1. (Modification of Royden, Chapter 3, problem 28.) Let $I = [0,1]$ and let $C \subset [0,1]$ be the usual Cantor set,
$$C = \{\sum_{i=1}^{\infty} \frac{a_i}{3^i} \text{ where } a_i = 0 \text{ or } 2\}.$$
Let $a_i = 2b_i$. Then $b_i = 0$ or 1 and we can define a function $\varphi : C \to I$ by $\varphi(\sum \frac{a_i}{3^i}) = \sum \frac{b_i}{2^i}$. It is clear that φ is continuous and monotone: $x < y \Rightarrow \varphi(x) \leq \varphi(y)$. Since every $x \in I$ has a binary expansion, φ is surjective. But φ is *not* injective. In fact, two binary expansions give the same number if and only if they are of the form
$$\{b_1, \ldots, b_n, 0, 1, 1, 1, \ldots\} \text{ and } \{b_1, \ldots, b_n, 1, 0, 0, 0, \ldots\}.$$
This means that for $x, y \in C$, $x < y$, $\varphi(x) = \varphi(y)$ if and only if x, y correspond to sequences of the form
$$x \leftrightarrow \{a_1, \ldots, a_n, 0, 2, 2, 2, \ldots\} \text{ and } y \leftrightarrow \{a_1, \ldots, a_n, 2, 0, 0, 0 \ldots\}.$$
But the intervals (x, y) with x, y if this form are exactly the intervals that are deleted at the $(n+1)$-st step of the usual inductive construction of C.

This means that φ can be extended to a continuous function $\varphi : I \to I$ which is constant on each connected component (x, y) of $I \setminus C$: for $z \in (x, y)$, define $\varphi(z) = \varphi(x)$, so φ is constanat on (x, y) and agrees with the previous definition on $[x, y] \cap C$, so it is continuous.

With this background in mind, answer the following questions:

   (a) Define $f : [0,1] \to [0,2]$ by $f(x) = x + \varphi(x)$. Prove that f is a homeomorphism (continuous bijection with continuous inverse). Let $g : [0,2] \to [0,1]$ be f^{-1}.

   (b) Let $F = f(C)$. Prove that F has measure 1.

   (c) Use the result of problem 16, Chapter 3 of Royden (which follows easily from 15 which you have done) to conclude that F has non-measurable subsets.

   (d) Deduce that there are measurable subsets $E \subset C$ such that $g^{-1}(E)$ is not measurable.

   (e) Deduce from the last statement that there are measurable sets that are not Borel sets.

2. Royden, Chapter 3, problem 23 (c). You may assume 23 (a,b) and that f is real valued (does not take on $\pm\infty$).

3. Royden, Chapter 3, problem 30.

4. Royden, Chapter 4, problem 3.