Definitions

1. Let \( \{X_\alpha\}_{\alpha \in A} \) be a collection of topological spaces indexed by a set \( A \). The product of this family is defined as

\[
\prod_{\alpha \in A} X_\alpha = \{x : A \rightarrow \bigcup_{\alpha \in A} X_\alpha : x(\alpha) \in X_\alpha \text{ for all } \alpha \in A\}
\]

2. For each \( \beta \in A \) the projection \( p_\beta : \prod_{\alpha \in A} X_\alpha \rightarrow X_\beta \) is defined by \( p_\beta(x) = x(\beta) \).

3. The product topology on \( \prod_{\alpha \in A} X_\alpha \) is defined as the smallest topology that makes all the projections \( p_\alpha \) continuous. A basis for this topology is

\[
\{p_\alpha^{-1}(U_\alpha_1) \cap \ldots \cap p_\alpha^{-1}(U_\alpha_n) : \{\alpha_1, \ldots, \alpha_n\} \text{ a finite subset of } A \text{ and } U_\alpha_i \text{ is open in } X_\alpha_i\}
\]

4. The Cantor set is the subset \( C \subset [0,1] \) consisting of all numbers \( x \in [0,1] \) that can be represented by an expansion to the base 3 with only zeros and twos:

\[
x = \sum_{i=1}^{\infty} \frac{a_i}{3^i} \quad \text{where } a_i = 0 \text{ or } 2.
\]

(This is easily seen to be equivalent to the usual middle thirds construction.)

5. A topological group is a group \( G \) that is also a topological space and where the group operations \( G \times G \rightarrow G \) (group multiplication) and \( G \rightarrow G \) (inversion) are both continuous. Here \( G \times G \) is given the product topology.

6. Some examples of topological groups. Here \( M_n \) is the space of \( n \times n \) real matrices, topologized in the natural way by the bijection with \( \mathbb{R}^{n^2} \) given by the matrix entries:

   (a) \( O(n) = \{A \in M_n : A^T A = AA^T = I\} \)
   
   (b) \( SO(n) = \{A \in O(n) : \det(A) = 1\} \)
   
   (c) \( E(n) \) is the group of Euclidean motions of \( \mathbb{R}^n \). It consists of all transformations of \( \mathbb{R}^n \) of the form \( x \rightarrow Ax + b \), where \( A \in O(n) \) and \( b \in \mathbb{R}^n \), topologized by the natural bijection with \( O(n) \times \mathbb{R}^n \).

7. An \( n \)-dimensional manifold is a Hausdorff space \( X \) with a countable basis that has the property that every \( x \in X \) has a neighborhood \( U \) which is homeomorphic to an open set in \( \mathbb{R}^n \).

8. An \( n \)-dimensional manifold with boundary is a Hausdorff space \( X \) with a countable basis that has the property that every \( x \in X \) has a neighborhood \( U \) that is homeomorphic to an open set in \( (\mathbb{R}^n)^+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n \geq 0\} \).

9. A surface is a 2-dimensional manifold, a surface with boundary is a 2-dimensional manifold with boundary.
10. Examples of surfaces:
   (a) The sphere, torus, Klein bottle have been previously defined. These are surfaces.
   (b) The Möbius strip has been previously defined. This is a surface with boundary.
   (c) The projective plane $P^2$ is the quotient of the sphere $S^2$ by the equivalence relation $x \sim -x$ for all $x \in S^2$. This is a surface. The projective n-space $P^n$ is defined in the same way as the quotient of $S^n$ by $x \sim -x$, and is an n-dimensional manifold.

11. Triangulations of surfaces.
   (a) The standard n-simplex $s^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_i \geq 0, \sum x_i = 1\}$
   (b) An n-simplex in a topological space $X$ is a map $\sigma^n : s^n \to X$ which is a homeomorphism onto its image. A 2-simplex is called a triangle, its three edges are obtained by setting $x_i = 0$, its three vertices by setting $x_i = x_j = 0$.
   (c) A triangulation $T$ of a compact surface $X$ is a decomposition $X = T_1 \cup \ldots \cup T_n$ where each $T_i$ is the image of a 2-simplex in $X$ and each intersection $T_i \cap T_j$ is either empty, or a common vertex of $T_i$ and $T_j$, or a common edge of $T_i$ and $T_j$.
   (d) The Euler characteristic $\chi(T)$ of a triangulation $T$ of a surface is the number $\chi(T) = V - E + F$ where $V =$ number of vertices, $E =$ number of edges, $F =$ the number of triangles.

Theorems

1. If $X_\alpha \neq \emptyset$ for all $\alpha \in A$, then $\prod X_\alpha \neq \emptyset$; if $X_\alpha$ is compact for all $\alpha \in A$, then $\prod X_\alpha$ is compact. (Both of these theorems were accepted without proof.)
2. The map $f : \prod_{i=1}^{\infty}\{0, 1\} \to C$ defined by $f(\{a_i\}) = \lim_{n \to \infty} \sum_{i=1}^{n} a_i$ is a homeomorphism. Here $C$ is the Cantor set, $\{0, 1\}$ has the discrete topology and $\prod_{i=1}^{\infty}\{0, 1\}$ has the product topology.
3. The Cantor set $C$ is a topological group. In particular, $C$ is homogeneous: given any $x, y \in C$ there exists a homeomorphism $\phi : C \to C$ with $\phi(x) = y$.
4. $O(n)$, $SO(n)$, $E(n)$ are topological groups.
5. $SO(3)$ is homeomorphic to the projective space $P^3$.
6. $SO(2)$, $SO(3)$ are connected, while each of $O(2)$, $O(3)$, $E(2)$ and $E(3)$ has two connected components.
7. Every compact surface has a triangulation. (Accepted without proof.)
8. Explicit triangulations can be given for $S^2$, $P^2$, $T$, $K$ (where $T$ is the torus and $K$ is the Klein bottle). Know how to triangulate these surfaces.
9. If $T_1$ and $T_2$ are two triangulations of a compact surface $X$, then $\chi(T_1) = \chi(T_2)$. Thus can define $\chi(X)$ as $\chi($any triangulation$)$.
10. If $X$ and $Y$ are homeomorphic compact surfaces, then $\chi(X) = \chi(Y)$.
11. $\chi(S^2) = 2$, $\chi(P^2) = 1$, $\chi(T) = 0$, $\chi(K) = 0$. 