Introduction to Algebraic and Geometric Topology
Week 1

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Fall 2017

Topics

- Metric spaces, isometries, Lipschitz mappings.
- Groups of isometries of the plane and sphere.
- Topological spaces and continuous mappings.
- Construction of topological spaces, identification topology.
- Compact spaces, connected spaces.
- Surfaces as identification spaces.
- Surfaces as metric spaces: Riemannian metrics, geodesics, Gaussian curvature.

Web - page and Notes

- Web - page for the course:

  `http://www.math.utah.edu/~toledo/5510.html`

  Look there for syllabus, homework, etc.

  In particular

  - Notes for the course also available there.
  - Notes will be updated as course goes on. Look for the version number.
  - Notes of the daily lectures also available there. Notes as projected will be posted every week.

Homework, tests, grading

- Homework to be handed in roughly every other week.
- Two midterm exams
  - September 27
  - November 8
- Final Exam: December 14, 10:30 - 12:30
- Grading:

Grading:	Homework , drop lowest 2:	35 %
	Midterm Exams:	40 %
	Final Exam:	25 %

Overview

Topological Spaces

generalize ↗

Metric Spaces

specialize ↘

Riemannian Metric ..

Let's start:

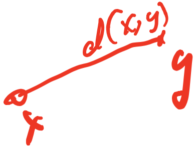
X

Definition

A *metric space* (X, d) is a non-empty set X and a function $d : X \times X \to \mathbb{R}$ satisfying

1. For all $x, y \in X$, $d(x, y) \geq 0$ and $\boxed{d(x, y) = 0 \text{ if and only if } x = y}$.
2. For all $x, y \in X$, $d(x, y) = d(y, x)$.
3. For all $x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$ (called the *triangle inequality*).

The function d is called the *metric*, it is also called the *distance function*.



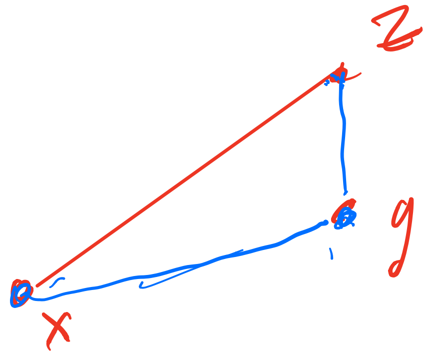
$$d \geq 0$$

$$X \times X = \{(x,y) : x \in X, y \in X\}$$

$$X \times X \longrightarrow \mathbb{R}$$

$$(x,y) \longrightarrow d(x,y) \in \mathbb{R}$$

$$d(x,y) = d(y,x)$$

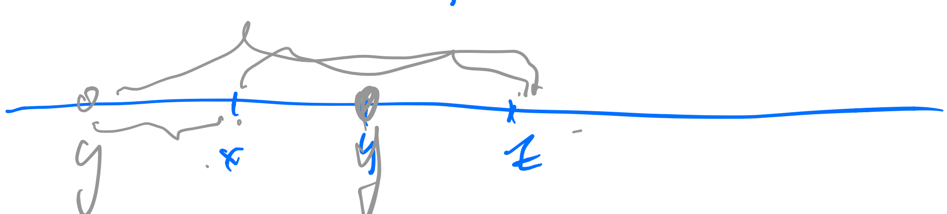


$d(x,z) \le d(x,y) + d(y,z)$

$d(x,y)$ = distance between x & y

Ex: $X = \mathbb{R}$

$$\begin{cases} d(x,y) = |y-x| \\ |y-x| \ge 0, \quad = 0 \Leftrightarrow x = y, \\ |y-x| = |x-y| \end{cases}$$



$d(x,z) = d(x,y) + d(y,z)$

$d(x,z) \le d(x,y) + d(y,z)$ $<$

Two notable properties of this definition are:

- Its simplicity.
- Its wide applicability:
  - large number of examples.
  - great variety of examples

Examples of Metric Spaces

Next, look at examples.

To verify that a given (X, d) is a metric space,

main point usually is:

Verify the triangle inequality

The other properties are usually much easier to verify.

u, v same sign (both ≥ 0 or both ≤ 0)

u, v opposite signs (one ≥ 0, other ≤ 0)

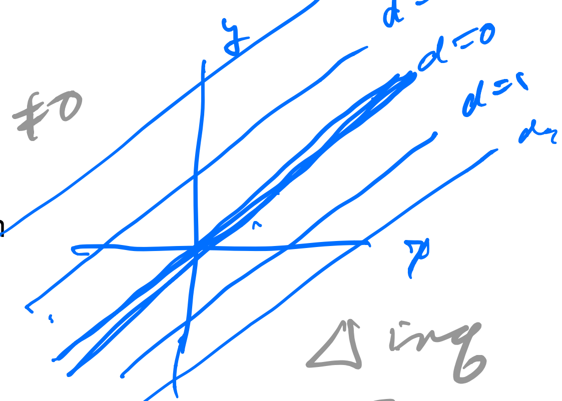
$d = 2$

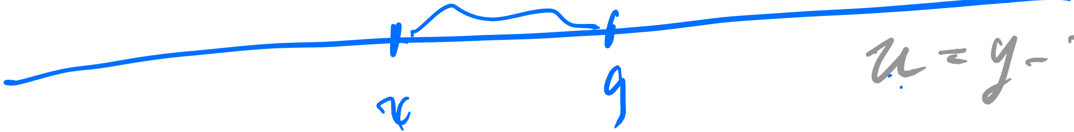
u, v same sign $\quad |u+v| = |u| + |v|$

u, v opposite sign $\quad |u+v| \leq |u| + |v|$ strict if both $\neq 0$

Example

Let $X = \mathbb{R}$ with the usual distance function $d(x, y) = |x - y|$.





Δ ineq

$u = y - x$

$v = z - y$

$u + v = z - x$

$|u+v| \leq |u| + |v|$

$d(x,y) = |y - x|$

$d(x,y) \geq 0$, $= 0 \Leftrightarrow x = y$

$d(x,y) = d(y,x)$

△ ineq: $u = y - x$

$\underline{v = z - y}$

$\forall u, v \in \mathbb{R}$ $\quad u + v = z - x$

$|u + v| \leq |u| + |v|$

Case: $u > 0$, $v > 0$ ✓

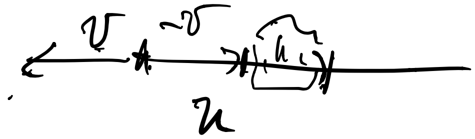
$|u| = u$, $|v| = v$ $\quad |u + v| = u + v \leq$

$u > 0$, $v < 0$ $\quad |u| = u$, $|v| = -v$

$|u + v|$ $\quad u + v < u$

$u - v$

$= |u| + |v|$



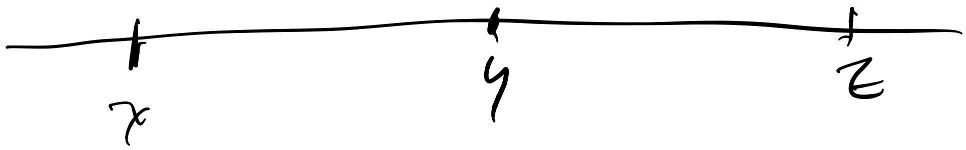
$|u + v| < u$

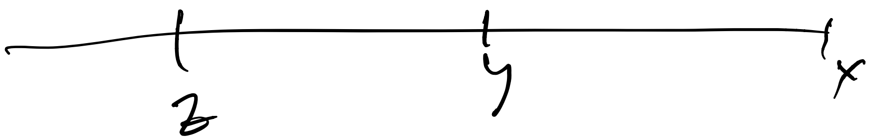
$|u| + |v| > u$

$\underline{|u + v| < |u| + |v|}$

$$d(x, z) = d(x,y) + d(y,z)$$

Ans: y between x & z





Example

Let $X = \mathbb{R}^2$ with the usual distance function

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2},$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$.

$\vec{u} = y - x$

$v = z - y$

$\|(u_1, u_2)\| = \sqrt{u_1^2 + u_2^2} = |u|$

$\|u + v\| \le \|u\| + \|v\|$

$|u + v| \le$

Triangle Inequality

Given 3 points $x, y, z \in \mathbb{R}^2$, let $u = x - y$ and $v = y - z$.

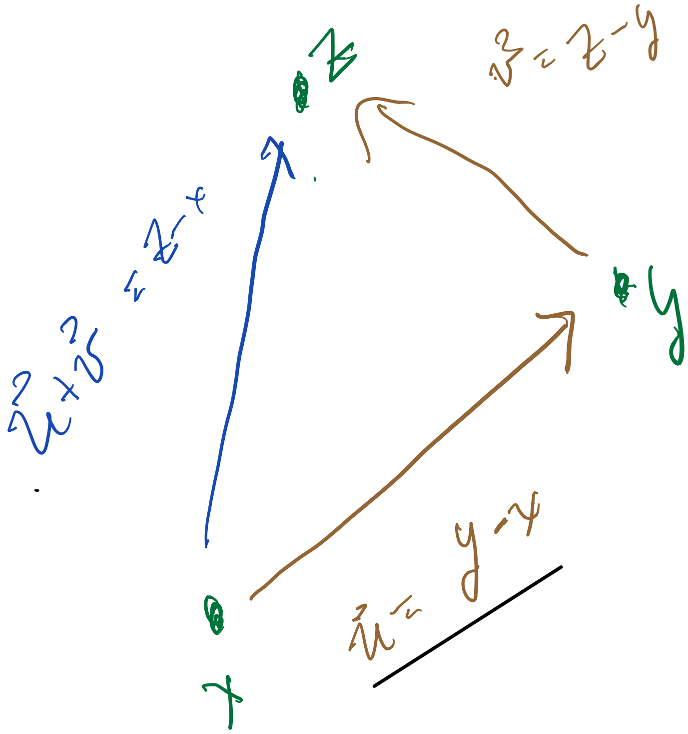
Then $u + v = x - z$.

so $d(x, z) = |u + v|, d(x, y) = |u|, d(y, z) = |v|$.

Therefore the triangle inequality is equivalent to

$$\boxed{|u + v| \leq |u| + |v|} \text{ for all } u, v \in \mathbb{R}^2.$$

$|u|$



$$|u+v| \le |u| + |v|$$

$$(y - x) + (z - y)$$

$$z - x$$

$$u, v \in \mathbb{R}^2$$

$$\overset{\shortparallel}{(u_1, u_2)} \quad \overset{?}{(v_1, v_2)}$$

$$|u| = \sqrt{u_1^2 + u_2^2}$$

$$u \cdot v = u_1 v_1 + u_2 v_2$$

$$|u| = \sqrt{u \cdot u}$$

$$|u+v| \leq |u| + |v|$$

$$\Longleftrightarrow |u+v|^2 \overset{?}{\leq} (|u| + |v|)^2$$

$$\underbrace{(u+v)\cdot(u+v)}_{u \cdot u + \boxed{2u \cdot v} + v \cdot v} \overset{?}{\leq} \cancel{|u|^2} + \boxed{2|u||v|} + \cancel{|v|^2}$$

$$\overset{\shortparallel}{\cancel{|u|^2}} \qquad \overset{\shortparallel}{\cancel{|v|^2}} \qquad u \cdot v \leq |u||v|$$

squaring both sides this is equivalent to

$$\boxed{|u+v|^2 \le |u|^2 + 2|u||v| + |v|^2.}$$

Using the properties of the dot product, we see that we want

$$|u+v|^2 = (u+v)\cdot(u+v) = u\cdot u + 2\boxed{u\cdot v} + v\cdot \le u\cdot u + 2\boxed{|u||v|} + v\cdot v,$$

which is equivalent to

$$\boxed{u\cdot v \le |u||v|}$$

Familiar?

Δ ineq

Cauchy

-Schwarz

$\le$

C-S ineq

$$|u \cdot v| \leq |u|\,|v|$$

$$-|u|\,|v| \leq \boxed{u \cdot v \leq |u|\,|v|}$$

Question

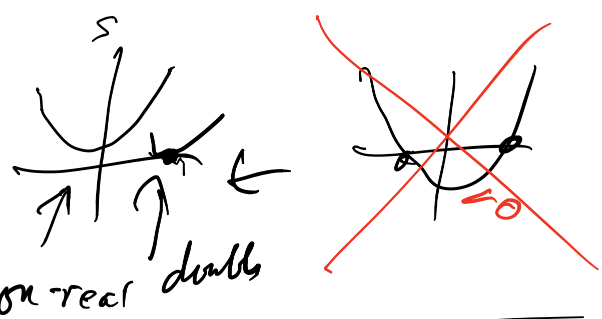
- When does equality hold?
- When is $u \cdot v = |u||v|$?
- When is $d(x, z) = d(x, y) + d(y, z)$?

$d(x,z) = d(x,y) + d(y,z)$?

One pf: given $u, v \in \mathbb{R}^2$, $t \in \mathbb{R}$

$$(tu+v)\cdot(tu+v) \geq 0$$

$$\underbrace{\boxed{t^2 (u\cdot u) + 2t(u\cdot v) + (v\cdot v) \geq 0}}_{at^2+bt+c \geq 0}$$



$$\sqrt{b^2-4ac}$$ img $(a=0)$

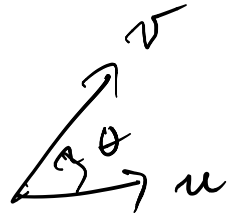
$$b^2-4ac \leq 0$$

$$(2(u\cdot v))^2 - 4(u\cdot u)(v\cdot v) \leq 0$$

$$\cancel{4}(u\cdot v)^2 \leq \cancel{4}(u\cdot u)(v\cdot v)$$

Aside

$u \cdot v = |u||v| \cos\theta$



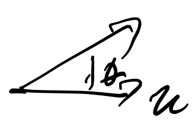
$|\cos\theta| \le 1$

$=$ $\quad \cos\theta = \pm 1$

$=$

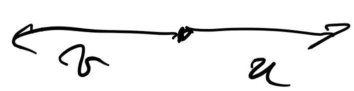
$\iff u \cdot v = |u||v|$

θ



$\cos\theta = 1$

$\cos\theta = -1$



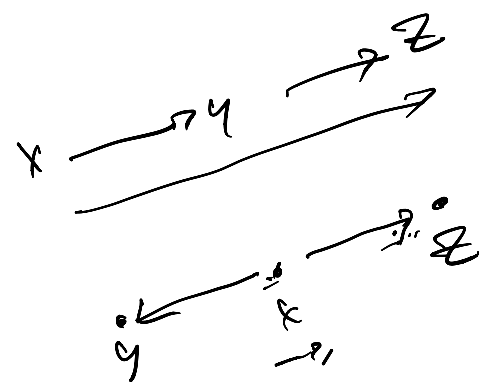
$=$ in C-S $\iff \{u, v\}$ linearly dependent

$$d(x,z) = d(x,y) + d(y,z)$$

$$u = y - x$$

$$v = z - y$$

$$\iff \boxed{u \cdot v = |u|\,|v|}$$



$\iff$ y in the line segment $\overline{xz}$

$$u \cdot v = |u||v|$$

$$\forall x, y \in \mathbb{R}^2$$

$$\Rightarrow \quad \boxed{d(x,z) \le d(x,y) + d(y,z)}$$

Example

Let $X = \mathbb{R}^n$ with the usual distance function

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2},$$

where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots y_n)$. The verifications are exactly as for the case $n = 2$ just discussed.

$d_{(Z,\,)}(x,y)$

Other metrics on $\mathbb{R}^n$

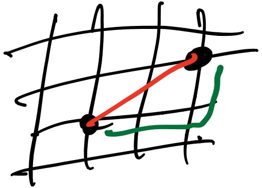
Manhattan

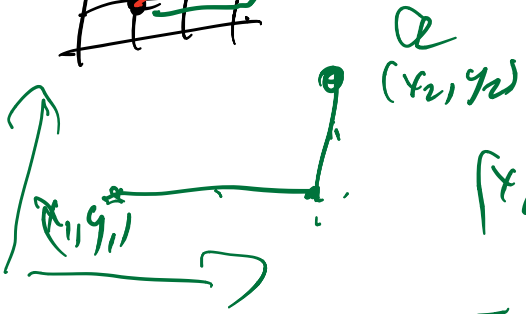
SLC

- The *Taxicab metric*

$$\boxed{d_{(1)}(x,y) = |x_1 - y_1| + \cdots + |x_n - y_n|}$$

- For $n = 2$ this is the usual way to measure distance when driving in Salt Lake City.
- Same applies to any city laid out in rectangular coordinates.





$$|x_2 - x_1| + |y_2 - y_1|$$

$$= d_{(1)}(x, y)$$

- Triangle inequality for $d_{(1)}$:

For each i, $1 \le i \le n$, apply the triangle inequality in $\mathbb{R}$:

$$\boxed{|x_i - z_i| \le |x_i - y_i| + |y_i - z_i|}$$

$i = 1$

$i = 2$

and sum over i:

L^1-metric

$$d_{(1)}(x, z) = \sum_{i=1}^{n} |x_i - z_i| \le \sum_{i=1}^{n} |x_i - y_i| + \sum_{i=1}^{n} |y_i - z_i|,$$

which is the same as $d_{(1)}(x, y) + d_{(1)}(y, z)$.

$$d(x,z) = |x_1 - z_1| + |x_2 - z_2|$$

$$\le |x_1 - y_1| + |y_1 - z_1| \qquad \le |x_2 - y_2| + |y_2 - z_2|$$

$d(x,y)$ $\qquad$ $d(y,z)$

- When does equality hold in

$$d_{(1)}(x,z) \le d_{(1)}(x,y) + d_{(1)}(y,z) \text{ ?}$$

- If and only if, for each i,

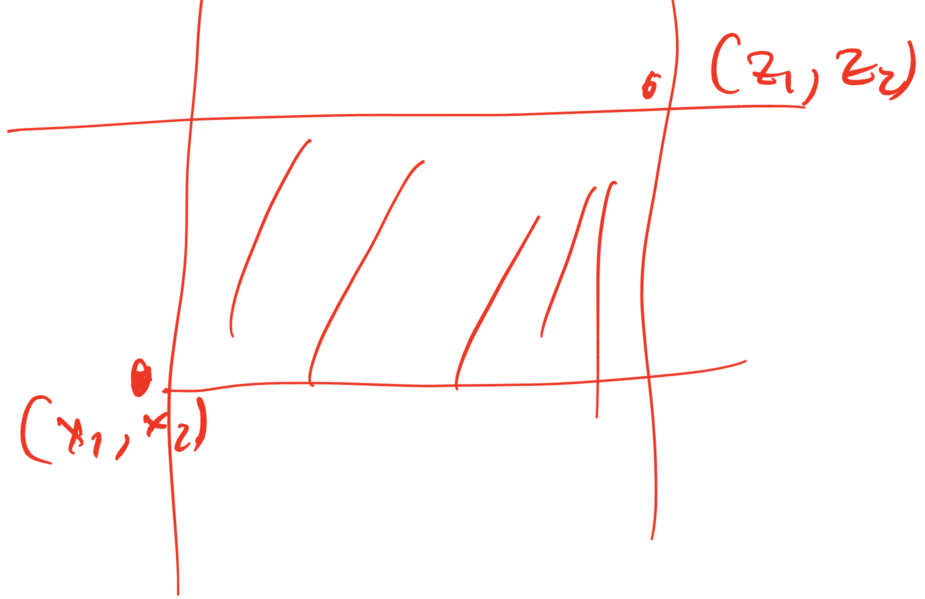
$$|x_i - z_i| = |x_i - y_i| + |y_i - z_i|$$

- Therefore, if and only if, for each i,

  y_i lies between x_i and z_i.

y_1 between x_1 & z_1

y_2 betw x_2 & z_2



(z1, z2)
(x1, x2)

- Picture for $n = 2$:

  Given $x = (x_1, x_2)$ and $z = (z_1, z_2)$,

  the set of all $y = (y_1, y_2)$ for which

  $d_{(1)}(x, z) = d_{(1)}(x, y) + d_{(1)}(y, z)$ looks like this:

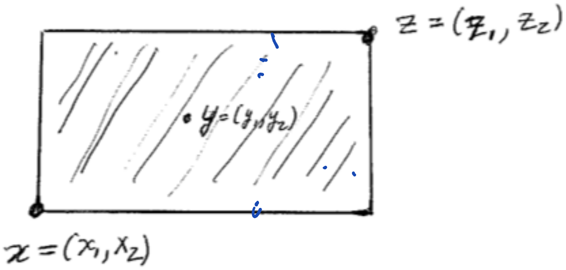


Figure: Equality Set for the Taxicab Metric

- Another useful metric on $\mathbb{R}^n$ is the *supremum metric* (or simply *sup metric*) defined by

$$\boxed{d_{(\infty)}(x, y) = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}.}$$

- Details left as exercises.
- These distances are all defined by *norms* on $\mathbb{R}^n$.

$u = y - x$

$|u|_{(1)} = |u_1| + |u_2|$

$v = z - y$

$d(0, u) = |u|$

Norm on $\mathbb{R}^n$ (on a vector space)

func $\overset{\in \mathbb{R}^n}{u} \longrightarrow |u| \in \mathbb{R}$

1) $|u| \geq 0$, $= 0 \Leftrightarrow u = 0$

2) $|\alpha u| = |\alpha|\,|u| \quad \forall u \in \mathbb{R}^n$
$\forall \alpha \in \mathbb{R}$

3) $|u + v| \leq |u| + |v|$

- One way to visualize metrics is by visualizing the shapes of balls. Terminology:
- Let (X, d) be a metric space, $x \in X$, $r \in \mathbb{R}$, $r \geq 0$
- The *ball (or open ball) of radius r centered at x* is

strict

$$B(x, r) = \{y \in X | d(x, y) < r\}.$$

- The *closed ball of radius r centered at x* is

$$\overline{B}(x, r) = \{y \in X | d(x, y) \leq r\}.$$

allow =

- The *sphere of radius r centered at x*

$$S(x, r) = \{y \in X | d(x, y) = r\}.$$

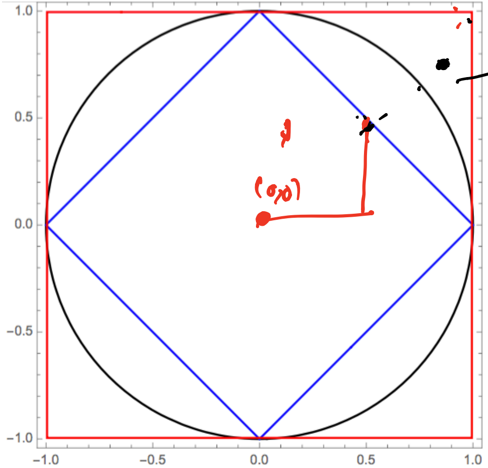
- The pictures for $n = 2$ of the unit spheres of the metrics defined so far:

L^p - metrics

L^p - norm

$$|u|_{(p)} = (|u_1|^p + |u_2|^p)^{1/p}$$

$$1 \le p \le \infty$$



$|u|_{(\infty)}$

$|u|_{(2)}$

$|u|_{(1)}$

Figure: Unit Spheres of $d_{(1)}, d_{(2)}, d_{(\infty)}$ (ordered from inner to outer).

- The pictures of the unit spheres illustrate (for $n = 2$) the following inequalities comparing the metrics:

1. $\boxed{d_{(2)}(x, y) \leq d_{(1)}(\overbrace{x, y}^{u = y - x}) \leq \sqrt{n}\, d_{(2)}(x, y).}$
2. $d_{(\infty)}(x, y) \leq d_{(2)}(x, y) \leq \sqrt{n}\, d_{(\infty)}(x, y).$
3. $d_{(\infty)}(x, y) \leq d_{(1)}(x, y) \leq n\, d_{(\infty)}(x, y).$

- Convince yourself that the pictures and inequalities correspond.

Proof of C-S ineq using $(tu+v)\cdot(tu+v) \geq 0$

$$\| u \|^2 t^2 + 2(u\cdot v)\, t + |v|^2$$

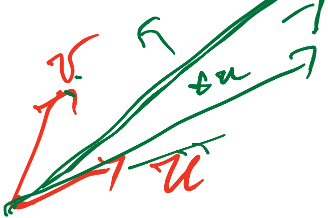
eq $\Leftrightarrow$ disc $= 0$ $\Leftrightarrow$ double root

but double root must be real $t_0 \in \mathbb{R}$

$\boxed{(t_0 u+v)\cdot(t_0 u+v) = 0} \Rightarrow \boxed{t_0 u+v = 0} \Rightarrow v = -t_0 u$

$x u + v$

lin dep.



$|u\cdot v|^2 \le |u||v$

discrimnar

$$4\left((u\cdot v)^2 - |u|^2|v|^2\right) \le 0$$

$=$ $\iff$ discr $= 0$

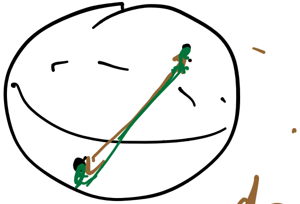
$\iff$ double root must be real: $t_0 \in \mathbb{R}$

Subspace of a Metric Space

- Let (X, d) be a metric space and $Y \subset X$.
- Let $d_Y = d|_{Y \times Y}$ be the restriction of the metric on X to a function on $Y \times Y$.
- Then d_Y is a metric on Y, called the *subspace metric*.
- The metric space (Y, d_Y) is called a *subspace* of (X, d).

S^2 = unit sphere in $\mathbb{R}^3$

$$d_e(x,y) = d_{\mathbb{R}^3,(2)}(x,y)$$



d_e = euclidean dist

- Example of a subspace:

  Let $S^2 \subset \mathbb{R}^3$ be the unit sphere $S(0, 1)$ in the $d_{(2)}$-metric (or *Euclidean metric*) in $\mathbb{R}^3$, centered at the origin $0 \in \mathbb{R}^3$:

$$S^2 = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$$

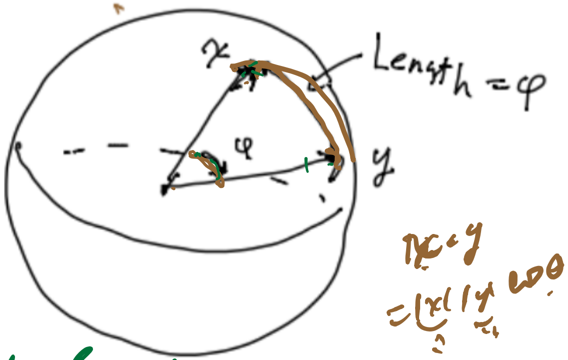
  and let d_e denote the subspace metric on S^2:

$$d_e(x, y) = d_{(2)}(x, y) \text{ for all } x, y \in S^2.$$

- (S^2, d_e) is a subspace of $(\mathbb{R}^3, d_{(2)})$.

- d_e is called the *extrinsic metric* on S^2.
- The *intrinsic metric* d_i on S^2 is the great-circle arc distance:

d_i



$x \cdot y$
$= |x| |y| \cos\theta$

$d_i(x, y)$
= length of shortest path in S^2 joining

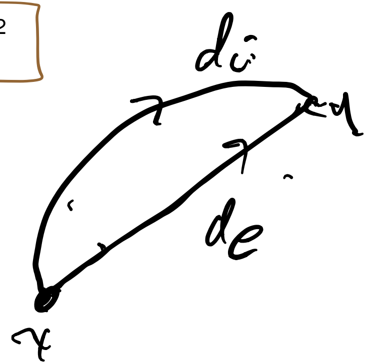
Figure: Intrinsic Distance on S^2

-x e.g. "Know" ! great circle arc

- Elementary geometry (or trigonometry) gives

$$\boxed{d_i(x, y) = \cos^{-1}(x \cdot y)} \; \boxed{\text{for all } x, y \in S^2}$$

  where $x \cdot y$ is the usual dot product.
- (S^2, d_i) is *not* a subspace of $\mathbb{R}^3$.
- $d_e(x, y) < d_i(x, y)$ if $x \neq y$



If know that L(great circle arc)

$= \inf \{ L(\gamma) : \gamma$ arc $\}$

then $\triangle$ area for the

"great circle arc distance" on

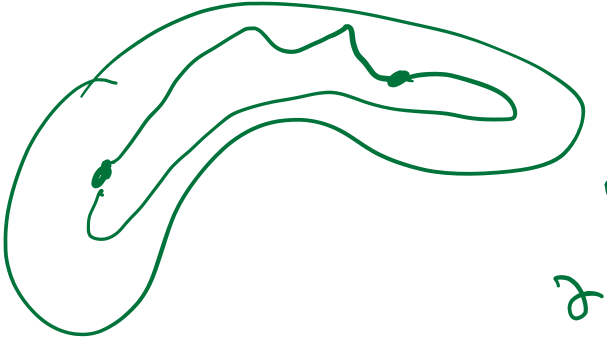
S^2 would be clear.

- One goal of this course:
  - Define (smooth) surface $S \subset \mathbb{R}^3$
  - Define intrinsic distance $d_{S,i}$ on any surface by

$$d_{S,i}(x,y) = \inf\{L(\gamma) | \gamma \in P(x,y)\}$$

    where

    - $P(x,y)$ is the collection of piecewise smooh curves in S from x to y
    - $L(\gamma)$ denotes the length of γ.
  - Triangle inequality is easy for $d_{S,i}$.
  - Prove that for S^2, d_i as before is same as $d_{S^2,i}$.



$\gamma : [0,1] \to S$

$\gamma(0) = x$

$\gamma(1) = y$

Length: $L(\gamma) = \int_0^1 |\gamma'(t)|\,dt$

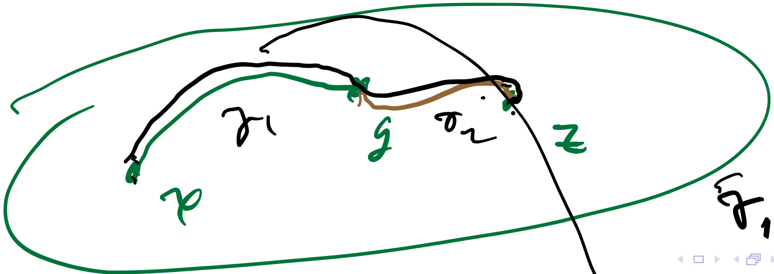
$$d_i(x,y) = \underline{\inf} \{ L(\gamma) \; ; \; \gamma \text{ path from } x \text{ to } y \}$$

($\underline{\min}$ need not exist)

$d_1(x,y) \geq 0$, $= 0 \iff x = y$ ✓

$d(x,y) = d(y,x)$ ✓

$$d(x,z) \leq d(x,y) + d(y,z)$$



γ_1 from x to y

γ_2 from y to z

$\gamma_1 \gamma_2$

Discrete Metric Space

"Concatenation"

is a path from x to z

$L(\gamma_1, \gamma_2)$

$= L(\gamma_1) + L(\gamma_2)$

- X any non-empty set
- Define $d : X \times X \to \mathbb{R}$ by

$$d(x, y) = \begin{cases} 0 \text{ if } x = y \\ 1 \text{ if } x \neq y. \end{cases}$$

choose γ_1, γ_2

$$d(x,z) \leq L(\gamma_1, \gamma_2) = \underbrace{L(\gamma_1)}_{= d(x,y)} + \underbrace{L(\gamma_2)}_{= d(y,z)}$$

ε

$L(\gamma_1) = d(x,y)$

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

$d(x,y) \geq 0$, $= 0 \Leftrightarrow x = y$

$d(x,y) = d(y,x)$ ✓

$d(x,z) \leq d(x,y) + d(y,z)$

0 or 1	
0	ok
1 $x \neq z$	$0 + 0$ impossible $\underbrace{x = y \quad y = z}$ $\Rightarrow x = z$ at least one $= 1$
1	1 or 2

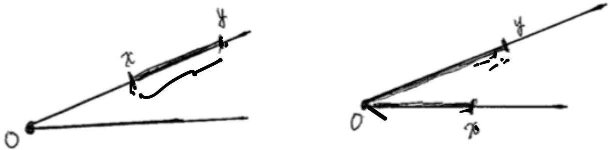
French Railway Metric

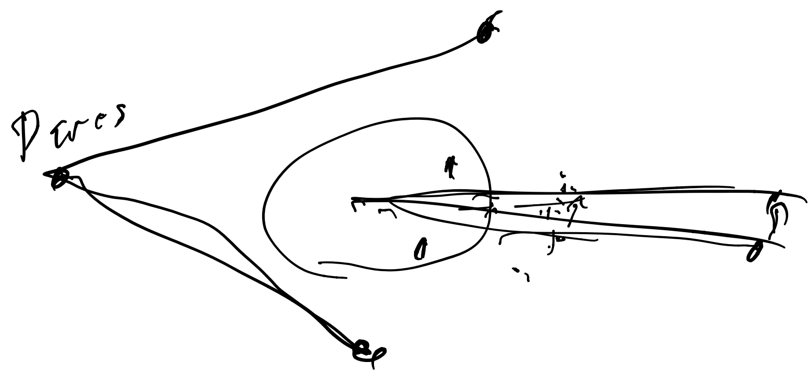
$L(\gamma_1) = d(y, z)$

- $X = \mathbb{R}^2$.
- $d : X \times X \to \mathbb{R}$ defined by

$$d(x,y) = \begin{cases} |x-y| & \text{if } x \text{ and } y \text{ are in same ray from } 0 \\ |x| + |y| & \text{otherwise,} \end{cases}$$

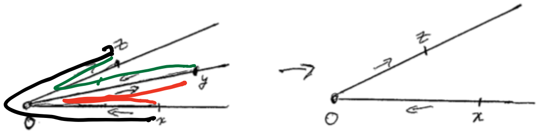
- Picture





- Given $x, y \in X$, define *path from x to y*.
- $d(x, y) =$ length of shortest path from x to y.
- Prove triangle inequality. Here's one case:

cases



p-adic metric p-adic norm

- Fix prime number p.
- If $x \in \mathbb{Z}$, $x \neq 0$, let $e_p(x)$ be the exponent of p in the prime factorization of x, that is,

$$x = k\, p^{e_p(x)} \text{ where } p \text{ does not divide } k.$$

- Let $X = \mathbb{Z}$ and let $d_p : X \times X \to \mathbb{R}$ be

$$\boxed{d_p(x,y) = \begin{cases} 0 \text{ if } x = y, \\ p^{-e_p(x-y))} \text{ if } x \neq y. \end{cases}}$$

$x = p_1^{k_1} p_2^{k_2} \cdots$ $\quad p \mid x \quad p^{e_p(x)}$

- d_p is called the p-adic metric on $\mathbb{Z}$
- Triangle inequality: Given $u, v \in \mathbb{Z}$, $u, v \neq 0$,

$$e_p(u + v) \geq \min\{e_p(u), e_p(v)\}$$

therefore

$$p^{-e_p(u+v)} \leq \max\{p^{-e_p(u)}, p^{-e(v)}\}$$

- Given $x, y, z \in \mathbb{Z}$, apply to $u = x - y$, $v = y - z$:

$$d_p(x, z) \leq \max\{d_p(x, y), d_p(y, z)\}.$$

called the *ultrametric inequality*, which $\Longrightarrow$ triangle inequality

$e_7(1) = 0$
$d_7(0,1) = 7^{-0} = 1$

- Example: If $p = 7$, then

$$d_7(0,1) = d_7(0,2) = \ldots d_7(0,6) = d_7(0,8) = \cdots = 1$$

7^{e_7}

while

$$d_7(0,7) = d_7(0,14) = d_7(0,21) = \ldots d_7(0,56) = \cdots = \frac{1}{7}$$

and

$$d_7(0,49) = d_7(0,98) = \cdots = \frac{1}{49}$$

$e_7(49) = 2$

7^{-2}

etc.

$d(x,y) \geq 0, \; = 0 \iff x = y$ ✓

$d(x,y) = d(y,x)$ ✓

$d(x,z) \leq d(x,y) + d(y,z)$

$$|x|_p = p^{-e_p(x)}$$

$$|u+v|_p \leq |u|_p + |v|_p$$

$p^{e_p(x)} \mid x \qquad u+v$

$e_p(v) \mid v$

$p^{\min e_p}$

- Extend d_p to $\mathbb{Q}$:
- Write

$$x = \frac{k}{l}\, p^{e_p(x)}$$

  where $k, l \in \mathbb{Z}$ no common factor, p does not divide k nor l.
- $e_p(x)$ may now be negative.
- Define $d_p(x, y)$ as before.
- Example:

$$d_7(0, \frac{2}{5}) = 1, d_7(0, \frac{2}{7}) = 7, d_7(0, \frac{10}{49}) = 49, \ldots$$

$p^{e_p(u)} \mid u \qquad p^{e_p(v)} \mid v$

$p^{\min(e_p(u),\, e_p(v))} \mid u+v$

$e_p(u+v) \geq \min\{e_p(u), e_p(v)\}$

$-e_p(u+v) \leq \max\{-e_p(u), -e_p(v)\}$

$p^{-e_p(u+v)} \leq \max\{p^{-e_p(u)}, p^{-e_p(v)}\}$

$$\boxed{|u+v|_p \leq \max\{|u|_p, |v|_p\}}$$ stronger than $\triangle$ ineq

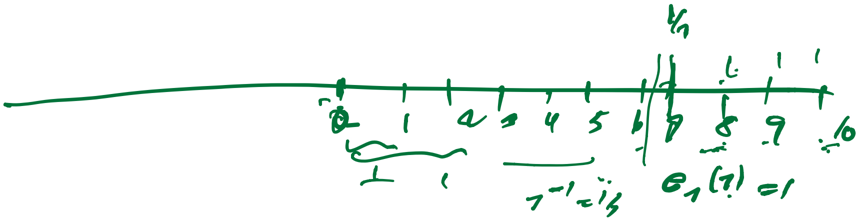
$\leq |u|_p + |v|_p$

"ultrametric ineq"

- $\boxed{\text{Example: the sequence } 7^n}$
    - Converges to 0 in d_7.
    - Is bounded in d_p for $p \neq 7$
- While the sequence $\frac{1}{7^n}$
    - $\to \infty$ in d_7
    - is bounded in d_p for $p \neq 7$.

$$d(0, 7^n) = 7^{-n} \downarrow 0$$

$$d_5(0, 7^n) = 1$$



Convergence

- Let $\{x_n\}$ be a sequence in (X, d).

1. Let $x \in X$. We say $\lim\{x_n\} = x$ iff for all $\epsilon > 0$ there is an $N (= N(\epsilon)) \in \mathbb{N}$ so that $d(x, x_n) < \epsilon$ for all $n > N$.
2. We say that $\{x_n\}$ *converges* iff there exists $x \in X$ so that $\lim\{x_n\} = x$.
3. We say that $\{x_n\}$ is a *Cauchy sequence* iff for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ so that $d(x_m, x_n) < \epsilon$ for all $m, n > N$.

Will continue with Convergence and completeness in metric spaces

Next Week