Definitions

1. Definitions concerning metric spaces. See Chapter 2 of the text. Whenever they are different from the ones in the text, we adopt the ones here.

(a) A metric space \((X, d)\) is a set \(X\) and a function \(d : X \times X \to \mathbb{R}\) satisfying

i. For all \(x, y \in X\), \(d(x, y) \geq 0\) and \(d(x, y) = 0\) if and only if \(x = y\).

ii. For all \(x, y \in X\), \(d(x, y) = d(y, x)\).

iii. For all \(x, y, z \in X\), \(d(x, z) \leq d(x, y) + d(y, z)\) (triangle inequality).

(b) Suppose \((X, d)\) is a metric space.

i. If \(x \in X\) and \(r > 0\), the set \(B(x, r) = \{y \in X : d(x, y) < r\}\) is called the ball of radius \(r\) centered at \(x\).

ii. A subset \(U \subset X\) is called an open set if and only if, for all \(x \in U\) there exists an \(r > 0\) such that \(B(x, r) \subset U\).

iii. A subset \(F \subset X\) is called a closed set if and only if its complement \(X - F\) is an open set.

(c) Suppose \((X, d)\) is a metric space and \(A \subset X\).

i. The interior of \(A\), denoted by \(A^\circ\) is defined by

\[ A^\circ = \bigcup\{U \subset X : U\text{ is open in }X \text{ and } U \subset A\}. \]

Thus \(A^\circ\) is the largest open set contained in \(A\).

ii. The closure of \(A\), denoted by \(\bar{A}\) is defined by

\[ \bar{A} = \bigcap\{F \subset X : F\text{ is closed and } A \subset F\}. \]

Thus \(\bar{A}\) is the smallest closed set containing \(A\).

(d) Let \((X, d)\) be a metric space and let \(\{x_n\}\) be a sequence in \(X\).

i. \(\lim\{x_n\} = x\) means that for all \(\epsilon > 0\) there exists a natural number \(N\) so that \(d(x, x_n) < \epsilon\) for all \(n > N\).

ii. The sequence \(\{x_n\}\) is a Cauchy sequence if for all \(\epsilon > 0\) there exists a natural number \(N\) so that \(d(x_m, x_n) < \epsilon\) for all \(m, n > N\).

iii. \((X, d)\) is called complete if every Cauchy sequence has a limit.

(e) Some metrics on \(\mathbb{R}^n\) and \(S^n\):

i. \(d(2)(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}\).

ii. \(d(\infty)(x, y) = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}\).

iii. \(d(1)(x, y) = |x_1 - y_1| + \cdots + |x_n - y_n|\).

iv. The “French railway metric” on \(\mathbb{R}^2\):

\[ d_{FR}(x, y) = \begin{cases} d(2)(x, y) & \text{if } x \text{ and } y \text{ are in the same ray from origin} \\ d(2)(0, x) + d(2)(0, y) & \text{otherwise} \end{cases} \]
v. On any set $X$, the *discrete metric* on $X$ is defined by
\[
d_{\text{disc}}(x, y) = \begin{cases} 
0 & \text{if } x = y \\
1 & \text{if } x \neq y
\end{cases}
\]

vi. On the unit sphere $S^n = \{x \in \mathbb{R}^{n+1} : d(2)(0, x) = 1\}$, the *spherical metric* is defined by
\[
d_S(x, y) = \cos^{-1}(x \cdot y).
\]

(f) Let $(X, d)$ and $(X', d')$ be metric spaces, and let $f : X \to X'$.

i. $f$ is *continuous* if for all $x \in X$ and all $\epsilon > 0$ there is a $\delta > 0$ such that $d'(f(x), f(y)) < \epsilon$ for all $y \in X$ with $d(x, y) < \delta$.

ii. $f$ is a *homeomorphism* if $f$ is continuous, $f^{-1}$ exists, and $f^{-1}$ is continuous.

iii. $f$ is *Lipschitz* if there is a constant $C > 0$ so that $d'(f(x), f(y)) \leq Cd(x, y)$ for all $x, y \in X$.

iv. $f$ is *bi-Lipschitz* if there exist constants $C_1, C_2 > 0$ so that
\[
C_1 d(x, y) \leq d'(f(x), f(y)) \leq C_2 d(x, y)
\]
for all $x, y \in X$.

v. $f$ is an *isometry* if $d'(f(x), f(y)) = d(x, y)$ for all $x, y \in X$.

(g) Equivalence relations between metric spaces

i. $(X, d)$ and $(X', d')$ are *homeomorphic* if there exists a homeomorphism $f : (X, d) \to (X', d')$.

ii. $(X, d)$ and $(X', d')$ are *bi-Lipschitz equivalent* if there exists a surjective map $f : (X, d) \to (X', d')$ which is bi-Lipschitz.

iii. $(X, d)$ and $(X', d')$ are *isometric* if there exists a surjective map $f : (X, d) \to (X', d')$ which is an isometry.

2. Definitions concerning topological spaces. See Chapter 3 of the text.

(a) A *topological space* $(X, T)$ is a set $X$ and a subset $T \subset 2^X$ satisfying:

i. $T$ is closed under arbitrary union: If $\{U_\alpha\}$ is any collection of elements of $T$, then $\bigcup U_\alpha \in T$.

ii. $T$ is closed under finite intersections: if $U_1, \ldots, U_n \in T$, then $U_1 \cap \ldots \cap U_n \in T$.

iii. $X \in T$ and $\emptyset \in T$.

(b) If $(X, T)$ is a topological space, the elements of $T$ are called the *open sets*.

(c) If $(X, T)$ is a topological space, a subset $F \subset X$ is called *closed* if its complement $X - F$ is open.

(d) If $(X, T)$ is a topological space and $A \subset X$, the interior $A^\circ$ and closure $\bar{A}$ of $A$ are defined by the same definition given above for subsets of metric spaces.

(e) If $(X, T)$ and $(X', T')$ are topological spaces a map $f : X \to X'$ is *continuous* if for all $U \in T'$ we have that $f^{-1}(U) \in T$ (the inverse image of each open set is open).
Theorems

1. Let \((X, d), (X', d'), \) etc, be metric spaces.
   (a) For all \(x \in X\) and \(r > 0\), \(B(x, r)\) is an open set.
   (b) The collection of open sets in \((X, d)\) is closed under arbitrary unions and finite intersections.
   (c) A function \(f : (X, d) \rightarrow (X', d')\) is continuous if and only if, for all open sets \(U \subset X'\), \(f^{-1}(U)\) is open in \(X\).
   (d) A function \(f : (X, d) \rightarrow (X', d')\) is continuous if and only if, for all closed sets \(F \subset X'\), \(f^{-1}(F)\) is closed in \(X\).
   (e) A Lipschitz function \(f : (X, d) \rightarrow (X', d')\) is continuous.
   (f) A surjective bi-Lipschitz map \(f : (X, d) \rightarrow (X', d')\) is a homeomorphism.
   (g) The metrics \(d(2), d(1), d(\infty)\) on \(\mathbb{R}^n\) are all bi-Lipschitz equivalent (meaning that the identity map is bi-Lipschitz). More precisely, the following inequalities are true:
      i. \(d(\infty)(x, y) \leq d(2)(x, y) \leq \sqrt{n} d(\infty)(x, y)\).
      ii. \(d(\infty)(x, y) \leq d(1)(x, y) \leq n d(\infty)(x, y)\).
      iii. \(d(2)(x, y) \leq d(1)(x, y) \leq \sqrt{n} d(2)(x, y)\).
   (h) If \(x, y \in X\) and \(x \neq y\), there exist open sets \(U, V \subset X\) such that \(x \in U\), \(y \in V\) and \(U \cap V = \emptyset\).
   (i) If \(\{x_n\}\) is a sequence in \(X\), \(\lim x_n = x\) and \(\lim x_n = y\), then \(x = y\).
   (j) If \(A \subset X\), then
\[
A^o = \{x \in A : \text{there exists an } r > 0 \text{ such that } B(x, r) \subset A\},
\]
and
\[
\bar{A} = \{x \in X : \text{for all } r > 0, B(x, r) \cap A \neq \emptyset\}
\]
   (k) The set \(Isom(X, d)\) of all isometries of \((X, d)\) forms a group under composition. Isometric spaces have isomorphic isometry groups.
   (l) \(f \in Isom(\mathbb{R}^n, d(2))\) if and only if there exists an orthogonal \(n\) by \(n\) matrix \(A\) and a vector \(b \in \mathbb{R}^n\) such that \(f(x) = Ax + b\).
   (m) \(f \in Isom(S^n, d_S)\) if and only if there exists an orthogonal \(n + 1\) by \(n + 1\) matrix \(A\) such that \(f(x) = Ax\).

2. Let \((X, \mathcal{T}), (X', \mathcal{T}'), \) etc, be topological spaces.
   (a) The composition of continuous maps is continuous.
   (b) If \(A \subset X\), there is a characterization of \(A^o, \bar{A}\) similar to the characterization for metric spaces, using neighborhoods instead of balls in the statement. (see section 4 of Chapter 3 of the text).
   (c) \(A^o = X - (X - A), \bar{A} = X - (X - A)^o\). (See same section.)