1. Recall that the *discriminant* of a polynomial $f(X) \in F[X]$ of degree n with roots $\alpha_1, \cdots, \alpha_n$ in some extension field K was defined to be

$$\Delta(f) = \prod_{i<j} (\alpha_i - \alpha_j)^2.$$

   It was outlined in class that $\Delta(f) \in F$ and clearly $\Delta(f) = 0$ if and only if $f(X)$ has a double root.

   Let $f(X) \in \mathbb{R}[X]$ be a cubic polynomial. Prove that $f(X)$ has 3 distinct real roots if and only if $\Delta(f) > 0$.

2. Let $f(X) = X^3 + pX + q \in \mathbb{R}[X]$. Recall from class the formula for the discriminant in terms of the coefficients: $\Delta = -(4p^3 + 27q^2)$. Use elementary calculus to prove from this formula that $f(X)$ has three distinct real roots if and only if $\Delta > 0$. Follow these steps:

   (a) Show that $f : \mathbb{R} \to \mathbb{R}$ has a local maximum and a local minimum if and only of $p < 0$. Compute where these are and argue that $f(X)$ has three distinct real roots if and only if $f(-\sqrt{\frac{-p}{3}}\,) > 0$ and $f(\sqrt{\frac{-p}{3}}\,) < 0$.

   (b) Show that these inequalities are equivalent to $4p^3 + 27q^2 < 0$.

3. In previous problems you have found that the minimal polynomial of $\sqrt{2} + \sqrt{3}$ is $X^4 - 10X^2 + 1$ and you have proved that this polynomial is irreducible. Prove now that $\mathbb{Q}(\sqrt{2} + \sqrt{3}) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$.

4. Prove that the splitting field of $f(X) = X^4 - 10X^2 + 1 \in \mathbb{Q}[X]$ is $\mathbb{Q}(\sqrt{2}, \sqrt{3})$, and find all the roots of $f(X)$.