Introduction to Algebraic and Geometric Topology

Week 2

posted every week

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Another Example of a Metric Space

- Here's a variation on example of unit sphere

$$S^2 = \{(x_1, x_2, x_2) \mid x_1^2 + x_2^2 + x_3^2 = 1\} \subset \mathbb{R}^3$$

- and of the intrinsic distance (great circle arc)

$$d_i(x, y) = \cos^{-1}(x \cdot y)$$

$= \inf\{L(\gamma) : \gamma \text{ path from } x \text{ to } y\}$

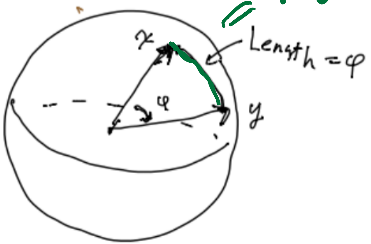

Figure: Intrinsic Distance on S^2

$\mathbb{R}^3$: Euclidean metric

$x \cdot y = x_1 y_1 + x_2 y_2 + x_3 y_3$

Minkowski Space

- ## Definition (Minkowsi Space)

  Minkowski Space is $\mathbb{R}^3$ with the *Minkowski inner product*, defined as follows: if $x, y \in \mathbb{R}^3$, then

  $$x \boxed{\diamond} y = \underbrace{x_1 y_1 + x_2 y_2}_{\text{space}} - \underbrace{x_3 y_3}_{\text{time}}.$$

  - The *Minkowski Length* of a vector $x \in \mathbb{R}^3$ is

    $$|x|_M = (x \diamond x)^{\frac{1}{2}}$$

  - Note that $|x|_M$ can be positive, zero, or imaginary, since $x \diamond x$ can be any real number.

ω_1 $\langle \omega_2, \omega_1 - f_4' \rangle$

$$|x|_M = (X \diamond X)^{1/2}$$

$$X \diamond X = \begin{cases} > 0 \\ = 0 \\ < 0 \end{cases}$$

$\langle \omega_1, \omega_2 \rangle$ $f_1 - f_2,\ f_1 - f_2$

$$|x|_M = \geq 0 \quad \mathbb{R}$$

ω_2

0

$i\mathbb{R}$

$\omega_1 - f_1$

$\omega_1 - f_1'$

$x_1^2 + x_2^2 - x_3^2 = 0$

one

$$\boxed{x_1^2 + x_2^2 - x_3^2 = 1}$$

$x_1^2 + x_2^2 = 1 + x_3^2 \geq 1$

ω_1

$x_1^2 + x_3^2 - x_3^2 = -1$

$\langle f_1 - f_2, f_1' - f_2' \rangle$

$= x_1^2 + x_2^2 + 1 = x_3^2$ $\quad |x_3| \geq 1$

≥ 1

- The level sets of the Minkowski squared norm $x \diamond x$: $x = (x_1, x_2, x_3)$

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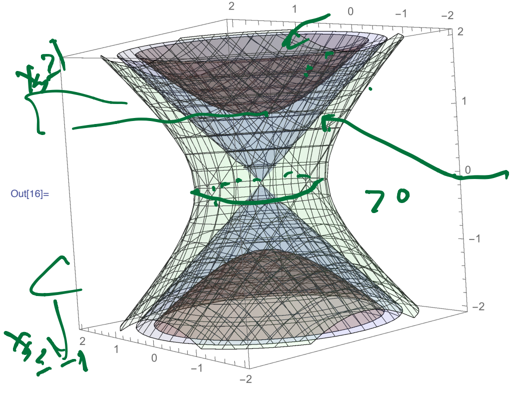


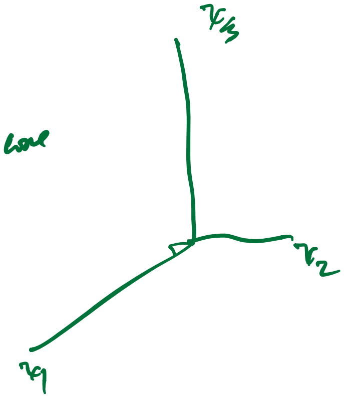


Figure: Level Sets of Minkowski Squared Norm

- $$x \diamond x = \begin{cases} = 0 & \text{if } x \in \text{ cone,} \\ -1 & \text{if } x \in \text{hyperboloid of two sheets} \\ 1 & \text{if } x \in \text{hyperboloid of one sheet} \end{cases}.$$

$x \triangle x = x_1^2 + x_2^2 - x_3^2$

$x \triangle y = x_1 y_1 + x_2 y_2 - x_3 y_3$

$x = y$

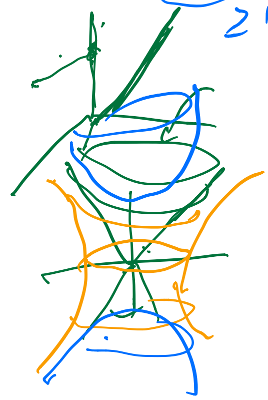
$$x_1^2 + x_2^2 - x_3^2 = \begin{cases} 1 \Leftarrow & x_1^2 + x_2^2 = 1 + x_3^2 \geq 1 \\ 0 & \\ -1 \Leftarrow & \underbrace{x_1^2 + x_2^2}_{\geq 1} + 1 = x_3^2 \end{cases}$$

$x_3^2 \geq 1$

$|x_3| \geq 1$

$x_3^2 = x_1^2 + x_2^2$

$x_3 = \pm\sqrt{x_1^2 + x_2^2}$



"light cone"

$+ \; + \; -$

$x_1^2 + x_2^2 - x_3^2$

$(0,0,1)$ < 0

$(x_1, x_2, x_3) \perp^M (0,0,1)$

$\cancel{= x_3 = 0}$

$$\boxed{\begin{matrix} (x_1, x_2, 0) & \perp & (0,0,1) \\ > 0 & & < 0 \end{matrix}}$$

if $x \triangle x < 0$

$x \triangle y = 0$

then $y \triangle y > 0$

Hyperbolic space

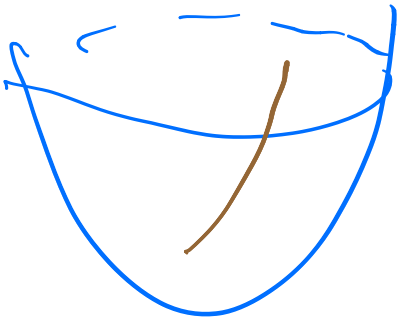
- The tangent vector $x'(t)$ to a curve $x(t)$ lying in the hyperboloid of two sheets has positive Minkowski length:
- Differentiate the equation $x(t) \diamond x(t) = -1$, get

$$\boxed{2x(t) \diamond x'(t) = 0}$$

- So $x'(t)$ is Minkowski orthogonal to $x(t)$.
- A non-zero vector Minkowski orthogonal to a negative vector is positive.

half of $x \diamond x = -1$

Tube



$X(t)$ curve in t of half.

$X(t) \triangleleft x\langle t\rangle = - \ )'$

$x'(t)$ position

$\partial\, \underbrace{x''(t)} \triangleleft x(t) = 0$

$\perp$

- Let

$$X = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 - x_3^2 = -1,\ x_3 > 0\}$$

  be the top half of the hyperboloid of one sheet.
- Can define the length of a piecewise smooth curve $\gamma : [0,1] \to X$:

  ≥ 0 real numbers

$$L(\gamma) = \int_0^1 |\gamma'(t)|_M \, dt = \int_0^1 (\gamma'(t) \diamond \gamma'(t))^{\frac{1}{2}} \, dt$$

- If $x, y \in X$ can define their *intrinsic distance*

$$\boxed{d_i(x,y) = \inf\{L(\gamma) \mid \gamma \text{ p-wise smooth curve from } x \text{ to } y\}}$$

- Turns out that

$$\boxed{d_i(x,y) = cosh^{-1}(x \diamond y)}$$

- Compare with formula for sphere:

$$d_i(x,y) = \cos^{-1}(x \cdot y)$$

- To make formula plausible, compute the length of the curve

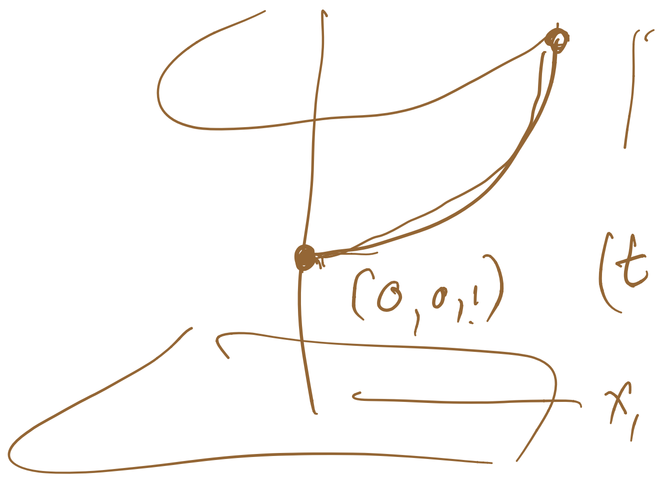
$$\gamma(t) = (t, 0, \sqrt{1+t^2}\,), \quad 0 \le t \le x_1)$$

from $(0,0,1)$ to $(x_1, 0, \sqrt{1+x_1^2}\,)$.

$\cosh(t) = \frac{e^t + e^{-t}}{2}$

$\cos(t) = \frac{e^{it} + e^{-it}}{2}$

- Compute:
$$\gamma'(t) = (1, 0, \frac{t}{\sqrt{1+t^2}}),$$
and
$$(\gamma'(t) \diamond \gamma'(t))^{\frac{1}{2}} = \frac{1}{\sqrt{1+t^2}}$$
- So
$$L(\gamma) = \int_0^{x_1} \frac{1}{\sqrt{1+t^2}} dt = \sinh^{-1}(x_1) = cosh^{-1}(\sqrt{1+x_1^2})$$
(by making the substitution $t = \sinh(u), dt = \cosh(u)$)
- The answer is the same as
$$cosh^{-1}(\sqrt{1+x_1^2}) = \cosh^{-1}(x \diamond y)$$



$(t, 0, \sqrt{1+t^2}) = \gamma(t)$

$t^2 + 0^2 - (\sqrt{1+t^2})^2$

$= t^2 - 1 - t^2 = -1$

$$L(\gamma) = \int_0^{x_1} \sqrt{1^2 + 0^2 - \left(\frac{t}{\sqrt{1+t^2}}\right)^2}$$

$\gamma'(t) = \left(1, 0, \frac{t}{\sqrt{1+t^2}}\right)$

$$= \int_0^{x_1} \sqrt{\frac{1+t^2-t^2}{1+t^2}}\, dt$$

$$= \int_0^{x_1} \frac{1}{\sqrt{1+t^2}}\, dt$$

ϕ

$\int_0^{x} \frac{dt}{\sqrt{1-t^2}}$

$= \arcsin(x)$

$\operatorname{arsinh}(x_1)$

$= \operatorname{arcosh}(\sqrt{1+x_1^2})$

[illegible]

$(0, 0, 1)$

$(x_1, 0, \sqrt{1+x_1^2})$

[illegible]

$-\sqrt{1+x_1^2}$

Hyperbolic Geometry

- The geometry of the space X just defined is called *Hyperbolic Geometry*
- Take any formula in spherical geometry involving trigonometric functions.
- Write down the same formula changing trigonometric functions to hyperbolic functions.
- For example, $\cos \to \cosh$, $\tan \to \tanh$, etc.
- Then you have the correct formula in Hyperbolic Geometry.

why: complex analysis

"analytic continuation"

Back to Metric Space Theory

- (X, d) will denote a metric space.
- Recall the concept of *convergence* ∞ # of terms
- Let $\{x_n\}$ be a sequence in (X, d).

1. Let $x \in X$. We say $\lim\{x_n\} = x$ iff for all $\epsilon > 0$ there is an $N(= N(\epsilon)) \in \mathbb{N}$ so that $d(x, x_n) < \epsilon$ for all $n > N$.
2. We say that $\{x_n\}$ *converges* iff there exists $x \in X$ so that $\lim\{x_n\} = x$.
3. We say that $\{x_n\}$ is a *Cauchy sequence* iff for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ so that $d(x_m, x_n) < \epsilon$ for all $m, n > N$.

$\{x_n\}$ sequence in X

$$\mathbb{N} \longrightarrow X$$
$$n \longmapsto x_n$$

$\forall \varepsilon > 0$ $\exists N$ s.t.

$n > N(\varepsilon)$ $\Rightarrow$ $d(x, x$

$x = \lim \{x_n\}$

$$n \geq N(\varepsilon) \implies d(x, x_n) < \varepsilon$$

$\{x_n\}$ converges $\Longrightarrow$ $\{x_n\}$ Cauchy seq.

$\varepsilon > 0 \;\Big|\; N(\varepsilon)$, $m, n \geq N(\varepsilon)$: $d(x_m, x_n) < \varepsilon$

$d(x, x_m) < \varepsilon/2$ for $m \geq N(\varepsilon)$

$d(x, x_n) < \varepsilon/2$ $n \geq N(\varepsilon)$

$d(x_m, x_n)$

$$d(x_m, x_n) \leq \underbrace{d(x_m, x)}_{<\varepsilon} + \underbrace{d(x, x_n)}_{<\varepsilon}$$

for $n \geq N(\varepsilon)$

$< \boxed{2\varepsilon}$ $\quad \varepsilon$

Thm $\lim\{x_n\} = x$, $\lim\{x_n\} = y$ $\Longrightarrow$ $x = y$

$d(x, y)$

$\leq \underbrace{d(x, x_n)}_{<\varepsilon} + \underbrace{d(x_n, y)}_{<\varepsilon}$

$\Rightarrow d(x, y) < 2\varepsilon \quad \forall \varepsilon > 0$

Completeness

conv $\Rightarrow$ Cauchy

$\Rightarrow d(x_n, y) = 0$

- (X, d) is called *complete* if every Cauchy sequence converges.
- $(\mathbb{R}, \text{usual } d)$ is complete. Known fact
- $(\mathbb{R}^n, d)$ is complete, where d is any one of $d_{(1)}, d_{(2)}, d_{(\infty)}$
- Given the first statement (completeness of $\mathbb{R}$) how would you prove the second?

$\mathbb{R}$ complete

$\Rightarrow$ $\mathbb{R}^2, d_{(1)}$ complete

Eg (X, d) not complete

(1)

Need seq $\{a_n\}$ in $\mathbb{Q}$, Cauchy, not

convergent in $\mathbb{Q}$

$\{a_n\} \to x \in \mathbb{R}/\mathbb{Q}$

- ($\mathbb{Q}$, usual d) is *not* complete.

- ~~($\mathbb{Q}$, d_p) is not complete.~~

- Let $\mathbb{R}^\infty = \{x = (x_1, x_2, \ldots, x_n, \ldots) | x_i \in \mathbb{R}$

  and $\exists N(x)$ such that $x_i = 0$ for $i > N(x)\}$.

  $(\mathbb{R}^\infty, d)$ is not complete, d any one of $d_{(1)}, d_{(2)}, d_{(\infty)}$.

in $\mathbb{R}$;

$$\sum_{n=1}^{\infty} a_n$$

$|a_i| \le c_i$

$\sum c_i$ conv

$$\sum_{i=1}^{\infty} a_i \quad \text{conv} \quad \text{iff}$$

$$\Longleftrightarrow S_n = \left\{ \sum_{i=1}^{n} a_i \right\} \text{ conv.} \qquad m < n$$

$$S_n - S_m = \sum_{i=m+1}^{n} a_i$$

$\forall \varepsilon > 0 \ \exists N$ s.t. $m, n > N \Rightarrow$

$|S_n - S_m| < \varepsilon$

$|a_i| < c_i, \geq 0$ $\quad \sum c_i < \infty \Rightarrow$ Cauchy seq

$$\left|\sum_{m+1}^{n} a_i\right| \leq \sum_{m+1}^{n} |a_i| \leq \sum_{m+1}^{n} c_i$$

$\forall \varepsilon > 0 \ \exists N$ s.t. $N \leq m < n$

$$\sum_{m+1}^{n} c_i < \varepsilon$$

Basis of Comparison the ~~th~~ Completeness

$\mathbb{R}$ complete

$\overset{?}{\Longrightarrow}$ $(\mathbb{R}^2, d_{(1)})$ (and $d_{(2)}$ and $d_{(\infty)}$)

~~$\{x_n\}$ in $\mathbb{R}^2$~~ $\{(x_n, y_n)\}$ sequence $\mathbb{R}^2$

Suppose Cauchy in $d_{(1)}$

Want: $\{x_n\}$, $\{y_n\}$ Cauchy in $\mathbb{R}$

$\forall \varepsilon > 0\ \exists N$ s.t. $m, n > N \Rightarrow$

$$\underbrace{|x_m - x_n| + |y_m - y_n|}_{d_{(1)}((x_m, y_m), (x_n, y_n))} < \varepsilon$$

$\geq |x_m - x_n|$; both Cauchy

$\geq |y_m - y_n|$ ...

$x_n \to x$

$y_n \to y$

$n > N_1$

$\forall \varepsilon > 0 \quad \exists \; N_1 \quad |x_n - x| < \varepsilon$

$\exists N_2 \quad n > N_2, \; |y_n - y| < \varepsilon$

$N = \max(N_1, N_2) \qquad n > N$

$|x_n - x| + |y_n - y| < 2\varepsilon$

$\mathbb{Q} \longrightarrow \mathbb{R} \longrightarrow$ Cauchy seqs

~~Dedekind~~

- Any metric space (X, d) has a *completion* $(\bar{X}, \bar{d})$.
- Means:
  1. $(\bar{X}, \bar{d})$ is a *complete* metric space.
  2. (X, d) is a *dense subspace* of $(\bar{X}, \bar{d})$.
- (X, d) *dense* in $(\bar{X}, \bar{d})$ means that every $\bar{x} \in \bar{X}$ is the limit of some sequence $\{x_n\}$ in X.
- Construct $(\bar{X}, \bar{d})$ as equivalence classes of Cauchy sequences in (X, d)
- Model: Construction of $\mathbb{R}$ from $\mathbb{Q}$ by Cauchy sequences.

- If (X, d) is complete, then it is its own completion.
- If $d_{\mathbb{Q}}$ is the usual distance in $\mathbb{Q}$, then $(\bar{\mathbb{Q}}, \bar{d}) = (\mathbb{R}, d_{\mathbb{R}})$, where $d_{\mathbb{R}}$ is the usual distance on $\mathbb{R}$
- Fix a prime number p.

  The completion $(\bar{\mathbb{Z}}, \bar{d}_p)$ of $(\mathbb{Z}, d_p)$ is called the ring of *p-adic integers*

  The completion $(\bar{\mathbb{Q}}, \bar{d}_p)$ of $(\mathbb{Q}, d_p)$ is called the field of *p-adic numbers*.

Equivalence Relations: three equivalent formulations

- X set, $R \subset X \times X$ equivalence relation.

- X set, partition of X into subsets E_y, $y \in Y$

- $f : X \to Y$ surjective

Refl $\quad x \sim x$

Sym $\quad x \sim y \iff y \sim x$

Trans $\quad x \sim y \ \& \ y \sim z \implies x \sim z$

$d(x,y) \geq 0, \ = 0$ iff $x = y$

ace

$- \ d(x,y) = d(y,x)$

Completion of a metric space

- Let (X, d) be a metric space.
- Let $\mathcal{C}(X)$ denote the set of all Cauchy sequences in (X, d).
- Define an equivalence relation on $\mathcal{C}(X)$ by

$$\{x_n\} \sim \{y_n\} \iff \lim\{d(x_n, y_n)\} = 0.$$

- Denote by $[\{x_n\}]$ the equivalence class of $\{x_n\}$
- Let $\bar{X} = \mathcal{C}(X)/\sim$ be the set of equivalence classes.
- Define the distance between equivalence classes by

$$\boxed{\bar{d}([\{x_n\}], [\{y_n\}]) = \lim\{d(x_n, y_n)\}}$$

$d(x_n, y_n) \to 0$

$d(y_n, z_n) \to 0$

$x_n \sim y_n \qquad y_n \sim z_n \qquad \Rightarrow \qquad x_n \sim z_n$

representatives for
$[\{x_n\}], [\{y_n\}]$

$d(x_n, z_n)$.

$\leq d(x_n, y_n) + d(y_n, z_n)$

- Identify (X, d) with a subspace of $(\bar{X}, \bar{d})$ by

$$x \in X \mapsto \text{ constant sequence } \{x, x, x, \ldots\}$$

- Check that everything is well defined.
- Check that (X, d) is dense in $(\bar{X}, \bar{d})$.
- This is a construction of the completion.

Need: $\exists \lim(d(x_n, y_n))$ Exists

& indep of rep. $x_n \sim x_n'$

$y_n \sim y_n'$

$\Rightarrow \lim d(x_n, y_n)$

$= \lim d(x_n', y_n')$

$$\boxed{|u|^2|v|^2-(u\cdot v)^2}$$

$$= (u_1v_2-u_2v_1)^2$$

$n=2$ $\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}^2$

$\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} = \pm$ area

$\pm$ area $= \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$

$$u\cdot v = |u||v|\cos\theta$$

---

any n $\quad |u|^2|v|^2-(u\cdot v)^2 = \sum_{i<j} \begin{vmatrix} u_i & u_j \\ v_i & v_j \end{vmatrix}^2$

$n=3$

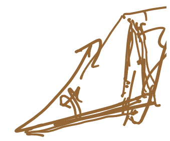
$\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$ $\quad \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}^2 + \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}^2$

$+ \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}^2$

1) 3
2) 1
3) 3
4) 5

$$\|u \times v\|^2$$

$$\|u \times v\| = \underline{|u|\,|v| \operatorname{sen}\theta}$$



$= \underline{\text{área}}$

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f ↗, — $f(s+t) \le f(s) + f(t)$

(X,d) métrica $\Rightarrow$ $(X, f \circ d)$ métrica

$$\left(X, \frac{d}{1+d}\right) \qquad f(s) = \frac{s}{1+s}$$

Check prop ≤

$f(0) = 0$ ✓

f ↗ $\quad f' = \dfrac{1}{(1+s)^2} > 0$

$$f(s+t) \le f(s) + f(t)$$

$$\frac{s+t}{1+s+t} \overset{?}{\le} \frac{s}{1+s} + \frac{t}{1+t}$$

$s, t \geq 0$

$$\frac{s}{1+s+t} + \frac{t}{1+s+t}$$

$$\leq \frac{s}{1+s} \qquad \leq \frac{t}{1+t}$$

$$\frac{s}{1+s+t} \leq \frac{s}{1+s}$$

because $1+s+t \geq 1+s$

$a < b$

$c < d$

$\Downarrow$?

$\frac{a}{c} < \frac{b}{d}$

$a < b$

$c > d$ $\to$ $\frac{a}{c} < \frac{b}{d}$

$d_{(2)}(x,y) \le d_{(\infty)}(x,y) \le \sqrt{n}\, d_{(2)}(x,y)$

$n=2$ $\quad x_i - y_i = u_i$

$\sqrt{u_1^2+u_2^2} \le |u_1|+|u_2| \le \sqrt{2}\sqrt{u_1^2+u_2^2}$

$\sqrt{u_1^2+\cdots+u_n^2} \le |u_1|+\cdots+|u_n| \le \sqrt{n}\sqrt{u_1^2+\cdots+u_n^2}$

$u_1^2+\cdots+u_n^2 \le (|u_1|+\cdots+|u_n|)^2 = (|u_1|+\cdots+|u_n|)(|u_1|\cdots|u_n|)$

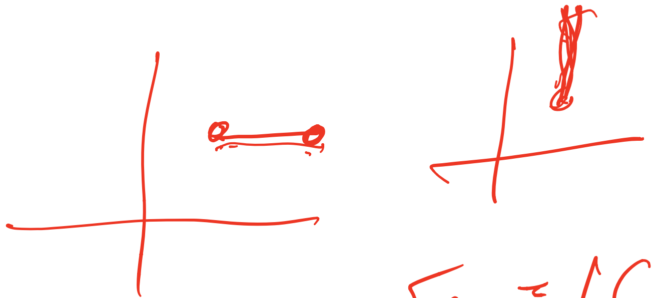
$u_1 \quad |u_1||u_2| \cdots$

$|u_2||u_1| \quad u_n^2$

diag $\le$ all

$=$ all off diag ≥ 0

$|x_1||x_2| = 0 \Rightarrow |x_1| = 0$ or $|x_2| = 0$



$\sqrt{n} = \|(1,1,\dots,1)\|_{(2)}$

$|x_1| + \dots + |x_n| = (\underbrace{1,1,\dots,1}_{n}) \cdot (|x_1|, |x_2|, \dots, |x_n|)$

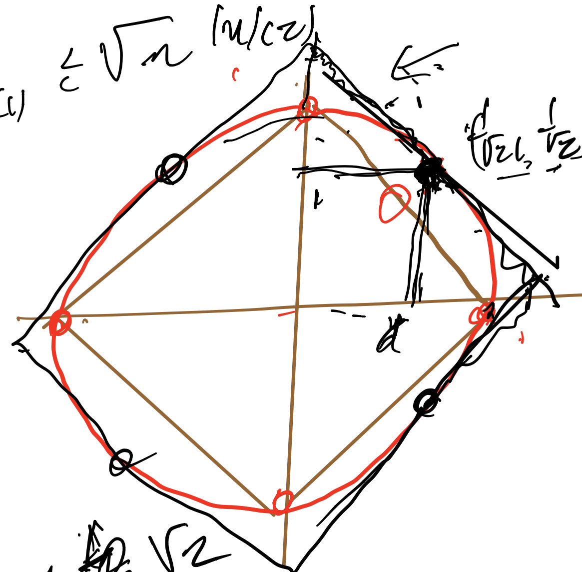
$\le \sqrt{\underbrace{1^2+\dots+1^2}_{n}}\sqrt{x_1^2+\dots+x_n^2}$

$= \iff$ all x_i are equal

$\|x\|_{(2)} \le \|x\|_{(1)}$

$\frac{\|x\|_{(2)}}{\|x\|_{(1)}} \le 1$

$\|x\|_{(1)} \le \sqrt{n}\,\|x\|_{(2)}$



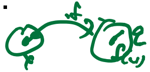
$\|x\|_{(1)} \le \sqrt2$

$\|x\|_{(2)}$

Maps between metric spaces

- Let (X, d) and (Y, d') be metric spaces.
- Let $f : X \to Y$.
  - Let $x \in X$. The map f is *continuous* at x iff for all $\epsilon > 0$ there exists a $\delta > 0$ so that for all $y \in X$, if $d(x, y) < \delta$, then $d'(f(x), f(y)) < \epsilon$.

$\exists \delta$



  - The map f is *continuous* iff it is continuous at all $x \in X$.

    Explicitly, f is continuous iff for all $x \in X$ and $\epsilon > 0$ there exists a $\delta \boxed{(= \delta(x, \epsilon))}$ so that $d'(f(x), f(y)) < \epsilon$ for all $y \in X$ with $d(x, y) < \delta$.

- The map f is *uniformly continuous* iff for all $\epsilon > 0$ there exists a $\delta(= \delta(\epsilon))$ so that $d'(f(x), f(y)) < \epsilon$ for all $x, y \in X$ with $d(x, y) < \delta$

- The map f is called *Lipschitz* iff there exists a constant $C > 0$ so that

$$d'(f(x), f(y)) \leq \boxed{C} d(x, y)$$

holds for all $x, y \in X$.

  - The constant C is called *a Lipschitz constant* for f.
  - If a smallest Lipschitz constant exists, then it is called *the* Lipschitz constant for f.

Conts not unif cont

$f(x) = x^2$ on $\mathbb{R}$

$\mathbb{R} \xrightarrow{f} \mathbb{R}$

$(x^2 - y^2) = (x+y)(x-y)$

$\boxed{|x^2 - y^2| = |x+y|\,|x-y|}$ cont.

Cont e.g. $\underset{\text{fix}}{x}$ y $|x-y| < 1$

$|x^2 - y^2|$ $\quad |x+y| < (2|x|+1)$

$|x^2 - y^2| \le (2|x|+1)\,\underbrace{|x-y|} < \varepsilon$

$|x-y| < \boxed{\dfrac{\varepsilon}{2|x|+1} = \delta}$

$|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

Cont.

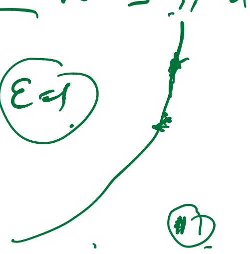
is it unif cont?

$\frac{|x^2 - y^2|}{|x-y|} = \boxed{|x+y|}$ big.

Not $\left(\forall \varepsilon > 0 \ \exists \delta \text{ s.t. } \forall x, y,\ d(x,y) < \delta \Rightarrow d(f(x), f(y)) \right)$

$\exists \varepsilon > 0 \ \forall \delta \ \exists x, y \ d(x,y) < \delta$ and $d(f(x), f(y)) > \varepsilon$

$\varepsilon = 1$



$|x^2 - y^2| = |x+y|\,|x-y| \ge 1$

$|x+y| > 1/\delta$

Ex of Lipschitz

$(X, d) \xrightarrow{f} (X', d')$ $d'(f(x), f(y)) \le C\, d(x,y)$

$\mathbb{R} \to \mathbb{R}$ $f(x) = 3x$

$|f(x) - f(y)| = 3|x-y|$

"linear distortion"

$\boxed{\text{Lipschitz} \Rightarrow \text{unif cont.}}$

$d'(f(x), f(y)) \le C\, d(x,y)$

given $\varepsilon > 0$, $\boxed{\delta = \varepsilon / C}$

$d(x,y) < \delta \Rightarrow d'(f(x), f(y)) < \varepsilon$

Lipschitz $f : (X, d) \to (X', d')$

$\boxed{d'(f(x), f(y)) \le C\, d(x,y)}$

$\exists\, C \;\; \forall\, x, y$

Lipschitz $\Rightarrow$ Unif cont

~~$\forall \varepsilon > 0$~~

Suppose $f : \cancel{(X, d)} \to (X', d')$

Lipschitz

C

given $\varepsilon > 0$ $\exists \delta$ st $\forall x, y$

$d(x, y) < \delta \Rightarrow d'(f(x), f(y)) < \varepsilon$

$$\boxed{\delta = \varepsilon / C}$$

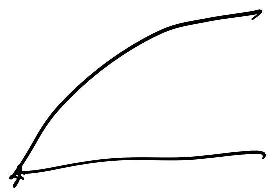
Lipschitz $\Rightarrow$ Unif Cont

$\Leftarrow$?

NO!

$f(x) = \sqrt{x} \qquad x \geq 0$

$f : [0, \infty) \to [0, \infty)$



$\sqrt{\cdot} \quad \underline{0, x}$

$|\sqrt{x} - \sqrt{0}| \leq C|x - 0|$

$\sqrt{x} \leq C|x|$?

$\exists C$ s.t.

$\frac{\sqrt{x}}{|x|} \leq C \quad \forall x \in [0, \infty)$

" $\frac{1}{\sqrt{x}} \to \infty$ as $x \to 0$

<u>**Not**</u> Lipschitz

This satisfies "Hölder Condition" $1/2$

$0 < \alpha \leq 1$

Hölder α

$f : (X, d) \to (X', d')$

is α-Hölder cont

$\exists\ C > 0$

$d'(f(x), f(y)) \leq \underline{C\, d(x,y)^{\alpha}}$

$\alpha = 1$: Lipschitz

$\alpha < 1$ (or $1/2$) Less than Lin

Lipschitz

- Source of Lipschitz maps:

  Differentiable maps with bounded derivative.

first $f : \mathbb{R} \to \mathbb{R}$

(then $\mathbb{R}^m \to \mathbb{R}^n$)

So suppose $f: \mathbb{R} \to \mathbb{R}$ diff, and $\exists\, C > 0$ s.t. $|f'(x)| \leq C \;\; \forall x \in \mathbb{R}$

(bounded derivative)

$\forall x, y \in \mathbb{R}$

$$|f(y) - f(x)| = |f'(\xi)(y - x)|$$

for some ξ between x & y

So $|f(y) - f(x)| = \underbrace{|f'(\xi)|}_{\leq C}\, |y - x|$

$$\leq C\, |y - x|$$

$\Rightarrow f$ is Lipschitz

More any bound C for $|f'(x)|$: Constant $C \geq 0$ s.t. $|f'(x)| \leq C$ $\forall x \in \mathbb{R}$ works as Lipschitz constant

In part

$$C = \inf\{C : |f'(x)| \leq C \ \forall x \in \mathbb{R}\}$$

is a Lipschitz constant for f.