Introduction to Algebraic and Geometric Topology
Week 12

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Recall: Implicit Function Theorem in the Plane

Theorem

- *Let* $U \subset \mathbb{R}^2$ *be a nbd of* $(0,0)$
- *Let* $f : U \to \mathbb{R}$ *be of class* C^1.
- *Suppose*
  - $f(0,0) = 0$
  - $\frac{\partial f}{\partial y}(0,0) \neq 0$
- *Then there exist*
  - $\epsilon, \delta > 0$
  - *a continuous function* $\phi : (-\delta, \delta) \to (-\epsilon, \epsilon)$
- *so that for all* $(x,y) \in (-\delta,\delta) \times (-\epsilon,\epsilon)$

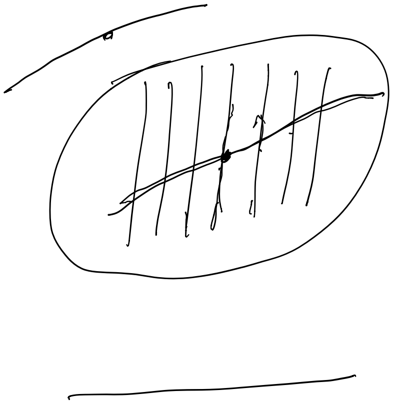
$$\{(x, \phi(x))\} = \{(x,y) \mid f(x,y) = 0\}$$

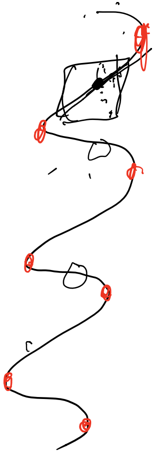
$f(a,b) = 0$

$\frac{\partial f}{\partial y}(a,b) \neq 0$

Picture

$f(x,y) = \sin y - x$





$$\frac{d}{dt}\left(f(t, \varphi(t)) = 0 \right)$$

if diff'ble φ'

$$\frac{\partial f}{\partial x}(t, \varphi(t)) + \frac{\partial f}{\partial y}(t, \varphi(t))\, \varphi'(t) = 0$$

$$\varphi'(t) = - \frac{\frac{\partial f}{\partial x}(t, \varphi(t))}{\frac{\partial f}{\partial y}(t, \varphi(t))}$$

$\neq 0$

need: $\frac{\Delta \varphi}{\Delta t} \to ?$ $\Delta t \to 0$

$\Delta \varphi \to 0$ as $\Delta t \to 0$ continuity

- Proved existence and continuity of ϕ
- Need to prove differentiability
- Start from: f of class C^1 $\Longrightarrow$

diff

$$f(x+\Delta x, y+\Delta y) - f(x,y) = \underbrace{\frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y}_{\text{linear approx}} + E$$

- The error E satisfies

$$E = E(x, y, \Delta x, \Delta y) = \epsilon_x \Delta x + \epsilon_y \Delta y$$

where $\epsilon_x, \epsilon_y \to 0$ as $\Delta x, \Delta y \to 0$

faster than linear

Evaluate at $(x, y) = (t, \phi(t))$ and $(\Delta x, \Delta y) = (\Delta t, \Delta\phi)$, where $\Delta\phi = \phi(t + \Delta t) - \phi(t))$:

$$\underbrace{f(t+\Delta t, \phi(t+\Delta t))}_{=0} - \underbrace{f(t, \phi(t))}_{=0} = \frac{\partial f}{\partial x}(t, \phi(t))\Delta t + \frac{\partial f}{\partial y}(t, \phi(t))\Delta\phi + E$$

$f(t, \varphi(t)) = 0$ by def of φ

where

$$E(t, \phi(t), \Delta t, \Delta\phi) = \epsilon_x \Delta t + \epsilon_y \Delta\phi,$$

where $\epsilon_x, \epsilon_y \to 0$ as $\Delta t \to 0$. (Note that the continuity of ϕ implies that $\Delta\phi \to 0$ as $\Delta t \to 0$)

By definition of ϕ each term on left hand side is 0, so we get

$$0 = \left(\frac{\partial f}{\partial x}(t, \phi(t)) + \epsilon_x\right)\Delta t + \left(\frac{\partial f}{\partial y}(t, \phi(t)) + \epsilon_y\right)\Delta\phi,$$

$\Delta t \to 0 \Rightarrow \Delta\phi \to 0$

Solving :

$$\boxed{\frac{\Delta\phi}{\Delta t} = -\frac{\frac{\partial f}{\partial x}(t, \phi(t)) + \epsilon_x}{\frac{\partial f}{\partial y}(t, \phi(t)) + \epsilon_y}}$$

$\to 0$ $\quad$ $\Delta t \to 0$

thus

$$\boxed{\frac{\Delta\phi}{\Delta t} \to -\frac{\frac{\partial f}{\partial x}(t, \phi(t))}{\frac{\partial f}{\partial y}(t, \phi(t))}} \text{ as } \Delta t \to 0,$$

cont

proving that ϕ is differentiable and ϕ' has the expected value.

$f \quad C^1$: cont diff $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ exist & cont

C^2

C^3 — order 3

$\vdots$

C^∞ : all derivatives exist & cont

$f \; C^k \Rightarrow \varphi$ is C^k

$f(t, \varphi(t)) = 0$

$\frac{\partial}{\partial t}$

$f_x(t, \varphi(t)) + f_y(t, \varphi(t))\,\varphi' = 0 \Rightarrow \varphi' = \cdots$

φ' $\quad$ φ' $\quad$ $\neq 0$

$\boxed{f_{xx}(t, \varphi(t))} + \boxed{f_{xy}(t, \varphi(t))\,\varphi'(t) + f_{xy}(t, \varphi(t))\,\varphi'(t)}$ φ'

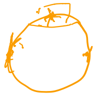
$+ \boxed{f_y(t, \varphi(t))\,\varphi''}$

$\neq 0$

$\varphi'' = -\dfrac{(\quad)}{f_y(t, \varphi(t))}$

$\varphi''' = \dfrac{\quad}{\quad} \cdots$

Example



$f(x, y) = x^2 + y^2 - 1$ $y = \sqrt{1-x^2}$

or $y = -\sqrt{1-x^2}$

with [illegible]

$(\pm 1, 0)$

$f(x, y)$ $\frac{\partial f}{\partial y}$

Make $\vec{x}$, $\vec{y}$ vectors

$\in \mathbb{R}^m$ $\in \mathbb{R}^n$

Higher Dimensions

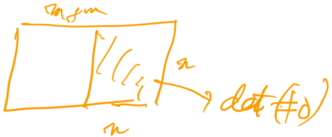
$\mathbb{R}^m \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$\vec{x}, \vec{y} \longrightarrow \vec{f}(\vec{x}, \vec{y})$

Theorem

- $U \subset \mathbb{R}^{m+n}$ *open*, $\mathbf{f} = (f_1, \dots f_n) : U \to \mathbb{R}^n$ *smooth*
- $Z = \{\mathbf{x} \in R^{m+n} \mid f(\mathbf{x}) = 0\}$, $\mathbf{x}_0 = (x_1^0, \dots, x_{m+n}^0) \in Z$
- *Suppose the matrix*

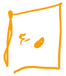
$$\left(\frac{\partial f_i}{\partial x_j}(\mathbf{x}_0)\right)_{i=1,\dots,n,\ j=1,\dots,n}$$



is invertible.

- *Then there exists a nbds* N_1 *of* $(x_1^0, \dots, x_m^0)$ *and* N_2 *of* $(x_{m+1}^0, \dots, x_{m+n})$ *and a smooth map* $\phi : N_1 \to N_2$ *such that* $N_1 \times N_2 \subset U$ *and*

$$Z \cap (N_1 \times N_2) = \{(x, \phi(x) \mid x \in N_1\}$$



in some nbhd

$\forall x_1, \dots, x_n \; \exists! \; x_{n+1}$ s.t. $f(x_1, \dots, x_n, x_{n+1}) = 0$

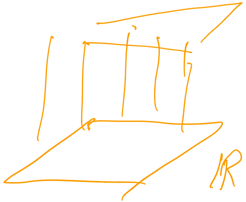
Proof.

- For $n = 1$, same argument as before.
- *Picture*

$\mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$

$f(\vec{0}, 0) = 0$

$\frac{\partial f}{\partial x_{n+1}}(\vec{0}, 0) \neq 0$



$\mathbb{R}^n$ $(n=2)$

$x_{n+1} = \varphi(x_1, \dots, x_n)$

- For $n > 1$ more subtle topological invariants are needed.

smooth: C^∞ □

(don't worry $C^1, C^2, \dots$)

Surfaces in $\mathbb{R}^3$

- We will need mostly the following version $(m = 2, n = 1)$

Theorem

- *$U \subset \mathbb{R}^3$ open, $f : U \to \mathbb{R}$ smooth, $\mathbf{x}_0 \in U$.*
- *$Z = \{\mathbf{x} \in U \mid f(\mathbf{x}) = 0\}$, $\mathbf{x}_0 \in Z$*
- *Suppose the gradient $\nabla f(\mathbf{x}_0) \neq 0$*
- *Then there is nbd N_2 in the space of one of the variables, a nbd N_1 in the remaining variables so that*
  - *$N_1 \times N_2$ is a nbd of $\mathbf{x}_0$*
  - *There is a smooth function $\phi : N_1 \to N_2$ so that*

$$Z \cap (N_1 \times N_2) = \{(x, \phi(x) \mid x \in N_1\}$$

$\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$

Proof.

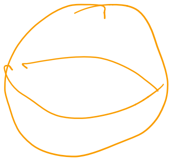
By definition, the gradient $\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$.

If $\nabla f(\mathbf{x}_0) \neq 0$, then at least one of the partial derivatives does not vanish at $\mathbf{x}_0$, say $\frac{\partial f}{\partial z}(\mathbf{x}_0) \neq 0$.

Now apply the previous version of the implicit function theorem for $m = 2, n = 1$. $\square$

Example

$f(x, y, z) = x^2 + y^2 + z^2 - 1$ with zero set the unit sphere in $\mathbb{R}^3$.



$\nabla f = (2x, 2y, 2z) \overset{?}{=} (0,0,0)$

exactly at $(0,0,0)$

but $f(0,0,0) = -1 \neq 0$

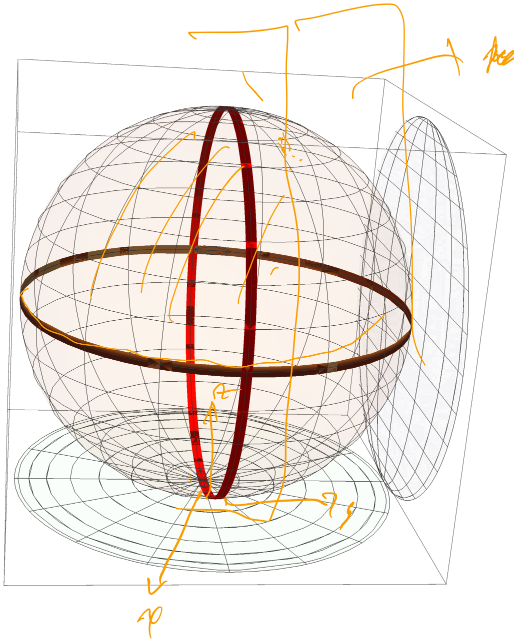
$\left\{ \begin{array}{l} f = 0 \\ \nabla f = \vec{0} \end{array} \right\}$ has no solutions

loc solve for one variable in terms of the other 2

$(f = 0)$ is locally the graph of a func upper hem

$z = \text{func } x,y$

$z = \sqrt{1-x^2-y^2}$



hemisphere $y > 0$

$y = \text{func}(x,z)$

$y = \sqrt{1-x^2-z^2}$

⋮

Topological Surface

- Hausdorff space S with a countable basis
- Every $x \in S$ has a nbd U homeomorphic to an open set in $\mathbb{R}^2$,
- So $\exists$ open cover $\{U_\alpha\}_{\alpha \in A}$ for some index set A,
- For each $\alpha \in A$ there exists a homeomorphism $\phi_\alpha : U_\alpha \to V_\alpha$, where $V_\alpha \subset \mathbb{R}^2$ is open.
- These homeomorphisms are called *coordinate charts* or simply *charts*.
- The collection of charts is called an *atlas*.

$\phi_\alpha : U_\alpha \to V$

Smooth Surfaces

- A topological surface S.
- The atlas $\{U_\alpha, \phi_\alpha\}_{\alpha \in A}$ can be chosen so that
- Whenever $U_\alpha \cap U_\beta \neq \emptyset$,

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$$

is *smooth*.

$\forall \alpha, \beta$

$\varphi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{R}^2$

$\varphi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{R}^2$

$\varphi_\alpha \varphi_\beta^{-1}$

$(\varphi_\alpha \circ \varphi_\beta^{-1})^{-1}$

$\varphi_\beta \circ \varphi_\alpha^{-1}$

Smooth,

Smooth

C^∞

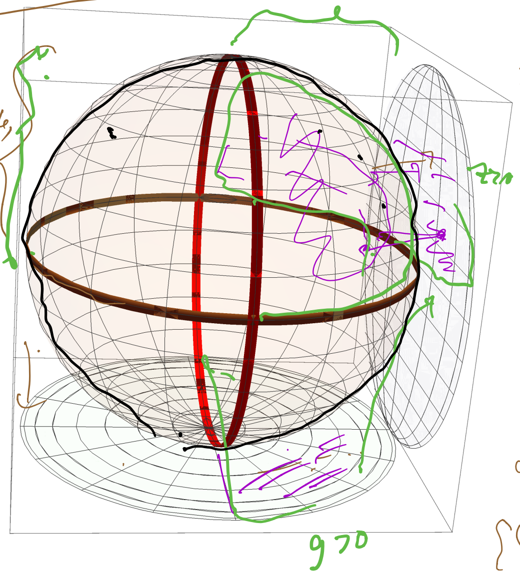
Again, look at S^2:

$\implies \varphi_\alpha \circ \varphi_\beta^{-1}$ are all invertible, smooth inverse.

diffeomorphism

C^∞, invertible, inverse C^∞.

U_z^+



$U_x^\pm, U_y^\pm, U_z^\pm$ cover S^2

$x>0$, $x<0$ $\quad$ $y>0$, $y<0$ $\quad$ $z>0$, $z<0$

Atlas with 6 charts.

$\varphi_x^\pm$ proj on (y,z) plane

$\varphi_y^\pm$ ---- xz-plane.

$\varphi_z^\pm$ ---- xy plane

$U_z^+ : \{(x,y,z) : x^2+y^2+z^2=1$ & $z>0\}$

$\varphi_z^+ \downarrow$

$\{(x,y) : x^2+y^2<1\}$

$U_y^+ : \{x^2+y^2+z^2=1,\ y>0\}$
$\downarrow$
$\{x^2+z^2<1\}$

$U_z^+ \cap U_y^+ : z>0, y>0$

$\varphi_z^+ \downarrow \qquad \searrow \varphi_y^+$

$\{x^2+y^2<1,\ y>0\} \qquad \{x^2+z^2<1,\ z>0\}$

~~$\{\xi,\eta<1,\ \eta>0\}$~~

why?

$\boxed{z \to \sqrt{1-x^2-z^2}}$

$(\varphi_z^+ \circ (\varphi_y^+)^{-1})$

$\boxed{(x, z)}\ z>0$
$\downarrow (\varphi_y^+)^{-1}$
$(x, \sqrt{1-x^2-z^2}, z)$
$\downarrow \varphi_z^+$
$(x, \sqrt{1-x^2-z^2})$

$\boxed{(x, z) \to (x, \sqrt{1-x^2-z^2})}$

$(z>0) \qquad C^\infty,\ C^\infty$ [illegible]

Cont funcs

$\uparrow$

Topological space

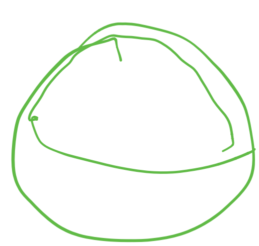
Diff funcs

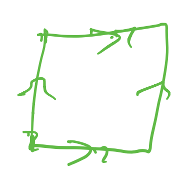
Need Vector space

$$\frac{f(x+\Delta x) = f(x)}{\Delta x}$$

$$f(x+\Delta x, y+\Delta y) - f(x,y)$$

$$= \underbrace{\frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y}_{\text{linear function}}$$





$$\varphi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^2$$

$f : X \to \mathbb{R}$ diff $\Leftrightarrow$ $f \circ \varphi_\alpha^{-1} : V_\alpha \to \mathbb{R}$ is diff $\forall \alpha$. ($V_\alpha \subset \mathbb{R}^2$)

Smooth functions

Definition

- S smooth surface, atlas $\{U_\alpha, \phi_\alpha\}$,
- $f : S \to \mathbb{R}$ a function.
- f is *smooth* if and only if, for all α

$$f \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha) \to \mathbb{R}$$

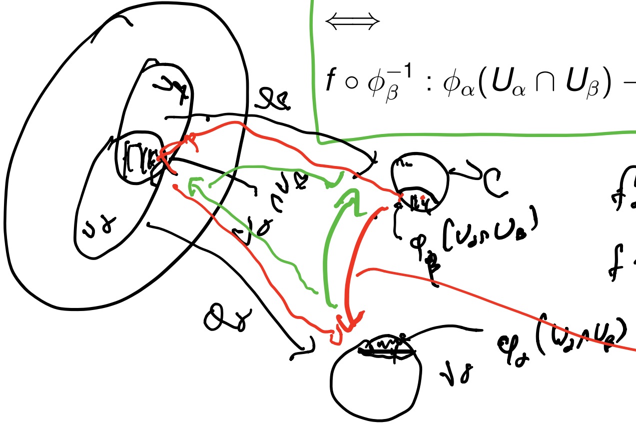
is a smooth function.

- Need to know: whenever $U_\alpha \cap U_\beta \neq \emptyset$, then

$$f \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \mathbb{R} \text{ is smooth}$$

$$\iff$$

$$f \circ \phi_\beta^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \mathbb{R} \text{ is smooth.}$$



$$\left.\begin{array}{l} f \circ \varphi_\alpha^{-1} \\ f \circ \varphi_\beta^{-1} \end{array}\right\} \text{smooth}$$

$\varphi_\beta \circ \varphi_\alpha^{-1}$

$\varphi_\alpha \circ \varphi_\beta^{-1}$

$$\varphi_\alpha \varphi_\beta^{-1} = \varphi_{\alpha\beta}$$

inverse of each other

$$\varphi_\beta \varphi_\alpha^{-1} = \varphi_{\beta\alpha}$$

$$\underbrace{(\varphi_\alpha \underbrace{\varphi_\beta^{-1})(\varphi_\beta}\varphi_\alpha^{-1})}_{\text{id}}$$

$$\boxed{\varphi_{\alpha\beta} \overset{\text{def}}{=} \varphi_\alpha \circ \varphi_\beta^{-1}}$$

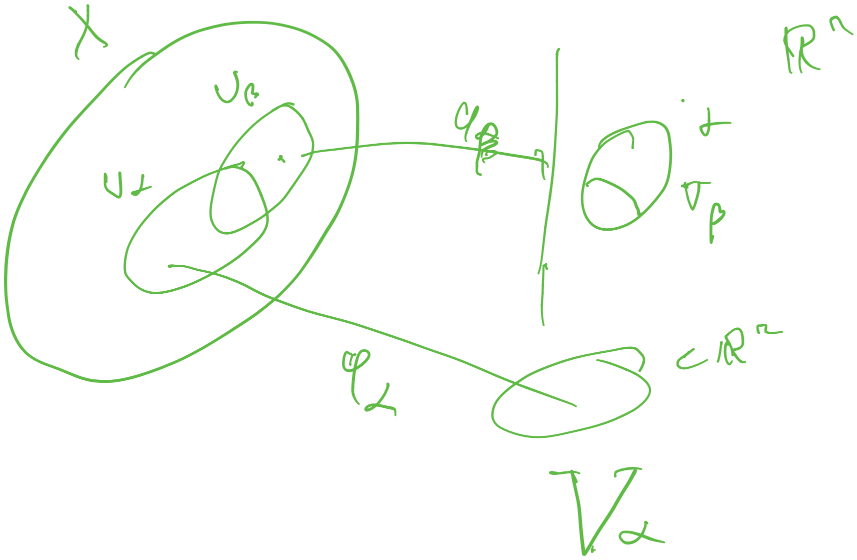
called the transition functions

An atlas for a smooth surface has to have all transition functions smooth

$\varphi_{\alpha\beta}$, $\varphi_{\beta\alpha}$ both smooth

: $\varphi_{\alpha\beta}$ are diffeomorphisms

smooth, invertible, smooth inverse.



$\varphi_\alpha \circ \varphi_\beta^{-1}$: open set in $\mathbb{R}^n$

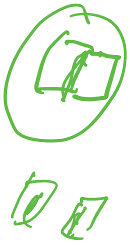
$\to$ open set in $\mathbb{R}^n$

$$\begin{array}{c} \varphi_\beta(U_\alpha \cap U_\beta) \\ \downarrow \varphi_\alpha \circ \varphi_\beta^{-1} \\ \varphi_\alpha(U_\alpha \cap U_\beta) \end{array}$$

- This follows from smoothnes of all $\phi_\alpha \circ \phi_\beta^{-1}$
- $$\boxed{f \circ \phi_\alpha^{-1}} = f \circ (\phi_\beta^{-1} \circ \phi_\beta \circ \phi_\alpha^{-1}) = (f \circ \phi_\beta^{-1}) \circ (\phi_\beta \circ \phi_\alpha^{-1})$$
- By assumption
$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$
is a smooth bijection with smooth inverse
- Therefore $f \circ \phi_\alpha^{-1}$ is smooth on $\phi_\alpha(U_\alpha \cap U_\beta)$
$$\iff$$
$f \circ \phi_\beta^{-1}$ is smooth on $\phi_\beta(U_\alpha \cap U_\beta)$.

Notion of smooth func

is well-defined



Terminology $\varphi_{\alpha\beta} \overset{\text{def}}{=} \varphi_\alpha \circ \varphi_\beta^{-1}$

are called the

Transition Functions

Examples of Smooth Surfaces

- Open subset of $\mathbb{R}^2$

  Atlas one chart: $id : U \to U$

- $S^2 \subset \mathbb{R}^3$

- If $U \subset \mathbb{R}^3$ open, $f : U \to \mathbb{R}$ smooth, $S = \{f = 0\}$, and

$$\forall p \in S, \ \nabla f(p) \neq 0$$

  then S is a smooth surface.

$f = 0$
$\nabla f = 0$ } no common sol

$\Rightarrow f = 0$ defines a smooth surf

Ex: $x^2+y^2+z^2-1 = f(\vec{x})$

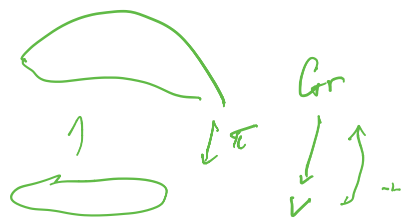
(implicit func thm)

graphs of funs

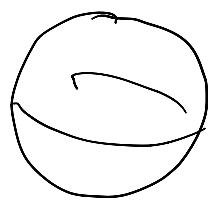
$U \cong Gr_\varphi \to V \subset \mathbb{R}^2$ proj

$\leftarrow$

$\{(x, \varphi(x)) : x \in V\}$



Exs in HW 6



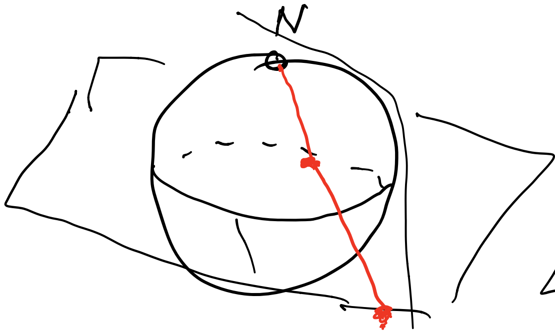
$\nexists$ Atlas with one chart
means X homeo to open set $\subset \mathbb{R}^2$

S^2 <u>not</u> homeo to open set $\subset \mathbb{R}^2$

[(visually "obvious") not easy to prove.]

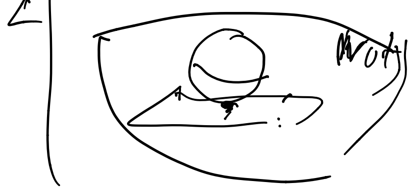
Looks need at least two charts

Can you do it with 2?



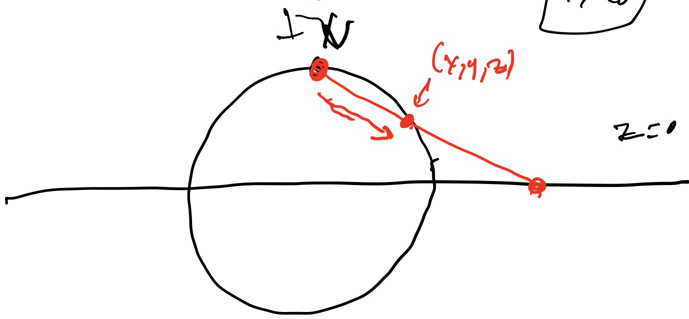
$S^2 = \{x^2+y^2+z^2=1\}$

$\{z=0\}$



$\varphi_N : S^2 - \{N\} \to \mathbb{R}^2$

$\boxed{x=0}$



$z=0$ $(x=0)$

homeo $S^2 - \{N\} \xrightarrow{\approx} \mathbb{R}^2$

$\varphi_N(x,y,z)$

param eq: $(0,0,1)$ to (x,y,z)

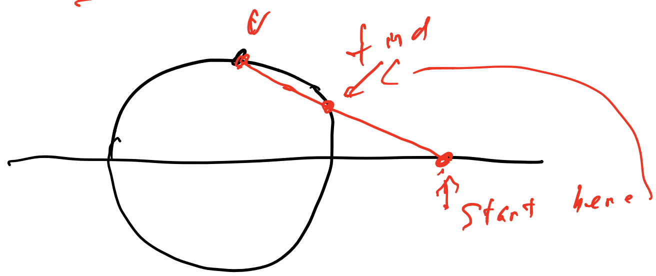
$(0,0,1) + t\big((x,y,z) - (0,0,1)\big)$

$= (0,0,1) + t(x,y,z-1)$

$= \big(\underbrace{tx,\ ty,\ t(z-1)+1}\big)$ $z=0$

$t(z-1)+1=0$

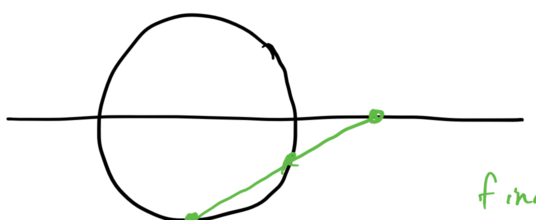
Inverse Map



$S^2 - \{N\}$ homeo to $\mathbb{R}^2$

What missing N

Stereo proj from $S' = (0,0,-1)$



find formulas

$S' = (0,0,-1)$

$S^2 - \{S'\}$ is homeo to $\mathbb{R}^2$

$U_N = S^2 - \{N\} \xrightarrow{\phi_N} \mathbb{R}^2$

$U_S = S^2 - \{S\} \xrightarrow{\phi_S} \mathbb{R}^2$

$\underbrace{U_N \cap U_S}_{\text{true}} = S^2 - \{N, S\}$