X_n - X_m \ (0, \ldots 0, \underbrace{\tfrac{1}{(m+1)^2}, \tfrac{1}{(m+2)^2} \ldots 1/n^2}, 0 \ldots 0)$$

$$|X_m - X_n|_1 = \sum_{m+1}^{n} 1/i^2$$

$\forall \varepsilon > 0 \ \exists N$ s.t. $m, n > N$

$\Rightarrow \sum_{m+1}^{n} 1/i^2 < \varepsilon$

$\{X_n\}$ is Cauchy,

<u>not</u> convergent

$$X = (1, 1/2^2, \ldots 1/n^2, \ldots \ldots)$$

Completion:

$X = \{a_1, a_2, \ldots a_n, \ldots \ldots\}$ ∞ seq of $\mathbb{R}$

$$\boxed{\sum |a_i| < \infty} \quad \boxed{\ell^1}$$

Groups of isometries

- If (X, d) is a metric space, the collection of all surjective isometries $f : (X, d) \to (X, d)$ forms a *group*
- Operations: Composition, inversion.
- We will compute some of these groups.

- Group:
  - A set G
  - An element $e \in G$.
  - A map $G \times G \to G$ (group multiplication):
$$(g, h) \to gh$$
  - A map $G \to G$ (inversion):
$$g \to g^{-1}$$
- Satisfying:
  - For all $g_1, g_2, g_3 \in G$, $(g_1 g_2) g_3 = g_1 (g_2 g_3)$ Assoc.
  - For all $g \in G$, $eg = ge = g$ id.
  - For all $g \in G$, $gg^{-1} = g^{-1} g = e$ inv.

- Main example for us: groups of isometries.
- Let (X, d) be a metric space.
- Let $\boxed{\mathcal{I}(X)}$ be the set of all surjective isometries $f : (X, d) \to (X, d)$.
- If $f, g \in \mathcal{I}(X)$, let $fg = f \circ g$, the composition of f and g:
$$(f \circ g)(x) = f(g(x))$$
- If $f \in \mathcal{I}(X)$, let f^{-1} be the usual inverse function. It exists because f is both injective and surjective.
- Let e be the identity map $id : X \to X$.

f, g isom

$\Rightarrow f \circ g$ isom

$d(f(g(x)), f(g(y))) = d(x, y)$

$\underline{d(f(g(x)), f(g(y)))}$ f isom

$= d(x, y)$

$\parallel$

$d(g(x), g(y))$

$\parallel$

$d(x, y)$

$d(fx, fy) = d(x, y)$

$u \quad v \qquad f^{-1}u, f^{-1}v$

$d(u, v) = d(f^{-1}u, f^{-1}v)$

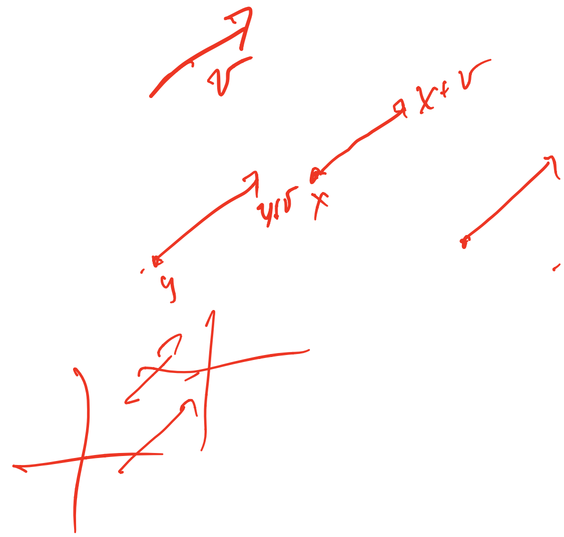
f^{-1} isom.

- Check that $\mathcal{I}(X)$ is a group:
- f, g isometries $\implies f \circ g$ isometry. ✓
- $(f \circ g) \circ h = f \circ (g \circ h)$ holds for maps of spaces.
- $f \in \mathcal{I}(X) \implies$ inverse map f^{-1} exists, and $f \circ f^{-1} = f^{-1} \circ f = id$ by definition of inverse map.

$e = id$ $id(x) = x$

- Any metric space has at least one isometry: *id*.
- In general, don't expect to have any others
- Let's look at $\mathbb{R}^n$ with any one of the $(1), (2)$, or (∞) metrics.
- There are always infinitely many isometries: Translations.
- Fix $v \in \mathbb{R}^n$. Define $t_v : \mathbb{R}^n \to \mathbb{R}^n$, translation by v, by

$$t_v(x) = x + v.$$



$d(x,y) = |x-y|$

$d(x+v, y+v) = |(x+\cancel{v}) - (y+\cancel{v})|$

$= |x-y|$

$\mathbb{R}^n, \; |\;|$

$|x| \geq 0$

$|\alpha x| = |\alpha|\,|x|$

$|x+y| \leq |x|+|y|$

$x \to -x$

$|x-(-y)| = |x-y|$

$|-x|$

$= |x|$

- $t_v(x)$ is an isometry. If $|x|$ is any norm on $\mathbb{R}^n$, the corresponding distance is

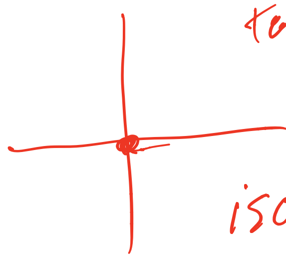
$$d(x, y) = |y - x|$$

- Therefore

$$d(t_v(x), t_v(y)) = |(y + v) - (x + v)| = |y - x| = d(x, y)$$

- Thus translations are isometries (for any distance given by a norm).

[illegible] in $\mathbb{R}$



isometry fixing $(0,0)$

$$(x,y) \to \begin{cases} (x,y) \\ (-x,y) \\ (x,-y) \\ (-x,-y) \\ (y,x) \\ (-y,x) \\ (y,-x) \\ (-y,-x) \end{cases}$$

- Are there any other isometries of R^n (in any of the norms?)
- Can you think of one other that always exists?
- Concentrate on $\mathbb{R}^2$ (just to draw pictures)
- Fix a norm on $\mathbb{R}^2$. Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is an isometry.
- Let $g = t_{-f(0)} \circ f$, so $g(x) = f(x) - f(0)$.
- Then $g(0) = 0$, that is, g fixes the origin, and

$$f = t_{f(0)} \circ g$$

$f: \mathbb{R}^2 \to \mathbb{R}^2$ isom

$0 \xrightarrow{f} f(0)$

$\boxed{t_{(-f(0))} \circ f}(x)$

$t_{(-f(0))}(f(x))$

$= f(x) - f(0) = g(x)$

$f \to g, \; g(0) = 0$

$g(0) = f(0) - f(0) = 0.$

$\mathcal{I}(\mathbb{R}^2, |-|) = \mathcal{I}$ group

$\{g \in \mathcal{I} : g(0) = 0\}$

a subgroup

$\mathcal{I}_0$

$f \to g = t_{(-f(0))} \circ f$

$t_{f(0)}\, g = \underbrace{t_{(f(0))} (t_{-f(0)}}_{= id} f)$

$\boxed{f = t_{f(0)}\, g}$ $g(0) = 0$

- Conclusion: Every isometry f of $\mathbb{R}^n$ is a composition $f = t_v \circ g$, or
$$f(x) = g(x) + v,$$
where g is an isometry *fixing the origin*.
- In practical terms:

To find all isometries of $\mathbb{R}^n$, enough to find all isometries fixing the origin

HW:

$\mathcal{I}_0$ for $\mathbb{R}^2$, taxicab

8 isometries

$\Longrightarrow$ all isometries

$\mathcal{I}$ = translation subgp $\mathcal{I}$

$\mathcal{I}_0$ = isoms fixing O.

Subgroups of $\mathcal{I}$

~~both~~

is either or

one normal?

- In the homework you'll work out the group $\mathcal{I}_0$ for the taxicab metric in $\mathbb{R}^2$. You'll see it's rather small.
- From this you'll know all isometries of $\mathbb{R}^2$ with the taxicab distance.
- Next we'll work out the group $\mathcal{I}_0$ for the usual Euclidean distance in $\mathbb{R}^2$.

- Observe that the set of all isometries fixing the origin is a *sub-group* of the group $\mathcal{I}(\mathbb{R}^n)$.
- We see two subgroups of $\mathcal{I}(\mathbb{R}^n)$:
  - $\mathcal{T}$, the subgroup of translations.
  - $\mathcal{I}_0$, the subgroup fixing the origin
- All isometries are obtained by combining these two types (in a very precise way).

- In the homework you'll work out the group $\mathcal{I}_0$ for the taxicab metric in $\mathbb{R}^2$. You'll see it's rather small.
- From this you'll know all isometries of $\mathbb{R}^2$ with the taxicab distance.
- Next we'll work out the group $\mathcal{I}_0$ for the usual Euclidean distance in $\mathbb{R}^2$.