Introduction to Algebraic and Geometric Topology Week 13

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Recall: Defined Smooth Surface

Examples

- Open subset of $\mathbb{R}^2$
- $\boxed{S^2 \subset \mathbb{R}^3}$
- If $U \subset \mathbb{R}^3$ open, $f : U \to \mathbb{R}$ smooth, $S = \{f = 0\}$, and

$$\forall p \in S, \nabla f(p) \neq 0$$

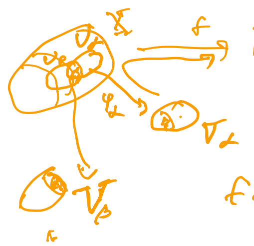
$f(p) = 0$

$\nabla_p f = 0$

no common sol's.

then S is a smooth surface.

- Some identification spaces: $\mathbb{R}^2/\mathbb{Z}^2$ (See homework 6)



$f\circ\varphi_\alpha^{-1} : V_\alpha \subset \mathbb{R}^2 \to \mathbb{R}$

diff

$f\circ\varphi_\beta^{-1} : V_\beta \subset \mathbb{R}^2 \to \mathbb{R}$

shaded $\to$ $\varphi_\alpha(U_\alpha \cap U_\beta) \subset V_\alpha$

$\searrow$

$\varphi_\beta(U_\alpha \cap U_\beta) \subset V_\beta$

no

$f\circ\varphi_\beta^{-1}$ is smooth on $\varphi_\beta(U_\alpha\cap U_\beta)$

but $f\circ\varphi_\alpha^{-1}$ not sm $\varphi_\alpha(U_\alpha\cap U_\beta)$

if pick all $\varphi_{\alpha\beta}$ smooth

: $\boxed{\varphi_\alpha\circ\varphi_\beta^{-1}} : \varphi_\beta(U_\alpha\cap U_\beta) \to \varphi_\alpha(U_\alpha\cap U_\beta)$

$\parallel$

$\varphi_{\alpha\beta}$

$\varphi_{\alpha\beta}$ smooth $\forall \alpha,\beta$

$\Rightarrow$ all diffeomorphisms

Another way

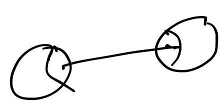
X covered by $\{U_\alpha\}$

open sets $\{V_\alpha\}$ in $\mathbb{R}^2$

how to construct X from $\{V_\alpha\}$?

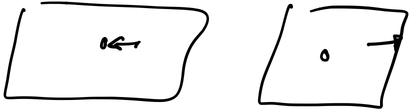
The $\varphi_{\alpha\beta}$ tell you how to glue:

$$\coprod_\alpha V_\alpha \Big/ \sim \qquad x\in V_\beta \sim \varphi_{\alpha\beta}(x)\in V_\alpha$$



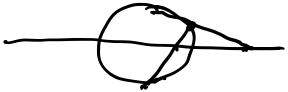
Ex S^2

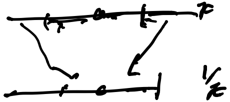
$$\frac{(\mathbb{R}^2-0)\amalg(\mathbb{R}^2-0)}{(x,y) \sim \left(\frac{x,y}{x^2+y^2}\right)}$$

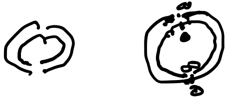


S^1 $\qquad \mathbb{R} \sqcup \mathbb{R}$

$x \sim 1/x$







$X/\sim$ points in $X/\sim$

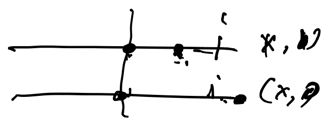
= equivalence classes of $\sim$ on X

$\mathbb{R} \sqcup \mathbb{R}$

$x \sim 1/x$

$\in \mathbb{R}-\{0\}$ $\mathbb{R}-\{0\}$

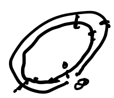
~~$x \sim y$~~ $\mathbb{C}$

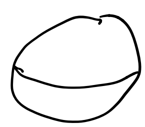


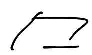
$(x,0) \sim (1/x, 1)$

$\underline{(x,0) \mid 1/x, 1)}$ $x \neq 0$

$(0,0)$ $(0,1)$







Examples of non-smooth surfaces

"Singular points" are exactly $f=0, \nabla f=0$

$f(x,y,z) = xyz$

$=0$

$x=0$

or $y=0$

or $z=0$

$\nabla f = (yz, xz, xy)$

$=(0,0,0)$

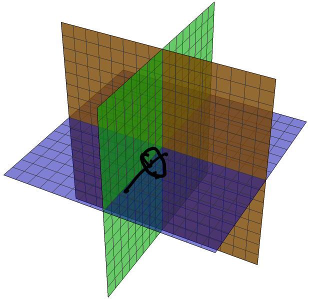
$yz=0$ $\quad y=0$ or $z=0$

$xz=0$

$xy=0$

at least

2 of $x, y, z = 0$



$z \neq 0$

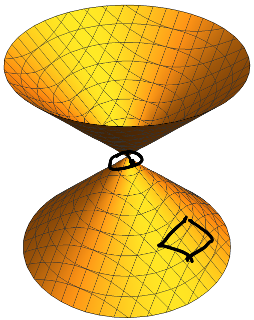
$x=0$

$y=0$

Figure: The Surface $xyz = 0$

$(x,0,0)$ or $(0,y,0)$ or $(0,0,z)$

or $(0,0,0)$



f

Figure: The Cone $\overbrace{x^2 + y^2 - z^2}^{f} = 0$

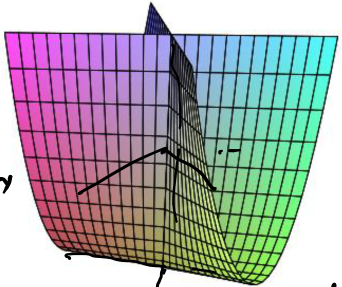
$\nabla f = (2x, 2y, -2z) = (0,0,0)$

at $(0,0,0)$

$(x+y)(x-y)=0$

$y=x$ or $y=-x$

$x^2-y^2=0$



$z\geq 0 \quad x^2 - zy^2 = (x-\sqrt{z}y)(x+\sqrt{z}y)$

$(0,0,z) \quad z\in\mathbb{R}$

Figure: The Whitney Umbrella

$f(x,y,z) = x^2 - y^2 z$ Then $\nabla f = (2x, -2yz, y^2) = (0,0,0)$

$z=0 \quad x^2-y^2=$

Smooth maps of smooth surfaces

S a smooth surface with atlas $\{U_\alpha, \phi_\alpha\}$. Then

- $f : S \to \mathbb{R}^n$ smooth $\iff$ $\forall \alpha,\ f \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha) \to \mathbb{R}^n$ is smooth.

$f : S \to \mathbb{R}$

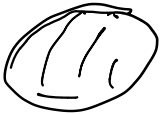
$f_1, \dots, f_n : S \to \mathbb{R}$

Example

$S = \{f(x, y, z) = 0\} \subset \mathbb{R}^3$, where $\nabla f \neq 0$ on S. Then the inclusion map $\iota : S \to \mathbb{R}^3$ is smooth.

coord fns

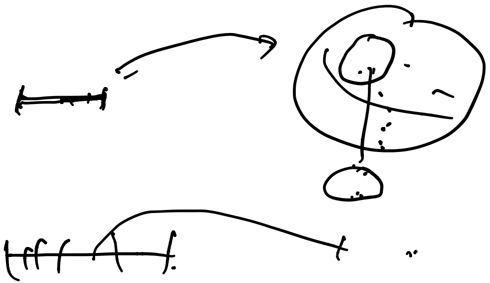
x, y, z are smooth



- $I \subset \mathbb{R}$ an interval. Then $\gamma : I \to S$ is smooth $\iff$ γ is continuous and

$$\forall \alpha, \quad \phi_\alpha \circ \gamma : (I \cap \gamma^{-1}(U_\alpha)) \to \phi_\alpha(U_\alpha)$$

is smooth.



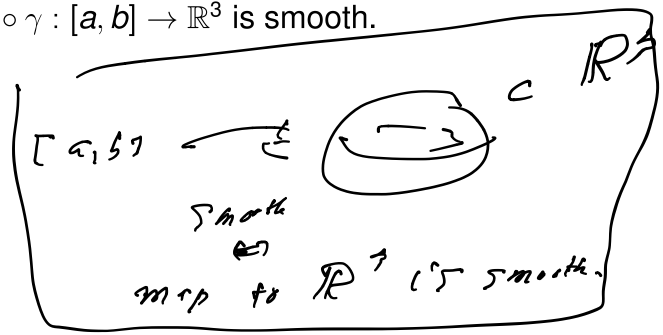
- Alternative definition: $\gamma : I \to S$ smooth $\iff$

  γ is continuous and I can be covered by intervals $\{I_\beta\}$ so that

  1. For each β there is an $\alpha(\beta)$ with $\gamma(I_\beta) \subset U_{\alpha(\beta)}$.
  2. $\forall \beta,\ \ \phi_{\alpha(\beta)} \circ \gamma : I_\beta \to \phi_{\alpha(\beta)}(U_{\alpha(\beta)})$ is smooth.

Example

Let $S = \{f(x,y,z) = 0\} \subset \mathbb{R}^3$, where $\nabla f \neq 0$ on S. Then a curve $\gamma : [a,b] \to S$ is smooth by the definition just given $\iff \iota \circ \gamma : [a,b] \to \mathbb{R}^3$ is smooth.



- S, T smooth surfaces with atlas $\{U_\alpha, \phi_\alpha\}, \{V_\beta, \psi_\beta\}$ respectively.
- $f : S \to T$ continuous.
- Refine the atlas on S as follows:
  1. Cover S by the connected components of the sets $U_\alpha \cap f^{-1}(V_\beta)$ (for $U_\alpha \cap f^{-1}(V_\beta) \neq \emptyset$)
  2. Call these sets W_γ. By definition, each $W_\gamma \subset U_{\alpha(\gamma)}$.
  3. Let $\eta_\gamma = \phi_{\alpha(\gamma)}|_{W_\gamma} : W_\gamma \to \mathbb{R}^2$.
- This atlas $\{W_\gamma, \eta_\gamma\}$ has the property that for each γ there is a $\beta(\gamma)$ so that $f(W_\gamma) \subset V_{\beta(\gamma)}$
- f is smooth $\iff$ for all γ the compositions

$$\psi_{\beta(\gamma)} \circ f \circ \eta_\gamma^{-1} : \eta_\gamma(W_\gamma) \to \psi_{\beta(\gamma)}(V_{\beta(\gamma)})$$

  are smooth.

- Simpler way: S, T smooth surfaces, $f : S \to T$ continuous.
- Assume have atlas $\{U_\alpha, \phi_\alpha\}, \{V_\beta, \psi_\beta\}$ on S, T respectively so that for each α there is a $\beta(\alpha)$ so that $f(U_\alpha) \subset V_{\beta(\alpha)}$.
- Then f is smooth $\iff$

$$\psi_{\beta(\alpha)} \circ f \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha) \to \psi_{\beta(\alpha)}(V_{\beta(\alpha)})$$

  are smooth.

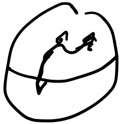
Theorem

Let S be a smooth surface.

1. *S is locally piecewise smoothly path connected.*
2. *If S is connected, then it is piecewise smoothly path connected.*

Surfaces in $\mathbb{R}^3$ as metric spaces

- $S \subset \mathbb{R}^3$ a smooth surface.
- $\gamma : [a, b] \to S$ a piecewise smooth curve.
- By composition, $\iota \circ \gamma : [a, b] \to \mathbb{R}^3$
- Write simply γ for $\iota \circ \gamma$
- The *length of* γ is by definition



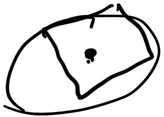
$$\boxed{L(\gamma) = \int_a^b |\gamma'(t)| dt}$$

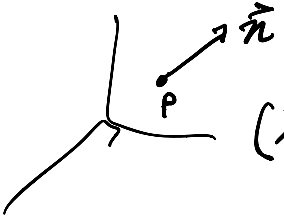
- The vector $\gamma'(t)$ is the tangent vector to γ in $\mathbb{R}^3$
- Note that γ' is actually tangent to S.
- $|\gamma'| = (\gamma' \cdot \gamma')^{1/2}$ is the length of γ' in $\mathbb{R}^3$.

$\gamma'(t) \in T_{\gamma(t)} S$

Tangent vector to $f=0$ at p.

$f(p)=0$ $(\nabla_p f \neq 0)$





$$(\vec{x}-p)\cdot\vec{n}=0$$

$$\nabla_p f \perp \boxed{T_p S}$$

$S \subset \mathbb{R}^3$

$\{f=0\}$, $\nabla f \neq 0$ where $f=0$

The tangent plane to S at p is the plane w/

$$(\vec{x}-\vec{p})\cdot \vec{\nabla}_p f = 0$$

($\perp$ to gradient f at p)

Ex: $f(x,y,z) = x^2+y^2+z^2-1$

$\nabla f = [2x_0, 2y_0, 2z_0]$

$\perp$

$x_0(x-x_0)+y_0(y-y_0)+z_0(z-z_0)=0$

$\gamma: [a,b] \to S$ piecewise C^1

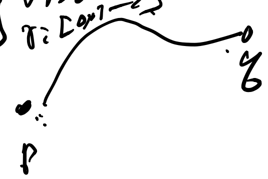
$$L(\gamma) = \int_a^b |\gamma'(t)|\,dt$$

define $d(p,q)$ $\quad p, q \in S$

$$= \inf\left\{ L(\gamma) : \begin{array}{l} \gamma:[0,1]\to S \\ \gamma(a)=p \\ \gamma(b)=q \end{array} \right\}$$

$[a,b]$

$\exists$ $\forall p,q$ $\gamma:[0,1]\to S$



Need S | path connected
piecewise smoothly

Remark pf of
Conn, loc path conn
$\Rightarrow$ path conn

---

loc p.s. path conn, conn
$\Rightarrow$ p.s. path conn

~~Open set~~ $\subset \mathbb{R}^2$
ball in $\mathbb{R}^2$ is
p.s. path conn

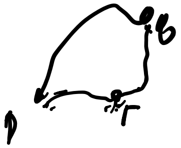
$\Rightarrow$ surfaces are loc
p.s. path conn

Summary:
if S is connected
~~then~~ $\Rightarrow$ piecewise smoothly
path conn

$\Rightarrow$ $d(p,q)$ is defined
& $(\neq \infty)$

$= d_S(p,q)$
called the
intrinsic metric

$d(p,q) \le d(p,r) + d(r,q) \quad \forall p,q,r \in S$



- Suppose S is connected.
- Then it is piecewise smoothly path connected
- Define the *intrinsic metric* d_S on S by

$$d_S(p,q) = \inf\{L(\gamma) \mid \gamma : [0,1] \to S \text{ p.s.}, \gamma(0) = p, \gamma(1) = q\}$$

$\inf\{\gamma : \text{path } p \to q\}$

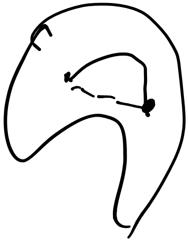
$\cup$

$\le \inf\{\gamma : p \to r \text{ concat } r \to q\}$

$\le \inf\{p \to r\} + \inf\{r \to q\}$

$d(p,q) \ge 0$ ✓

$d(p,q) = 0 \Rightarrow p = q$

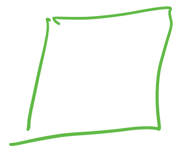


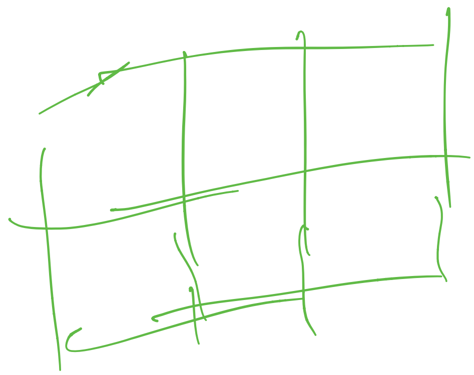
- Have checked d_S is a metric.
- $d_S(p,q) > 0$ if $p \neq q$ follows from

$$\boxed{d_S(p,q) \geq d_E(p,q)}$$

  where d_E is the Euclidean distance in $\mathbb{R}^3$.
- When is $d_S(p,q) = d_E(p,q)$?
- When can inf be replaced by min?
- When are there length minimizing curves?
- Length minimizers on S^2?

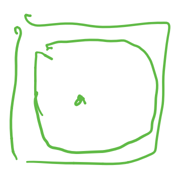
HWK → Monday

$T = $  $/\sim$

$=$ 

$[0,1] \times [0,1] /\sim$

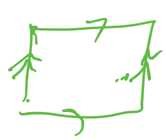
$0<x<1$
$0<y<1$
$[(x,y)] = (x,y)$



$[(x,0)] = \{(x,0),(x,1)\}$

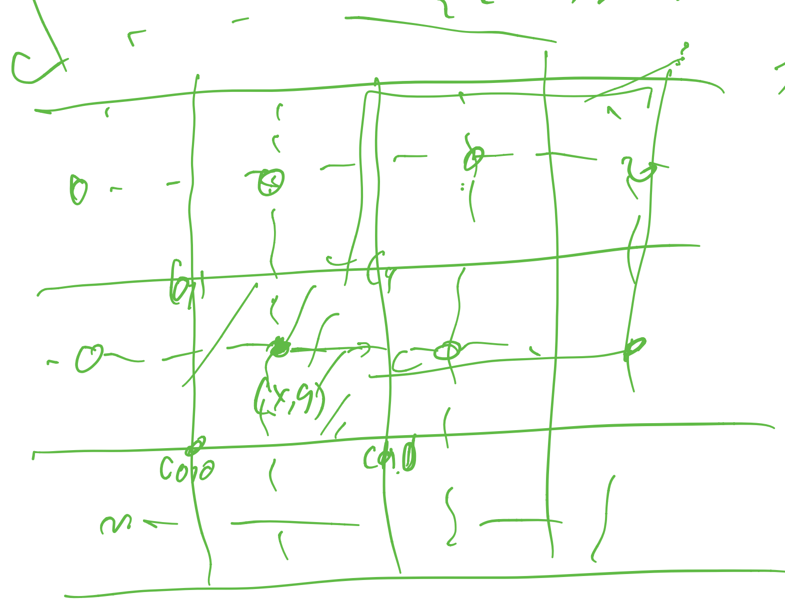
$[(0,y)] = \{(0,y),(1,y)\}$

$[(0,0)] = \{(0,0),(0,1),(1,0),(1,1)\}$



$\mathbb{R}^2/\mathbb{Z}^2$ $\quad \langle (x,y) \rangle$

$\approx$ $= \{ (x+m, y+n) : m, n \in \mathbb{Z} \}$



$x \in \mathbb{Z}$ or $y \in \mathbb{Z}$

Atlas for T^2

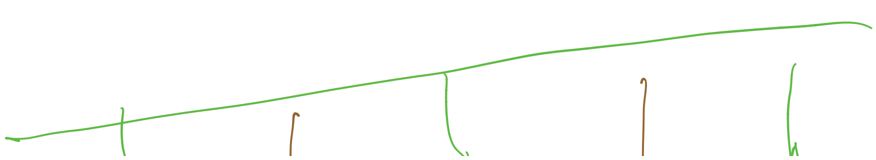
1) defined open sets in $\mathbb{R}^2$

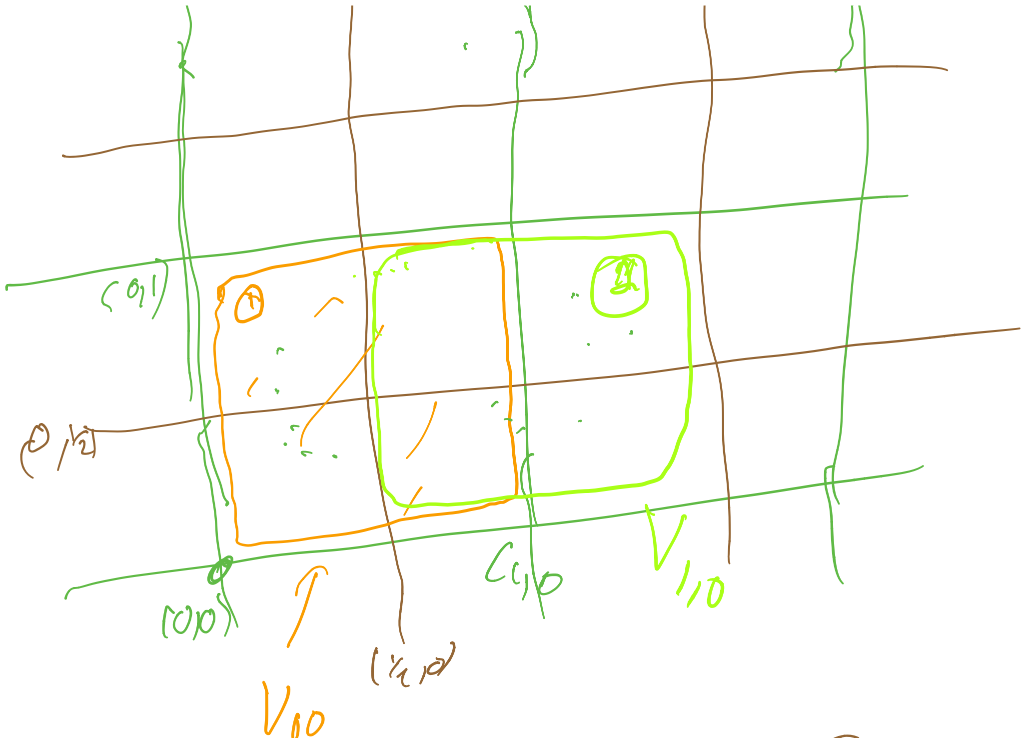
① $\varepsilon < x < 1-\varepsilon$
$\varepsilon < y < 1-\varepsilon$

② $1/2+\varepsilon < x < 3/2-\varepsilon$
$\varepsilon < y < 1-\varepsilon$

③ $\varepsilon < x < 1-\varepsilon$
$1/2-\varepsilon < y < 3/2-\varepsilon$

④ $1/2+\varepsilon < x < 3/2-\varepsilon$
$1/2+\varepsilon < y < 3/2-\varepsilon$



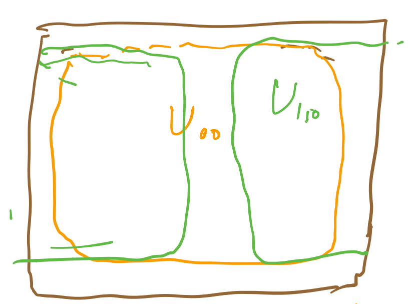


Let $U_{00} = \mathcal{G}(V_{0,0}) = \{V_{0,0} + m + n\}_{m,n}$

$U_{1,0}$ $\mathcal{G}(V_{1,0})$

$U_{00} \cap U_{1,0}$?

Picture it in $[0,1] \times [0,1]/\sim$



in mod

} Specific open sets of $\mathbb{R}^2$

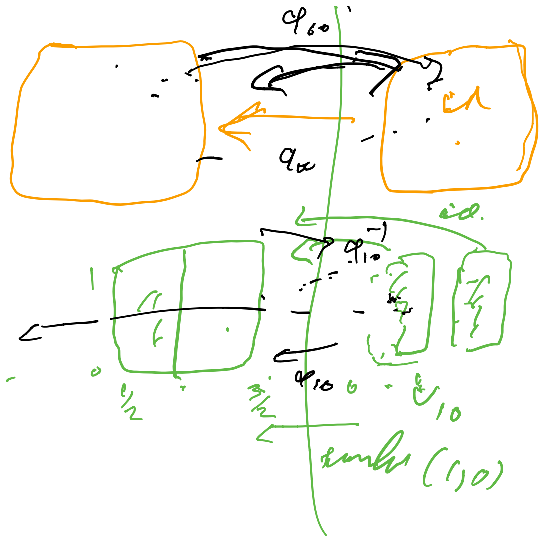
$\mathbb{R}^2$

$[0,1] \times [0,1]/\sim$

$V_{00} \simeq U_{00}$

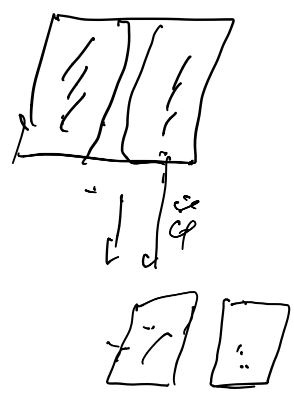
$V_{10} \simeq U_{10}$

$V_{0,0} \to$



Trans func ~~φ~~ from ~~id~~

$\varphi_{10}\varphi_{00}^{-1}$ on $\varphi_{00}(U_{00} \cap U_{10})$



$\varphi_{10}(U_{10})$

$\subset V_{1,0}$

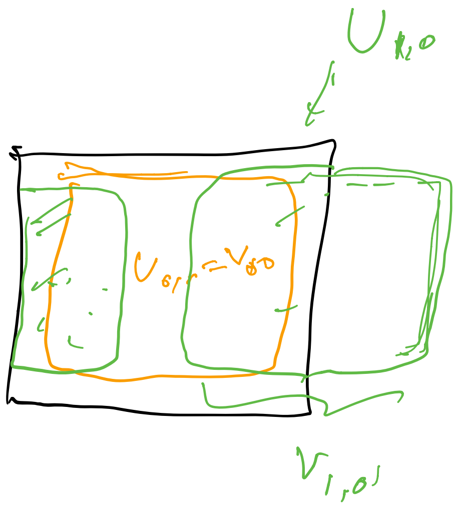
~~$\tfrac12+\varepsilon < x < \tfrac32+\varepsilon$~~

$\varphi_{10}\varphi_{00}^{-1}\Big|\begin{pmatrix} \frac12+\varepsilon < x < 1 \\ \varepsilon < y < 1-\varepsilon \end{pmatrix} = id$

$\begin{pmatrix} \varepsilon < x < \frac12-\varepsilon \\ \varepsilon < y \ldots \end{pmatrix} \ni \begin{pmatrix} (x,y) \\ (x+1,y) \end{pmatrix}$

$\downarrow \quad (x,y) \to (x+1,y)$

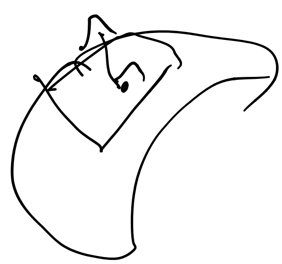
$\begin{pmatrix} 1+\varepsilon < x < \frac32-\varepsilon \\ < y \quad -- \end{pmatrix}$



Last time

$S \subset \mathbb{R}^3$ $\left(\begin{array}{l} f = 0 \\ \nabla_p f \neq 0 \text{ at all } p \in S \end{array} \right)$

$T_p S = (\nabla_p f)^{\perp}$

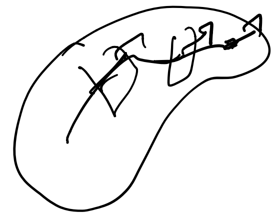


$\gamma : [0,1] \to S$ piecewise smooth

$[a,b] \to$ p.s.

$$L(\gamma) = \int_a^b |\gamma'(t)|\, dt$$

$$\gamma'(t) \in T_p S \subset \mathbb{R}^3$$



$$f(\gamma(t)) \equiv 0$$

$$\frac{d}{dt}\left(f(\gamma(t))\right) \equiv 0$$

$$\underbrace{\nabla_{\gamma(t)} f \cdot \gamma'(t) \equiv 0}$$

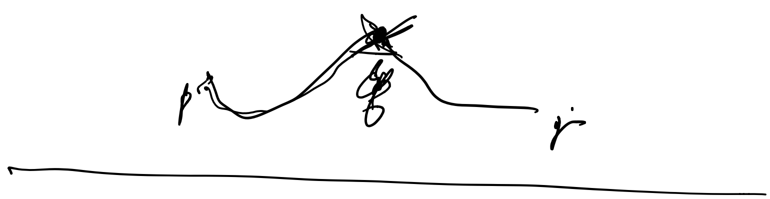
$$\iff \gamma'(t) \in T_{\gamma(t)} S$$

Assume S is connected

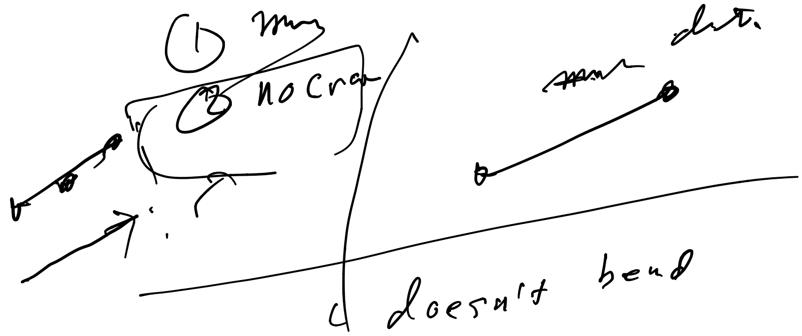
defined intrinsic dist if $p, q \in S$

$$d_S(p,q) = \inf \{ \underbrace{L(\gamma) : \gamma : [a,b] \to S}_{\neq \emptyset} \}$$

if S is
c c m.

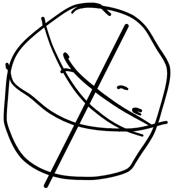


Geods $\longleftrightarrow$ straight lines

① min
② no curv

dist.

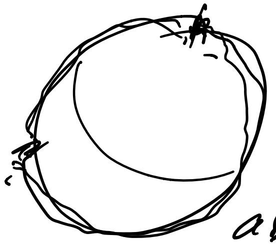
doesn't bend
curv $\equiv 0$
$\gamma'' \equiv 0$

Geodesics



- Temporary definition: length minimizing curves.
- Better: *locally* length minimizing (see S^2)
- Another issue: parametrized vs. unparametrized curves.
- The definition of length uses a parametrization:

$$L(\gamma) = \int_a^b |\gamma'(t)| dt = \int_\gamma ds$$



geod

not

abs

- But the value of $L(\gamma)$ is *independent of the parametrization*:
- This means: if $\alpha : [c, d] \to [a, b]$ is an increasing function (reparametrization),

$$\int_c^d \left|\frac{d(\gamma(\alpha(t)))}{dt}\right| dt = \int_a^b \left|\frac{d\gamma}{dt}\right| dt$$

- $\gamma \sim \gamma \circ \alpha$ is an equivalence relation on curves.
- Equivalence classes called *unparametrized curves*.

- $L(\gamma)$ depends only on the unparametrized curve γ
- Convenient to choose a distinguished representative

  called *parametrization by arc-length*
- Parameter denoted s, defined loosely as

$$s = \int_\gamma ds$$

- More precisely

$$s(t) = \int_a^t |\frac{d\gamma}{d\tau}| d\tau$$

- s is an increasing function of t, hence invertible, inverse function $t(s)$.
- Reparametrize $\gamma(t), \ a \leq t \leq b$ as

  $\gamma(t(s)), \ 0 \leq s \leq L(\gamma)$
- Call the reparametrized curve $\gamma(t(s))$ simply $\gamma(s)$.

$|\gamma'(s)| \equiv 1$

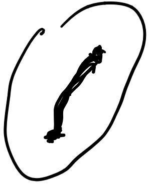
- Convention: s always means arclength.
- γ parametrized by arclength $\iff |\frac{d\gamma}{ds}| \equiv 1$.
- Convention:

$$\gamma' = \frac{d\gamma}{ds}, \quad \dot{\gamma} = \frac{d\gamma}{dt}$$

s = arclength $\iff |\gamma'(s)| = 1$

First Variation Formula for Arc-Length

- $S \subset \mathbb{R}^3$ a smooth surface (given by $f = 0, \nabla F \neq 0$)
- $\gamma : \boxed{[0, L_0]} \to S$ a smooth curve, parametrized by arclength, of length L_0
- endpoints $P = \gamma(0)$ and $Q = \gamma(1)$.
- Want necessary condition for γ to be shortest smooth curve on S from P to Q



$$\gamma : [0, L_0] \to S$$

$$\gamma(0) = P \ , \ \gamma(L_0) = Q$$

- Calculus: consider *variations of* γ,
- Meaning smooth maps

$$\tilde{\gamma} : [0, L_0] \times (-\epsilon, \epsilon) \to S \text{ with } \tilde{\gamma}(s, 0) = \gamma(s) \text{ for all } s \in [0, L_0].$$

with s being arclength on $\tilde{\gamma}(s, 0)$ but not necessarily on $\tilde{\gamma}(s, t)$ for $t \neq 0$.

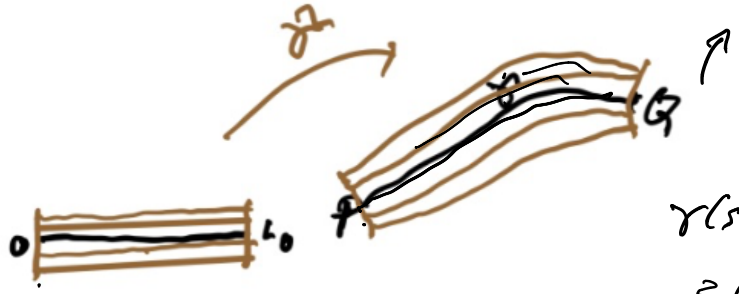


Figure: Variation with Moving Endpoints

$\gamma(s)$

$\tilde{\gamma}(s,t)$

$\gamma(s,0) = \gamma(s)$

- If, in addition, we have that

$$\tilde{\gamma}(0,t) = P,\ \tilde{\gamma}(L_0,t) = Q \text{ for all } t \in (-\epsilon,\epsilon),$$

.

we say that $\tilde{\gamma}$ is a *variation of* γ *preserving the endpoints*.

γ :

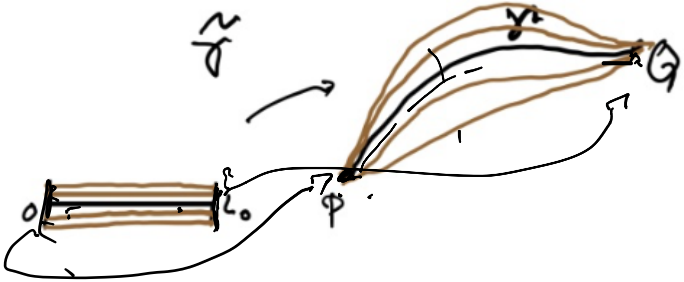


Figure: Variation with Fixed Endpoints

$$\int_0^{L_0} \left|\frac{\partial \gamma}{\partial s}(s,t)\right| ds$$

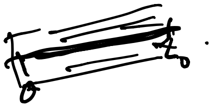
- Necessay condition for a minimum:
- Let $L(t) = \int_0^{L_0} |\frac{d\tilde{\gamma}}{ds}|\ ds$.
- Then $\frac{dL}{dt}(0) = 0$ for all variations $\tilde{\gamma}$ of γ with fixed endpoints P, Q.
- Let's compute $\frac{dL}{dt}(0)$ for arbitrary variations, then specialize to variations with fixed endpoints.

assume

$\gamma(s,0) = \gamma(s)$

s arclength

on $t=0$



- Begin with the formula for $L(t)$

$$\boxed{L(t) = \int_0^{L_0} (\tilde{\gamma}_s(s,t) \cdot \tilde{\gamma}_s(s,t))^{1/2}\, ds}$$

$$\frac{dL}{dt}\Big|_{t=0}$$

- Differentiate under the integral sign

$$\frac{dL}{dt} = \int_0^{L_0} \frac{1}{2}(\tilde{\gamma}_s(s,t) \cdot \tilde{\gamma}_s(s,t))^{-1/2}(2\,\tilde{\gamma}_{st}(s,t) \cdot \tilde{\gamma}_s(s,t))\, ds.$$

- Evaluate at $t = 0$ using that $\tilde{\gamma}_s(s,0) \cdot \tilde{\gamma}_s(s,0) = 1$

$$\boxed{\frac{dL}{dt}(0) = \int_0^{L_0} \tilde{\gamma}_{st}(s,0) \cdot \tilde{\gamma}_s(s,0)\, ds.}$$

- Integrate by parts, using equality of mixed partials and the formula

$$(\tilde{\gamma}_t(s,0)\cdot\tilde{\gamma}_s(s,0))_s = \boxed{\tilde{\gamma}_{ts}(s,0)\cdot\tilde{\gamma}_s(s,0)} + \boxed{\tilde{\gamma}_t(s,0)\cdot\tilde{\gamma}_{ss}(s,0)}$$

- Get

$$\frac{dL}{dt}(0) = \underbrace{(\tilde{\gamma}_t(s,0)\cdot\tilde{\gamma}_s(s,0))|_0^{L_0}}_{\text{End[illegible]}} - \int_0^{L_0} \underbrace{\tilde{\gamma}_t(s,0)\cdot\tilde{\gamma}_{ss}(s,0)\,ds}_{\nabla C(s)}$$

- Define a vector field $V(s)$ along γ by

$$V(s) = \tilde{\gamma}_t(s, 0).$$

- This is called the $\boxed{\textit{variation vector field.}}$
- $V(s)$ is the velocity vector of the curve $t \to \tilde{\gamma}(s, t)$ at $t = 0$.
- $V(s)$ tells us the velocity at which $\gamma(s)$ initially moves under the variation.
- If the variation preserves endpoints, then $V(0) = 0$ and $V(L_0) = 0$,

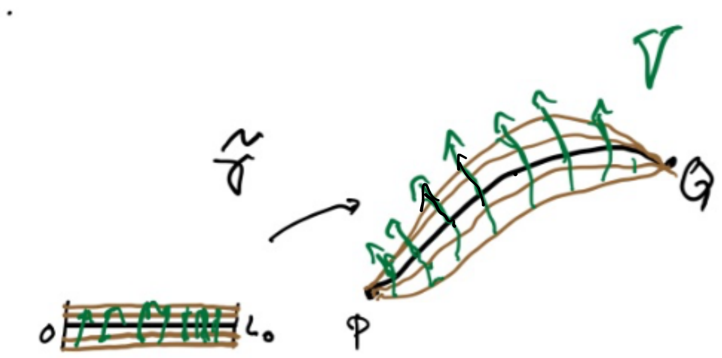


Figure: Variation Vector Field

- We can now write the final formula

$$\boxed{\frac{dL}{dt}(0) = V(s) \cdot \gamma'(s)\big|_0^{L_0} - \int_0^{L_0} V(s) \cdot \gamma''(s)^T \, ds.}$$

- Since $V(s)$ is tangent to S, we replaced $\gamma''(s)$ by its tangential component γ''^T
- Necessary condition for minimizer: $\frac{dL}{dt}(0) = 0$ for all variations $\tilde{\gamma}$of γ with fixed endpoints.
- equivalently

$$\int_0^{L_0} V(s) \cdot \gamma''^T = 0 \;\forall\; V \text{ along } \gamma \text{ with } v(0) = V(L_0) = 0$$

- Finally this means $\gamma''^T \equiv 0$.
- Reason: use "bump functions"

def

Geodesic

$$\int V(t)\cdot \gamma_{ss}^{T} = 0 \quad \forall V$$

$$\boxed{(\gamma_{ss})^{T} \equiv 0}$$

Definition

Let $\gamma : (a, b) \to S$ be a smooth curve and $V : (a, b) \to \mathbb{R}^3$ a smooth vector field along γ, meaning that V is a smooth map and for all $s \in (a, b)$, $V(s) \in T_{\gamma(s)}S$, the tangent plane to S at $\gamma(s)$.

1. The tangential component $V'(s)^T$ is called the *covariant derivative* of V and is denoted DV/Ds.
2. γ is a *geodesic* if and only if $D\gamma'/Ds = 0$ for all $s \in (a, b)$.