Introduction to Algebraic and Geometric Topology Week 15

Domingo Toledo

University of Utah

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LastTime:

- Very briefly explained that ODE's $\Longrightarrow$ existence of geodesic polar coordinates

$$\boxed{ds^2 = dr^2 + g(r,\theta)^2 d\theta^2}$$

P 

r = dist from P

- Stated the theorem

$$\boxed{g(r,\theta) \doteq r + cr^3 + O(r^4)}$$

Const indep of θ.

- Defined *Gaussian curvature* $K(P)$ by

$$\boxed{K(P)} = -6c$$

Exs: $dr^2 + r^2 d\theta^2$ Plane

$g(r,\theta) = r$

$r = \varphi$ on S^2

Sphere $dr^2 + \sin^2 r \, d\theta^2$ $\quad \sin r = r - \frac{r^3}{6} + \dots$ $\quad K = -6\left(-\frac{1}{6}\right) = 1$

$K = 0$ for $\mathbb{R}^2$, $= 1$ for S^2

- Geometric interpretation: If C_r is the geodesic circle of radius r, then

$$L(C_r) = 2\pi r - \frac{K(P)\pi r^3}{3} + O(r^4)$$

- Today: first prove $g(r, \theta) = r + cr^3 + O(r^4)$.
- There's detailed proof in V1 of the notes, just posted.
- Review for final also posted. Bring questions next time.
- If there's time, go back to geodesic equation and ODE's

$$L(C_r) = \int_0^{2\pi} g(r,\theta)\, d\theta$$

$$= \int_0^{2\pi} \left(r - \frac{K(p)}{6} r^3 + O(r^4) \right) d\theta$$

$$= 2\pi r - \frac{2\pi K(p)}{6} r^3 + O(r^4)$$

$$= \boxed{2\pi r - \frac{\pi K(p)}{3} r^3 + O(r^4)}$$

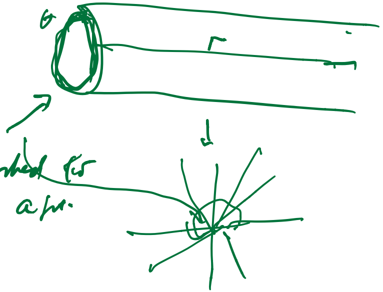
$K(p) > 0$: $L(C_r)$ smaller than in $\mathbb{R}^2$

$K(p) < 0$: — — bigger ..

$$ds^2 = dr^2 + g(r,\theta)^2 \, d\theta^2$$

$$g(r,\theta) < \cdots$$

Polar coords $\quad \underset{\substack{\uparrow \\ \text{func.} \\ \mathbb{R}}}{r}, \ \underline{\theta} \in \mathbb{R}/2\pi\mathbb{Z}$



Crushed to a pt.

Not every smooth func on $\mathbb{R} \times (\mathbb{R}/2\pi\mathbb{Z})$

comes from a smooth func on $\mathbb{R}^2$

Ex: r on $\mathbb{R}^2$ (u,v)

$r = \sqrt{u^2+v^2}$ cont, not C^1 at $(0,0)$

$$ds^2 = dr^2 + g(r,\theta)^2 d\theta^2$$

$u = r\cos\theta$

$v = r\sin\theta$

$$= g_{11}(u,v)\,du^2 + 2g_{12}(u,v)\,du\,dv + g_{22}(u,v)\,dv^2$$

where g_{11}, g_{12}, g_{22} are smooth at $(0,0)$

$$dr^2 + a r^2 d\theta^2$$

$$x = r\cos\theta \qquad dx = \cos\theta\, dr - r\sin\theta\, d\theta$$
$$y = r\sin\theta \qquad dy = \sin\theta\, dr + r\cos\theta\, d\theta$$

$$dr = \frac{x\,dx + y\,dy}{\sqrt{x^2+y^2}}$$

$$d\theta = \frac{y\,dx - x\,dy}{x^2+y^2}$$

$$dr^2 = \frac{(x\,dx + y\,dy)^2}{x^2+y^2}$$

$$d\theta^2 = \frac{(y\,dx - x\,dy)^2}{(x^2+y^2)^2}$$

$$dr^2 + a\,r^2\,d\theta^2$$

$$\frac{u^2\,du^2 + 2uv\,du\,dv + v^2\,dv^2}{u^2+v^2} + a\cdot\frac{v^2\,du^2 - 2uv\,du\,dv + u^2\,dv^2}{(u^2+v^2)^2}$$

$$= \frac{(u^2(u^2+v^2) + av^2)\,du^2 + 2uv(u^2+v^2 - 2auv)\,du\,dv + \ldots}{(u^2+v^2)^2}$$

$$\frac{u^2\,du^2 + 2uv\,du\,dv + v^2\,dv^2}{u^2+v^2} + a\,\frac{(u^2+v^2)\,v^2\,du^2 - 2uv\,du\,dv + u^2\,dv^2}{(u^2+v^2)^2}$$

$$= \frac{(u^2+av^2)\,du^2 + 2uv(1-a)\,du\,dv + (v^2+au^2)\,dv^2}{u^2+v^2}$$

$$\frac{uv}{u^2+v^2}$$

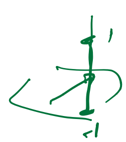
$u=0$ $\;\equiv 0$

v $\;\equiv 0$

$u=v$ $\;\frac{1}{2}$

$$\frac{r^2\cos\theta\sin\theta}{r^2} = \frac{\sin 2\theta}{2}$$

$$\frac{u^2 + a^{-1}v^2}{u^2+v^2} \qquad \frac{(u+v)(u-v)}{\frac{u^2-v^2}{u^2+v^2}}$$



homog of degree 0

f homog of deg d

$$f(tx, ty) = t^d f(x,y)$$

Ex: hom poly of deg d

$$u^d + c_1 u^{d-1} v + v^d$$

$f(u,v) \rightarrow$ homog

$$f(u,v) = \frac{uv}{u^2+v^2}$$

$$f(tu,tv) = \frac{(tu)(tv)}{(tu)^2+(tv)^2} = \frac{t^2 uv}{t^2(u^2+v^2)}$$

$$= \frac{uv}{u^2+v^2}$$

homog of deg 0

$\lim_{t \to 0} f(tu, tv)$ ~~indep~~ of (u,v) $= L$

$\lim_{(u,v)\to(0,0)} f(u,v)$ exists, $= L$

$f(u,v)$ indep of u,v constant.

f homog of deg d, smooth at $(0,0)$

$\Rightarrow$ homog polynomial of deg d.

~~Back to~~ d

$$dr^2+ar^2d\theta^2 = \frac{(u^2+av^2)\,du^2 \ldots 2(1-a)uv\,du\,dv}{u^2+v^2}$$

$a=1$

du^2+dv^2

$g(r,\theta) = r + cr^2 + O(r^3)$??

$$dr^2+g(r,\theta)^2\,d\theta^2 \longrightarrow g_{11}\,du^2+\ldots$$

$g_{11}(u,v), \ldots$ u,v smooth at 0

no

Taylor exp

$(\;)\,du + (\;)\,dv \ldots$

$\sum_{d=0} (\text{homog of deg } d)$

$$\underbrace{g(r,\theta) = c_0(\theta) + c_1(\theta) r + c_2(\theta) r^2 + \cdots}$$

$$dr^2 + g(r,\theta)^2 d\theta^2$$

$$g(r,\theta)^2 = (c_0^2) + (2c_0c_1) r + (2c_0c_2 + c_1^2) r^2$$

$$+ (2c_0c_3 + 2c_1c_2) r^3$$

$$\boxed{+ (2c_0c_4 + 2c_1c_3 + c_2^2) r^4}$$

$$0$$

$$dr^2 + \boxed{g(r,\theta)^2}\,\boxed{d\theta^2} \qquad -2$$

i wt	
$\cancel{-2}$	$C_0^2 d\theta^2$
$\cancel{-1}$	$\cancel{2C_0C_1 d\theta^2}$
0	$dr^2 + (c_0c_2 + c_1^2) r^2 d\theta^2$
1	
2	

$\therefore C_0 = 0$

$c_1^2 = 1 \qquad c_1 = 1$

$c_2 = 0$

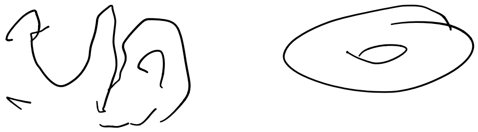
c_3

$$c_3 \underbrace{r^4 d\theta^2}$$

Smooth $c_3(\theta)\underbrace{(u\,dv - v\,du)^2}$

$\Rightarrow c_3(\theta)$ smooth homog deg 0

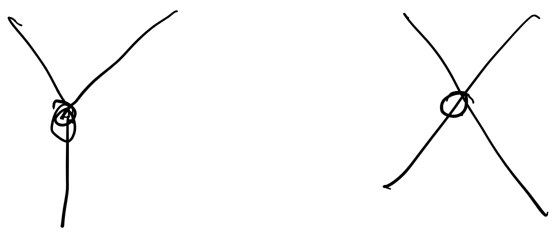
$\Rightarrow c_3(\theta) \equiv$ const indep of θ.



diff

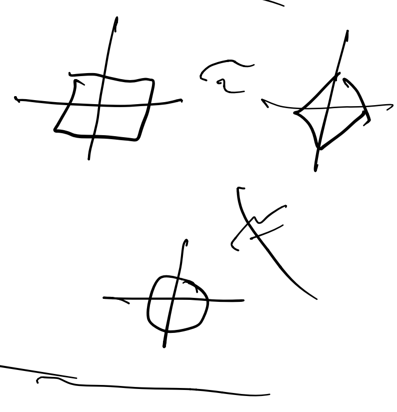
cont

not homeo



diffeo

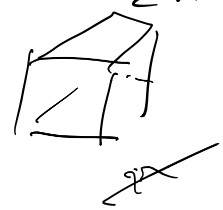
isom

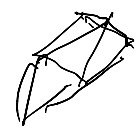


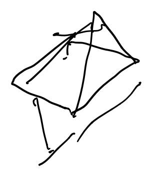
$\mathbb{R}^3$ $\quad \| \ \|_\infty$ not isom

$\qquad \| \ \|_1$

cub







$\mathbb{R}$ & $\mathbb{R}^2$ not homeo

$\mathbb{R}^2$ & $\mathbb{R}^3$ not homeo ??

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not diffeomorphic.

Thm. There is no

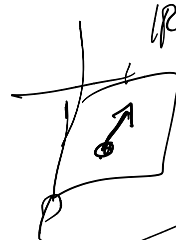
$f : \mathbb{R}^2 \to \mathbb{R}^3$

C^1, invertible, f^{-1} (C^1)

$\underline{df}$

$d_pf : \mathbb{R}^2 \to \mathbb{R}^5$

$\mathbb{R}^2$



$f(p)$

p

$\boxed{T_p\mathbb{R}^2} \xrightarrow{d_pf} \boxed{T_{f(p)}\mathbb{R}^5}$

$\xleftarrow[d(f^{-1})]{}$

$df : \mathbb{R}^2 \xrightarrow{\text{linear}} \mathbb{R}^5$

Surj cont $I \to I \times I$

space-filling curve

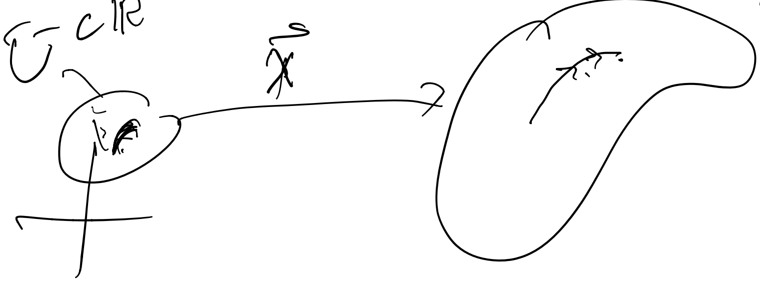
not homeo

$I \times I \qquad I \neq I \times I$

$$\boxed{ds^2 = d\vec{x}\cdot d\vec{x}}$$

$S \subset \mathbb{R}^3$

$U \subset \mathbb{R}^2$



$$\vec{x}(u,v) \qquad (u,v) \in U \subset \mathbb{R}^2$$

$$\frac{d}{dt}\vec{x}(u(t), v(t)) = \vec{x}_u u' + \vec{x}_v v'$$

$$\boxed{\begin{aligned}\frac{d\vec{x}}{dt}\cdot\frac{d\vec{x}}{dt} &= (\vec{x}_u u' + \vec{x}_v v')\cdot(x_u u' + x_v v') \\ \left|\frac{d\vec{x}}{dt}\right|^2 &= (x_u \cdot x_u)(u')^2 \\ &\quad + 2(\vec{x}_u \cdot \vec{x}_v)\, u'v' \\ &\quad + \vec{x}_v\cdot\vec{x}_v\,(v')^2\end{aligned}}$$

arc-length $s = \int \left|\frac{d\vec{x}}{dt}\right| dt$

$$ds = \left|\frac{d\vec{x}}{dt}\right| dt$$

$$ds^2 = \underbrace{\left|\frac{d\vec{x}}{dt}\right|^2}\, dt^2$$

$$\boxed{ds^2 = \vec{X}_u \cdot \vec{X}_u\, du^2 + 2\, \vec{X}_u \cdot \vec{X}_v\, du\,dv + \vec{X}_v \cdot \vec{X}_v\, dv^2}$$

mean:

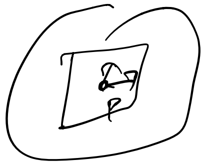
① a short-hand for

② go further:

du, dv are linear

$$ds^2 = g_{11}(u,v)\, du^2 + 2g_{12}(u,v)\, du\,dv + g_{22}(u,v)\, dv^2$$

g_{11}, g_{12}, g_{22} smooth function on U

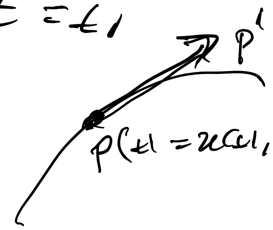


U

du : linear funcs
dv on $T_p U$

tang vectors: tang vecs
to a curve $(u(t), v(t))$ corresponds to
$(u'(t), v'(t))$ tang vec

but $t = t_1$ $p'(t) = (u'(t), v'(t))$



$p(t) = u(t), v(t)$

$du : (u', v') \to u'$
$dv : (u', v') \to v'$

$du^2, du\,dv, dv^2$ are
quadratic funcs on T_p

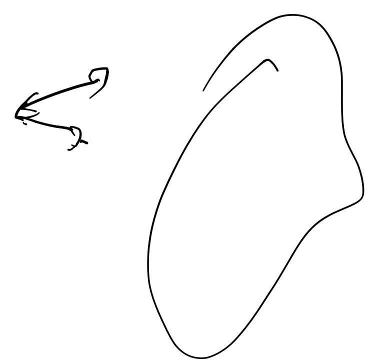
$du^2 : (u', v') \to (u')^2$
$\vdots$
ds^2 : quad func on $T_p U$

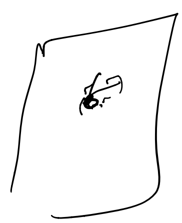
$\vec{v} \in T_p U$

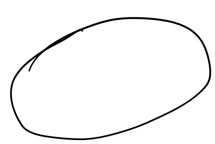
$d\vec{x}(\vec{v}) \in T_{\vec{x}(p)} S$

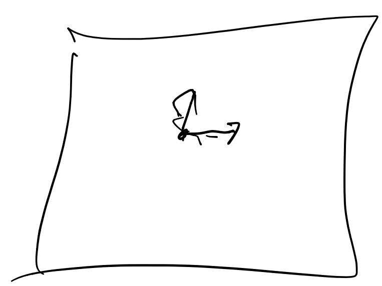
$d\vec{x}(\vec{v}) \cdot d\vec{x}(v)$

$= ds^2(\vec{v})$







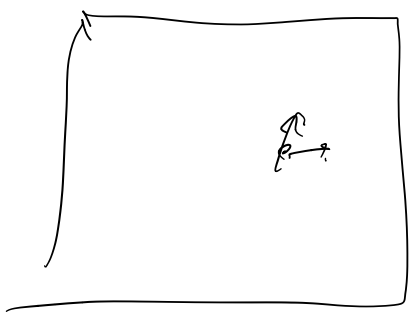


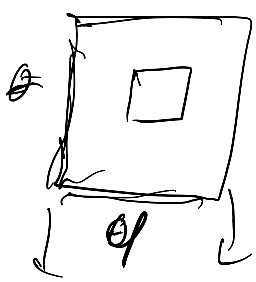
$\vec{v}, \vec{w}$

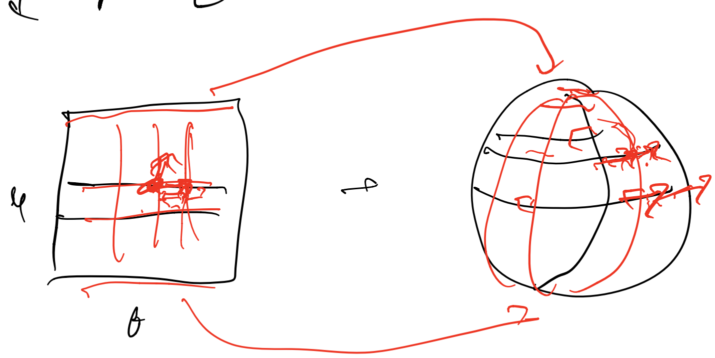
$d\vec{x}(\vec{v}) \cdot d\vec{x}(\vec{w}) = |d\vec{x}(\vec{v})||d\vec{x}(\vec{w})|$

$\cos\theta$

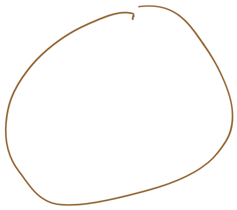
$$[w_1, w_2]\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$







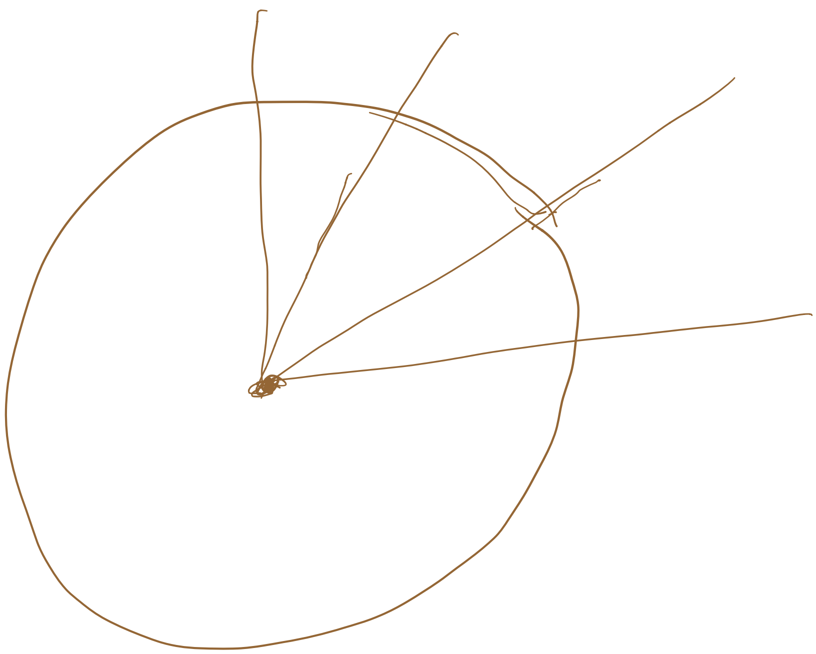
$$d\varphi^2 + \sin^2\varphi \, d\theta^2$$



$\sum g_{ij}$

$g_{11}\, du^2 + \cdots$

Ex $\boxed{dr^2 + \sinh^2 r\, d\theta^2}$



Hyperbolic Geometry

Gauss curv $K(0)$

$\sinh(r) = r +$

$\sin(r) = r - r^3/6 + \cdots$

$$\sinh r = \frac{e^r - e^{-r}}{2}$$

$$\frac{\left(\cancel{1} + r + \frac{r^2}{2} + \frac{r^3}{6} \cdots\right) - \left(\cancel{1} - r + \cancel{\frac{r^2}{2}} - \frac{r^3}{3!}\right)}{2}$$

$$\frac{2r + 2\frac{r^3}{6} + \cdots}{2}$$

$r + r^3/6 + \cdots$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

① $\sinh r = r + \frac{r^3}{6} + \cdots$

② $\sin r = r - \frac{r^3}{6} + \cdots$

$K = -6\left(\frac{1}{6}\right)$ for ① -1

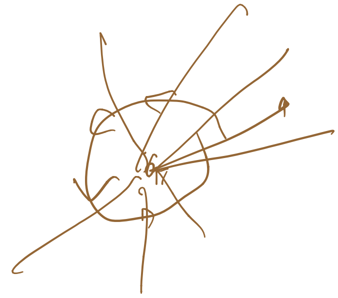
$K = -6\left(-\frac{1}{6}\right)$ for ② 1

$$L(C_r) = 2\pi r + 2\pi\left(\frac{-K(p)}{6}\right) r^3 \cdots$$

$$= 2\pi r - \frac{\pi K(p)}{3} r^3 \cdots$$

$K(p) > 0$: smaller than $\mathbb{R}^2$

$K(p) < 0$ larger than $\mathbb{R}^2$



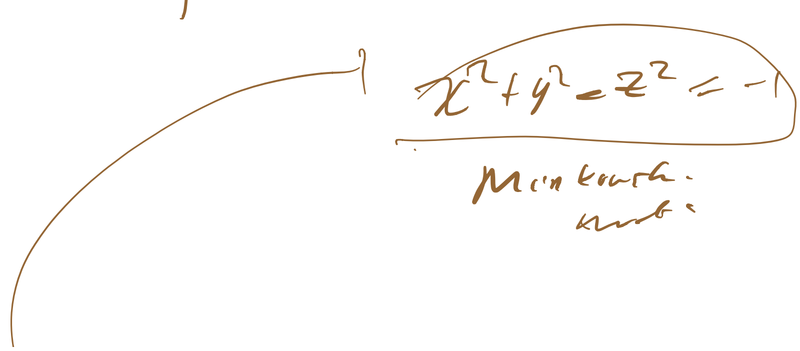
$\sinh^2 r$

Constant Gaussian curv

$>0, \quad 0, \quad <0$

$1 \quad 0 \quad -1$

Spherical Euclidean hyperbolic



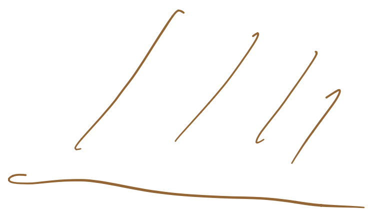
$x^2+y^2-z^2=-1$

Minkowski metric

$(\sinh(r)\cos\theta, \sinh(r)\sin\theta, \cosh$

$|z|<1$

$$\frac{|dz|^2}{(1-|z|^2)^2}$$



$$\frac{|dz|^2}{y^2}$$

Friday	10–12
Wed	10–12