Definitions

1. Let \((X, \mathcal{T})\) be a topological spaces. A subset \(B \subset \mathcal{T}\) is called a basis for \(\mathcal{T}\) if and only if every element of \(\mathcal{T}\) is a union of elements of \(B\).

2. Let \(X\) be a set. A subset \(B \subset 2^X\) is called a basis for a topology on \(X\) if and only if for all \(U, V \in B\), \(U \cap V\) is a union of elements of \(B\). The set \(T \subset 2^X\) consisting of all unions of elements of \(B\) is a topology on \(X\) with basis \(B\) and is called the topology generated by \(B\).

3. Let \((X_1, T_1)\) and \((X_2, T_2)\) be topological spaces. The product topology on \(X_1 \times X_2\) is the topology with basis \(B = \{U \times V : U \in T_1, V \in T_2\}\).

4. Let \((X, T)\) be a topological space, and let \(A \subset X\). The subspace topology, also called the relative topology on \(A\) is \(\{A \cap U : U \in T\}\).

5. Let \((X, T)\) be a topological space and let \(f : X \to Y\) be surjective. The quotient topology, also called the identification topology on \(Y\) is \(\{U \subset Y : f^{-1}(U) \in T\}\).

6. Let \(S = [0, 1] \times [0, 1]\) be the unit square.

   (a) The M"obius strip is the identification space of \(S\) by the equivalence relation \((0, y) \sim (1, 1 - y)\) for all \(y \in [0, 1]\), with the quotient topology.

   (b) The torus is the identification space of \(S\) by the equivalence relation \((x, 0) \sim (x, 1)\) and \((0, y) \sim (1, y)\) for all \(x, y \in [0, 1]\), with the quotient topology.

   (c) The Klein bottle is the identification space of \(S\) by the equivalence relation \((x, 0) \sim (x, 1)\) and \((0, y) \sim (1, 1 - y)\) for all \(x, y \in [0, 1]\), with the quotient topology.

7. A topological space \(X\) is connected if and only if, whenever \(X = U \cup V\), with \(U\) and \(V\) disjoint open sets, then either \(U = \emptyset\) or \(V = \emptyset\).

8. A topological space \(X\) is path connected if and only if, for all \(x, y \in X\) there exists a continuous map \(\phi : [0, 1] \to X\) with \(\phi(0) = x\) and \(\phi(1) = y\).

9. A topological space is locally connected if it has a basis of connected sets, and it is locally path connected if it has a basis of path connected sets.

10. If \(X\) is a topological space and \(x, y \in X\), we say that \(x\) and \(y\) are in the same component if there is a connected set \(A \subset X\) containing both \(x\) and \(y\). This is an equivalence relation on \(X\), and the equivalence classes are called the components of \(X\).

11. If \(X\) is a topological space and \(x, y \in X\), we say that \(x\) and \(y\) are in the same path component if there is a continuous map \(\phi : [0, 1] \to X\) with \(\phi(0) = x\) and \(\phi(1) = y\). This is an equivalence relation on \(X\), and the equivalence classes are called the path components of \(X\).
12. A topological space \( X \) is compact if and only if, whenever \( \{U_\alpha\}_{\alpha \in I} \) is a collection of open sets with \( X = \bigcup_{\alpha \in I} U_\alpha \), there is a finite subcollection \( U_{\alpha_1} \ldots U_{\alpha_n} \) with \( X = U_{\alpha_1} \cup \ldots \cup U_{\alpha_n} \).

13. A topological space \( X \) is a Hausdorff space if for all \( x, y \in X, x \neq y \), there exist disjoint open sets \( U, V \) with \( x \in U \) and \( y \in V \).

**Theorems** \((X, Y, X_1, X_2, \text{ etc, are topological spaces})\)

1. A map \( X \to X_1 \times X_2 \) if continuous if and only if \( p_1 f \) and \( p_2 f \) are continuous, where \( p_1 \) and \( p_2 \) are the projections of \( X_1 \times X_2 \) onto \( X_1 \) and \( X_2 \) respectively, and \( X_1 \times X_2 \) is given the product topology.

2. Suppose \( p : X \to Y \) is a surjective map and \( Y \) has the quotient topology from \( X \). Then \( f : Y \to Z \) is continuous if and only if \( fp : X \to Z \) is continuous.

3. A space \( X \) is connected if and only if the only subsets that are both open and closed are \( X \) and \( \emptyset \).

4. A space \( X \) is connected if and only if every continuous function \( f : X \to \{0, 1\} \) is constant, where \( \{0, 1\} \) is given the discrete topology.

5. If \( f : X \to Y \) is continuous, surjective, and \( X \) is connected, then \( Y \) is connected.

6. The unit interval \([0, 1]\) is connected.

7. If \( X \) is path connected, then \( X \) is connected.

8. If \( X \) is connected and locally path connected, then \( X \) is path connected.

9. If \( A \subset X \) and \( A \) is connected, then \( \overline{A} \) is connected.

10. The components of \( X \) are closed sets.

11. If \( f : X \to Y \) is continuous, surjective, and \( X \) is compact, then \( Y \) is compact.

12. If \( F \subset X \), where \( X \) is compact and \( F \) is closed, then \( F \) is compact.

13. If \( A \subset X \), where \( X \) is Hausdorff and \( A \) is compact, then \( A \) is closed.

14. If \( f : X \to Y \) is continuous, where \( X \) is compact and \( Y \) is Hausdorff, then \( f \) is a closed map \((f(F) \text{ is closed for all closed sets } F)\).

15. If \( f : X \to Y \) is a continuous bijection, where \( X \) is compact and \( Y \) is Hausdorff, then \( f \) is a homeomorphism.

16. The unit interval \([0, 1]\) is compact.

17. If \( X \) and \( Y \) are compact, so is \( X \times Y \).

18. Closed and bounded sets in \( \mathbb{R}^n \) are compact.