Foundations of Analysis II
Week 12

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$$\int_0^{\pi}\int_0^{2\pi} f(\varphi,\theta)\, d\varphi\, d\theta$$

int a function
$d\varphi \cdot d\theta$ "area"

int a 1-form.
$f(\varphi,\theta)\, d\varphi \wedge d\theta$

$f(t)\,dt$

$[c,d] \xrightarrow{g} [a,b] \xrightarrow{f}$

$t = g(s)$

$g^*(f(t)\,dt)$

$= f(g(s))\, g'(s)\, ds$

Change of var formula: $\int_c^d g^*(f(t)\,dt) = \int_a^b f(t)\,dt$

provided $g'(s) > 0$

$I \xrightarrow{g} J$

$f(t)\,dt$ ← infinitesimals

$$\int_I f(g(s))\,|g'|\,ds = \int_J f(t)\,dt$$

Leibniz

→ ε, δ ...

$f(t)\,dt$

$f : V \to U$, $V \subset \mathbb{R}^m$, $U \subset \mathbb{R}^n$

$f^*\left(a(x)\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}\right)$

$A^k(V) \xleftarrow{f^*} A^k(U)$

$= a(f(x))\, df_{i_1} \wedge \cdots \wedge df_{i_k}$

$a \in A^0(U)$

$f^* a = a(f(x))$

$f^*(dx_i) = df_i$

$\wedge$ mult

⇒ everything

$T \subset \mathbb{R}^3$

$R = [0,2\pi] \times [0,\pi]$

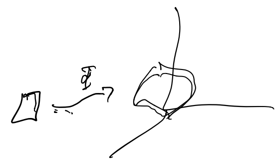
$R \to \mathbb{R}^3$

$\Phi^*(dx \wedge dy)$

$dx \wedge dy \in A^2(\mathbb{R}^3)$

$\subset \mathbb{R}^3$

$$\int_T dx\wedge dy = \int_R \Phi^*(dx\wedge dy)$$



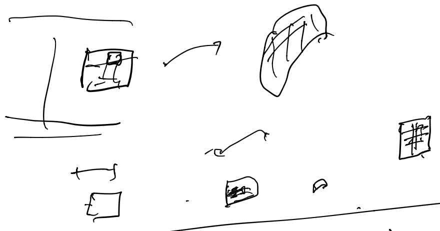
"k dim surf"

surf $\Phi : R \to \mathbb{R}^n$

$[\,]^k$

k-class

int in $\mathbb{R}^k$



Surf S

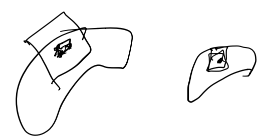
$p\in S$ $\quad T_pS$

= tang vector

Integrate

$\omega\in A^k(\mathbb{R}^n)$

U



$p\in S$

$p\in\mathbb{R}^n$

$T_p \cong \Lambda^k\mathbb{R}^n$

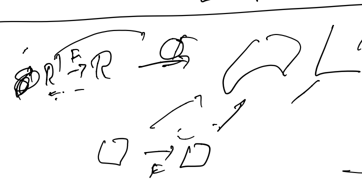
on param k-dim "manifold"

$\int \quad R \xrightarrow{\Phi} \mathbb{R}^n$, $R \subset \mathbb{R}^k$

C^1, $d\Phi$ injective $\forall x \in R$

$$\boxed{\int_\Phi \omega \overset{df}{=} \int_R \Phi^*\omega}$$

$\Phi:\ R' \xrightarrow{F} R \xrightarrow{\Phi}$



$\mathbb{R}^2$

$$\int_{\Phi\circ F}\omega \overset{?}{=} \int_\Phi \omega$$

$\forall \alpha\in A^k(R)$

$$\int_{R'} F^*\alpha = \int_R \alpha$$

$\alpha = \varphi(t)\,dt_1\wedge\cdots\wedge dt_k$

$$\int_{R'}(\Phi\circ F)^*\omega \qquad \int_R \Phi^*\omega$$

$\qquad\qquad\qquad /\!/$

$$\int_{R'} F^*\Phi^*\omega$$

$$\boxed{\int_{R'} \varphi(F(s))\,|\det(d_s F)|\, ds_1 \ldots ds_n = \int_R \varphi(x)\, dx_1 \ldots dx_n}$$

↑ Real Thm!

Rudin proves for φ cont. with compact support

$\int_R \varphi\, dx_1 \ldots dx_n$ = iterated integrals.

$$\left(\left(\int_{a_1}^{b_1} \varphi(x_1, \ldots, x_n)\, dx_1\right) dx_2\right) \ldots$$

Order?

$$\int_c^d \left(\int_a^b f(x,y)\, dx\right) dy = \int_a^b \left(\int_c^d f(x,y)\, dy\right) dx$$

$f(x,y) = g(x)\,h(y)$ obvious

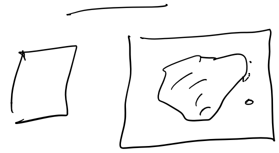
$$= \left(\int_a^b g(x)\, dx\right)\left(\int_c^d h(y)\, dy\right)$$

OK for $g(x)\,h(y)$

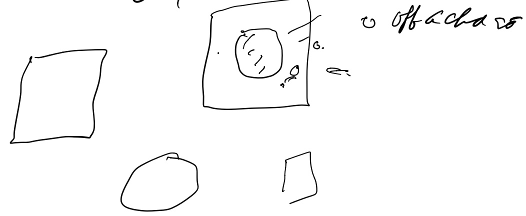
⇒ OK for sums & products.

Stone Weier → alg gen by products dense in $C([a,b] \times [c,d])$

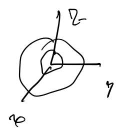
⇒ OK ∀ cont.



Let φ be continuous almost



Back $\int_T dx\,dy$



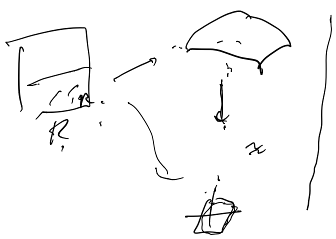
$dx\,dy$ on $\mathbb{R}^3$ = pull-back $dx\,dy$ in $\mathbb{R}^2$

$P(x,y,z) = (x,y)$

$$R \to T \xrightarrow{\pi} \mathbb{R}^2, \quad R_1 \subset R, \quad R_1 \xrightarrow{\pi\Phi} \mathbb{R}^2$$

$$\begin{array}{ccccc} R & \xrightarrow{\Phi} & T & \xrightarrow{\pi} & \mathbb{R}^2 \\ \cup & & \cup & & \\ R_1 & \longrightarrow & T_1 & \longrightarrow & \mathbb{R}^2 \end{array}$$

$$\int_{R_1} \Phi^*(dx\,dy) = \int_{\pi\Phi R_1} dx\,dy = \pm \text{Area}(\pi T_1)$$



$$\int_{R_1} \Phi^*\, dx\,dy = \pm \text{Area}(\pi T_1)$$

Orientation of $\mathbb{R}^2$: Choose a basis $e_1, \dots, e_n$ of $\mathbb{R}^k$

defining an orient.

=> The collection of all these has two connected comp.

$\underline{e} = A\,\underline{e}'$ A some $k \times k$ matrix

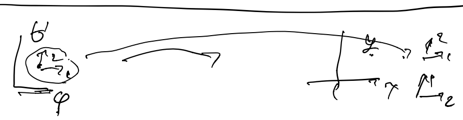
$\det(A) > 0 \Leftarrow e, e'$ same or.

$\det(A) < 0$ — e, e' diff. or.

Suppose.

Choose a component,

Call it positive



?

Rudin's pf of change of vars

1) if change F $\quad F(x_1,\dots,x_n) = (x_1,\dots,x_{m-1}, \alpha(x_1,\dots,x_n), \dots)$

$(x_1,\dots,x_{m-1}, \alpha(x_1,\dots,x_n), x_{m+1} \dots x_n)$

$$dF \begin{pmatrix} 1 \dots & & \\ \frac{\partial\alpha}{\partial x_1} \dots \frac{\partial\alpha}{\partial x_m} \dots & & \\ & \dots & \end{pmatrix} = \begin{pmatrix} 1 & & \\ \dots & \boxed{\frac{\partial\alpha}{\partial x_m}} & \dots \\ & & \dots \end{pmatrix}$$

$\Rightarrow$ OK $\int \varphi(F(x)) J(x) \dots = \int\int \varphi(x_1,\dots,\alpha(\dots),\dots) \frac{\partial\alpha}{\partial x_m} dx_1 \dots dx_n$

many 1-d

$=$

$\varphi(F(x)) |\det dF|$

1) $\int\dots\int \varphi(x_1,\dots,x_{m-1}, \alpha(x_1,\dots,x_n), x_{m+1} \dots x_n) \left|\frac{\partial\alpha}{\partial x_m}\right| dx_1 \dots \boxed{dx_m} \dots dx_n$

x_m'

$\left(\int \varphi(x_1 \dots x_n) dx_m \dots\right)$

by 1-variable thm..

2) if $F(x_1,\dots,x_n) = (x_1,\dots,x_j,\dots,x_i,\dots)$ OK

Let's prove that any F is locally a comp of (1)'s & (2)'s.

$F_1, F_2 \to F_1 \circ F_2$

~~$F(x,y) = ($~~

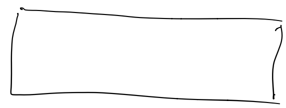
~~$F(x,y) = (f_1(x$~~

$F(x_1,x_2) = (f_1(x_1,x_2), f_2(x_1,x_2))$

$dF \to 0$, dF_0 invertible, one must be $\neq 0$

$$\begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{vmatrix} \neq 0 \quad \text{say } \frac{\partial f_1}{\partial x_1} \neq 0$$

$\frac{\partial f_1}{\partial x_1} \to (x_1, \tilde{f}_2(x_1,x_2))$



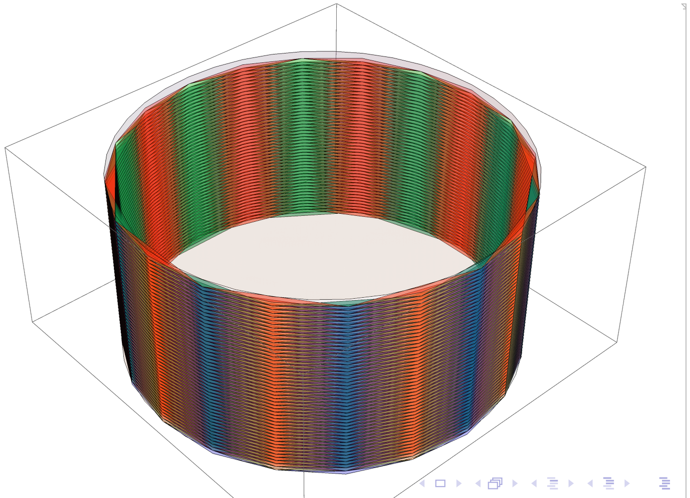
$\cancel{g(x_1, x_2}$

F_0

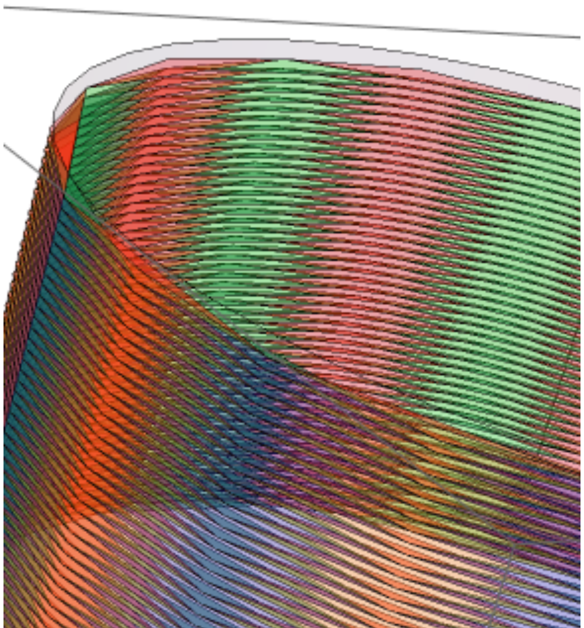
Schwarz Example

- H.A. Schwarz Example of a polyhedral surface approximating a torus uniformly, but far from smoothly,

Out[ ]=



Details



Observe

- The triangles have their vertices on the cylinder.
- The triangles are nearly horizontal
- Same: the triangles are nearly perpendicular to the cylinder
- The triangles are very thin: Their area much smaller than the square of their diameter.

Schwarz Inequality and Norm on $\Lambda^2(\mathbb{R}^n)$

$(u\cdot v)^2 \le (u,u)(v,v)$

consider $\quad (u,u)(v,v) - (u,v)^2$

$(u_1^2+\dots+u_n^2)(v_1^2+\dots+v_n^2) - (u_1v_1+\dots+u_nv_n)^2$

$= \quad u_i \dots -$

i.e. $\quad \sum \begin{vmatrix} u_i & u_j \\ v_i & v_j \end{vmatrix}^2 = \| u \wedge v \|^2$

$= \Leftrightarrow u\wedge v = 0 \Leftrightarrow u, v$ lin dep.

Change of variables formula

- Suppose E, D domains in $\mathbb{R}^k$, $\Phi : E \to D$ bijective, $d_s\Phi$ invertible for all $s \in E$. ($t = \Phi(s)$)
- Then for all continuous functions $f : D \to \mathbb{R}$

$$\int_E f(\Phi(s)) \left|\det(d_s\Phi)\right| ds_1 \ldots, ds_k = \int_D f(t) dt_1 \ldots dt_k$$

- Note how the absolute vaue $|\det(d\Phi)|$ appears, rathen than $\det(d\Phi)$. Results from orientation.

Next: prove for $k=2$ following Rudin's proof.

Integration in $\mathbb{R}^2$

$\mathbb{R} \to \mathbb{R}^2$ big jump

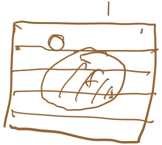
Rudin's

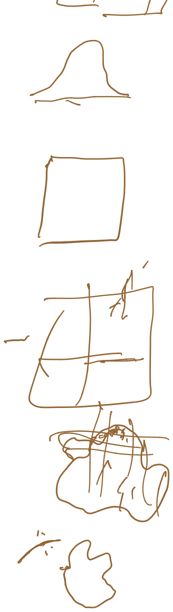
- $R = [a, b] \times [c, d]$ a rectangle in $\mathbb{R}^2$.
- $f : R \to \mathbb{R}$ continuous function
- Then $\int_R f(x, y)dxdy$ can be defined
  - By limit of Riemann sums.
  - As an iterated integral

Fubini's thm

$$\int_a^b \left(\int_c^d f(x, y)dy \right) dx = \int_c^d \left(\int_a^b f(x, y)dx \right) dy$$

  - This equality holds for $f \in \mathcal{C}(R)$. It's needed for the definition to make sense.
  - Rudin proves this quickly from special case $f(x, y) = g(x)h(y)$ and Stone-Weierstrass theorem





- Will take the iterated integral definition.
- Recall the *support of f* is

$$supp(f) = \overline{\{x : f(x) \neq 0\}}$$

- Let $\mathcal{C}_c(\mathbb{R}^2) \subset \mathcal{C}(\mathbb{R}^2)$ denote the space of continuous functions with *compact support*
- If $f \in \mathcal{C}_c(\mathbb{R}^2)$ can define

$$\int_{\mathbb{R}^2} f\, dxdy \quad \text{to be} \quad \int_R f\, dxdy$$

where R is any rectangle containing $supp(f)$ in its interior.
- Easily seen independent of R.

$\Sigma f(\xi_{ij})(x_i - x_{i-1})(y_j - y_{j-1})$

- Let $D \subset \mathbb{R}^2$ be a "domain of integration"
- A compact set with non-empty interior, boundary a finite union of $\mathcal{C}^1$ curves
- Examples:
  - Rectangle $[a, b] \times [c, d]$
  - Disk $x^2 + y^2 \leq r^2$
  - Triangle $0 \leq x \leq y \leq 1$
  - $x^2 + y^2 \leq 4, |x| \leq 1$

- If $supp(f) \subset D^0$ (interior of D) and R is a rectangle containing D, define

$$\int_D f \, dxdy = \int_R f \, dxdy$$

Cont

f ∈ compact support

- Good definition for $f \in \mathcal{C}_c(D^o)$.
- Too restrictive: want $\int_D f \, dxdy$ for all $f \in \mathcal{C}(D)$.
- A sensible definition: $\tilde{f}$ = extension of f to R by 0 on $R \setminus D$

replace by cont on

$$\int_D f \, dxdy = \int_R \tilde{f} \, dxdy$$

Riemann integral of the discontinuous function $\tilde{f}$ over R.

Change of variables formula in $\mathbb{R}^2$

- Suppose E, D domains in $\mathbb{R}^2$, $F : E \to D$ is of class $\mathcal{C}^1$, is bijective, and $d_{(u,v)}F$ invertible for all $(u, v) \in E$.
- (Thus $(x, y) = F(u, v) = (f(u, v), g(u, v))$ is a "change of variables")

$E \to D$

- Then for all continuous functions $\phi : D \to \mathbb{R}$

$E \to D$

$$\int_E \phi(F(u, v)) \, |\det(d_{(u,v)}F)| \, dudv = \int_D \phi(x, y) dxdy$$

- Note the absolute value $|\det(dF)|$ appears, rather than $\det(dF)$.

- Will only prove under the assumption $\phi \in \mathcal{C}_c(D^0)$
- Why is this difficult to prove?
- In one dimension, if $f : [c, d] \to [a, b]$ is strictly increasing, surjective, then

$$\int_c^d \phi(\overset{f}{\cancel{g}}(u))f'(u)du = \int_a^b \phi(x)dx$$

is easily proved from Riemann sums

$$\sum \phi(\xi_i)(x_i - x_{i-1}) = \sum \phi(\overset{f}{\cancel{g}}(\hat{\xi}_i))\overset{f}{\cancel{g}}'(\eta_i)(u_i - u_{i-1})$$

for some $\hat{\xi}_i, \eta_i \in [u_{i-1}, u_i]$

$\int_c^d \varphi(f(u))\, f'(u)\, du$

$f \nearrow$

$f(x) = x^2 \qquad f : [-1,1] \to [0,1]$

$$\int_{-1}^{1} \varphi(x^2)\,|2x|\,dx = \int_0^1 \varphi(y)\,dy$$

$\varphi(x) = x$

$$= \int_{-1}^{0} \varphi(x^2)\,|2x|\,dx + \int_0^1 \varphi(x^2)\,|2x|\,dx$$

$$\int_0^1 x\,dx = 1/2$$

$x = t^2$

$$\int_0^1 t^2 (2t)\,dt$$

$$2\,\frac{t^4}{4}\Big|_0^1 = 1/2$$

$x = t^2$

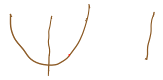
$$\int_{-1}^{0} -2t^3\,dt$$

$$-\frac{t^4}{2}\Big|_{-1}^{0} = 0 - (-1/2) = 1/2$$

$$\int -2t^3 = -\frac{t^4}{2}\Big|_{-1}^{0}$$

$$\int_{-1}^{1} t^2\,|2t|\,dt = \int_{-1}^{0} t^2(-2t)\,dt + \int_0^1 t^2(2t)\,dt$$

$\int 2i$



1 : mk $(-)$

2 : $f'(x) \neq 0$

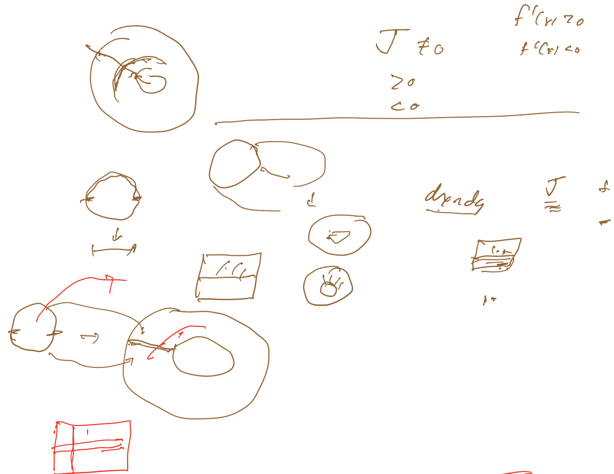
$f'(x) > 0$

$f'(x) < 0$

$J \neq 0$

> 0

< 0

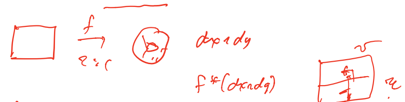


$dx \wedge dy$

J

$dx \wedge dy$ form

$|dx \wedge dy|$ "measure"



f, 2:1

$dx \wedge dy$

$f^*(dx \wedge dy)$

$= \varphi(u,v)\,du\,dv$

$\int \varphi(u,v)\,|\,du\,dv$

$[-1,1] \xrightarrow{x^2} [0,1]$ $f(x)=x^2$ $\int \varphi(x^2)$

$\int \varphi(x^2)|2x|\,dx = \int \varphi(y)\,dy$

$\int_d^c \varphi(f(x))\,f'(x)\,dx$

$= \int_d^c \varphi(f(x))(-f'(x))\,dx$ $|f'(x)|$

$\int_d^c f\,d$

- This is easy because g takes intervals to intervals.
- But in $\mathbb{R}^2$, F does not take rectangles to rectangles.
- F takes rectangular grids to “curvilinear” grids.
- Same for F^{-1}.

- A direct proof will have to take this into account.

- Two special transformations where change of variable formula is easy
- "Primitive transformation"

$$F(u, v) = (u, g(u, v)) \text{ or } F(u, v) = (f(u, v), v)$$

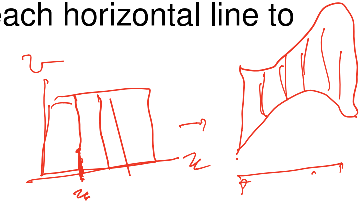
  moves at most *one* variable at a time (same in $\mathbb{R}^k$)
- "Flip" interchanges two coordinates

$$F(u, v) = (v, u)$$

Primitive Transformations

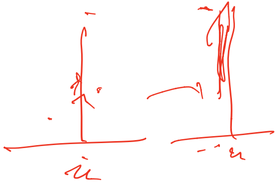
- Picture:
  $F(u, v) = (u, g(u, v))$ maps each vertical line to itself.

  $F(u, v) = (f(u, v), v)$ maps each horizontal line to itself.



$(u, v) \to (u, g(u, v))$

- Look at $(x, y) = F(u, v) = (u, g(u, v))$
-
$$dF = \begin{pmatrix} 1 & 0 \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{pmatrix}$$



- Jacobian determinant $\det(dF) = \frac{\partial g}{\partial v} \neq 0$.
- F^{-1} is also primitive:

$$F^{-1}(x, y) = (x, h(x, y))$$

where, for each $x = u$, $h(x, y)$ is the inverse of $g(x, v)$

$$g(x, h(x, y)) = y \quad \text{and} \quad h(u, g(u, v)) = v$$

$g_u(v)$

- If $E = [a, b] \times [c, d]$, then $D = F(E)$ is

$$a \leq x \leq b, \quad g(x, c) \leq y \leq g(x, d)$$

- If $F(E) \subset [a, b] \times [c', d']$ for some c', d'.
- If $supp(\phi) \subset D^0$, then

$$\int_{[a,b]\times[c',d']} \phi(x, y)\, dxdy = \int_a^b \left(\int_{g(x,c)}^{g(x,d)} \phi(x, y)\, dy \right) dx$$

- Finally, for each $u \in [a, b]$, the one-variable change of variables formula gives

$$\int_c^d \phi(u, g(u, v))|\frac{\partial g}{\partial v}|\ dv = \int_{g(u,c)}^{g(u,d)} \phi(u, y)dy$$

- Thus

$$\int_a^b \Big(\int_c^d \phi(u, g(u, v))|\det(d_{(u,v)}F)|dv\Big)du$$

equals

$$\int_a^b \Big(\int_{c'}^{d'} \phi(x, y)dy\Big)dx$$

which is the change of variable formula for F.

- Same argument gives change of variable formula for

$$F(u,v) = (f(u,v), v)$$

- Change of variables formula holds for all primitive transformations.
- It also holds for flips. It is equivalent to

$$\int_a^b \Big(\int_c^d \phi(x,y)dy \Big) dx = \int_c^d \Big(\int_a^b \phi(x,y)dx \Big) dy$$

The reduction to primitives and flips

- $F : E \to D$, $F(u,v) = (f(u,v), g(u,v))$ as above, $p \in E$.
- $$d_pF = \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{pmatrix}$$
- d_pF invertible $\Rightarrow$ at least one of $\frac{\partial f}{\partial u}, \frac{\partial g}{\partial u} \neq 0$
- Suppose $\frac{\partial f}{\partial u} \neq 0$.
- Let $G(u,v) = (f(u,v), v)$
  (a primitive map)

- Then
$$dG = \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ 0 & 1 \end{pmatrix}$$
- Inverse function theorem $\Rightarrow$ G invertible in a nbd of p, and
$$G^{-1}(x,y) = (h(x,y), y)$$
where $h(f(u,v),v) = u,\ f(h(x,y),y) = x.$
- Therefore
$$F(G^{-1}(x,y)) = (f(h(x,y),y), g(h(x,y),y)) = (x, g_2(x,y))$$
is a primitive map.

- Conclusion: Let $\tilde{F} = F \circ G^{-1}$.
- Then, restricted to a nbd of p,

$$F = \tilde{F} \circ G$$

is the composition of two primitive maps.

- Suppose $\frac{\partial f}{\partial u} = 0$.
- Then $\frac{\partial g}{\partial u} \neq 0$.
- Let $G(u, v) = (g(u, v), v)$

- As before G is invertible in nbd of p and

$$G^{-1}(x, y) = (h(x, y), y)$$

where $g(h(x, y), y) = x$
- Let B be the flip. Then

$$(B \circ F)(u, v) = (g(u, v), f(u, v))$$

- Let $\tilde{F} = B \circ F \circ G^{-1}$.
- Then

$$\tilde{F}(x, y) = (g(h(x, y), y), f(h(x, y, y))) = (x, func(x, y))$$

is primitive.

- Rewrite the last equation

$$F = B \circ \tilde{F} \circ G$$

where B is a flip, $\tilde{F}$ and G are primitive

- Put the two cases together:

$$F = B \circ \tilde{F} \circ G$$

where B is either the identity or a flip, $\tilde{F}$ and G are primitive.

Main Theorem

Theorem

Let E, D be domains in $\mathbb{R}^2$ and let $F : E \to D$ be $\mathcal{C}^1$, bijective, and d_pF invertible for all $p \in E^0$. Then every $p \in E^0$ has a nbd U_p such that there exist maps B, F_1, G_1 where

- *$B =$ the identity or a flip.*
- *F_1 and G_1 are primitive.*
- *$F|_{U_p} = B \circ F_1 \circ G_1$*

Change of Variables Theorem

Theorem

Let E, D be domains in $\mathbb{R}^2$ and let $F : E \to D$ be $\mathcal{C}^1$, bijective, and d_pF invertible for all $p \in E^0$. Let $\phi \in \mathcal{C}_c(D^0)$. Then

$$\int_E \phi(F(u,v))) \, |\det(d_{(u,v)}F)| \, dudv = \int_D \phi(x,y)dxdy$$

Proof

- Let $\{U_p\}_{p \in supp(F^*\phi)}$ be the open cover of $supp(F^*\phi)$ by the open sets U_p of the Main Theorem.
- Let $\{U_{p_i}\}$ be a finite subcover
- There are functions $\phi_i \in \mathcal{C}_c(F(U_{p_i}))$ such that $\phi = \sum \phi_i$. (Standard "partition of unity" argument.)
- Suffices to prove theorem for ϕ with $supp(\phi_i) \subset F(U_{p_i})$ for a single i.
- In U_{p_i} can write F as a composition of flips and primmitive mappings.
- Since the change of variables formula for two maps implies the fomula for their composition, we're done.

Summary of Differential Forms

- $U \subset \mathbb{R}^n$ open, $0 \leq k \leq n$
- $A^k(U) =$ differential k-forms on U
- Each $\alpha \in A^k(U)$ has unique expression

$$\alpha = \sum_I a_I(x) dx_I$$

over all stricltly increasing multi-indeces of length k.

  - $I = \{i_1, \ldots, i_k\}$ where $1 \leq i_1 < \cdots < i_k \leq n$.
  - $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$
  - $a_I : U \to \mathbb{R}$ smooth functions.

- May write simply $A(U)$ for the collection of all k-forms for all k.
- Say $deg(\alpha) = k \iff \alpha \in A^k(U)$.
- Mulriplication $A^k(U) \times A^\ell(U) \to A^{k+\ell}(U)$ as defined in class.
- Multiplication $\alpha \wedge \beta$ satisfies

$$\beta \wedge \alpha = (-1)^{deg(\alpha) deg(\beta)} \alpha \wedge \beta$$

Operations on forms

$a(x)$, dx_i 3 key properties

- Since $A(U)$ is generated by $A^0(U)$and the $dx_1 \in A^1(U)$, operations with reasonable multiplicative properties are uniquely determined by their values on $A^0(U)$ and the dx_i
- Two main examples:
  - Exterior derivative $d : A^k(U) \to A^{k+1}(U)$.
  - Pull-back $f^* : A^k(U) \to A^k(V)$ for a smooth map $f : V \to U$

Exterior Derivative $d : A^k(U) \to A^{k+1}(U)$

- Uniquely determined by requiring:
  - If $f \in A^0(U)$ a smooth function, then df is the usual derivative.

$$df = \sum_1^n \frac{\partial f}{\partial x_i} dx_i$$

  - $d(dx_i) = 0$
  - This version of the Leibnitz rule (product rule)

$$\boxed{d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{deg(\alpha)} \alpha \wedge d\beta}$$

- Must have for each stricly increasing multi-index I

$$d(a_I dx_I) = (da_I) \wedge dx_I$$

$d(a\, dx_{i_1} \cdots dx_{i_k}) = (da) \wedge dx_{i_1} \wedge \cdots \wedge$

$d^2 = 0$

$d(dx_i) = 0$

$+ a\, d(dx_{i_1} \wedge \cdots \wedge dx_{i_k})$

$d(a_1 dx_1 + a_2 dx_2)$

$= da_1\, dx_1 + da_2\, dx_2$

$\frac{\partial a_1}{\partial x_2} dx_2 \wedge dx_1 + \frac{\partial a_2}{\partial x_1} dx_1 \wedge dx_2$

$= \left(\frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2}\right) dx_1 \wedge dx_2$

- $d(df) = d(\sum_1^n \frac{\partial f}{\partial x_i} dx_i) = \sum_1^n d(\frac{\partial f}{\partial x_i}) \wedge dx_i$
- Expand

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_i} dx_j \wedge dx_i$$

- Group i, j and j, i together:

$$d^2 f = \sum_{i<j} \left(\frac{\partial^2 f}{\partial x_i \partial x_j} - \frac{\partial^2 f}{\partial x_j \partial x_i}\right) dx_i \wedge dx_j$$

- If f is of class $\mathcal{C}^2$, $d^2 f = 0$.

$d(\sum a_I\, dx_I)$

$= \sum \frac{\partial a_I}{\partial x_j} dx_j \wedge dx_I$

$= (\quad) da_I \wedge dx_I$

$\pm\left(\sum_{i<j}\left(\frac{\partial a_{j}}{\partial x_{i}}-\frac{\partial a_{i}}{\partial x_{j}}\right)dx_i\wedge dx_j\right)$

- If $f \in A^0(U)$, f is smooth, so $d^2 f = 0$.
- In other words, the map

$$d^2 : A^0(U) \to A^2(U)$$

  is the zero map.
- It follows that for all k

$$d^2 : A^k(U) \to A^{k+2}(U)$$

  is always zero
- If $a\,dx_I \in A^k(U)$, then
  $d(d(a\,dx_i)) = d(da \wedge dx_i) = \underline{\underline{d^2 a \wedge dx_i = 0}}$

Pull-back

- If $V \subset \mathbb{R}^m$ open and $f : V \to U$ smooth, where $x = f(t)$, written explicitly is

$$(x_1, \dots, x_n) = (f_1(t_1, \dots, t_m), \dots, f_n(t_1, \dots t_m)),$$

  then

$$f^* : A^k(U) \to A^k(V)$$

  is uniquely determined by
  - $(f^* a)(t) = a(f(t))$ for all $a \in A^0(U)$
  - $f^*(dx_i) = df_i$
  - $f^*(\alpha \wedge \beta) = f^*\alpha \wedge f^*\beta$
    for all $\alpha \in A^k(U), \beta \in A^\ell(U)$
- Must have

$$f^*(a\, dx_I) = (f^* a)\, df_{i_1} \wedge \cdots \wedge df_{i_k}$$

- Finally two important identities:
  - For all $\alpha \in A^k(U)$

    $$f^*(d\alpha) = d(f^*\alpha)$$

    briefly

    $$df^* = f^*d$$

  - If $W \subset \mathbb{R}^\ell$ is open and $g : W \to V$ smooth,

    $$(f \circ g)^*\alpha = g^*(f^*\alpha)$$

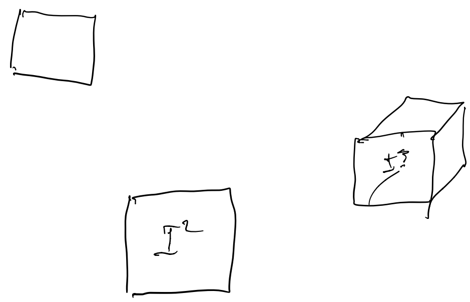
    for all $\alpha \in A^k(U)$
    briefly

    $$(f \circ g)^* = g^* \circ f^*$$

Cubical Chains

Rudin: simplicial chains
(Spivak: Calculus on Mfds: cubical)

- Let $I^k = \{(t_1, \dots t_k) \in \mathbb{R}^k \;:\; 0 \leq t_i \leq 1$ be the *standard cube*.
- If $U \subset \mathbb{R}^n$ is open, a *singular* k*- cube* in U is a smooth map
$$\sigma : I^k \to U$$
- Let $Q_k(U)$ denote the set of all singular k-cubes in U.
- Let $S_k(U)$ be the $\mathbb{R}$-vector space with basis $Q^k(U)$
- Thus the elements of $S^k(U)$ are finite linear combinations
$$\sum_{\sigma \in Q_k(U)} a_\sigma \sigma$$
with $a_\sigma \in \mathbb{R}$ and $a_\sigma \neq 0$ for only finitely many σ.



for each k one fixed k-dim cube

Standard cube $I^k = [0,1]^k$

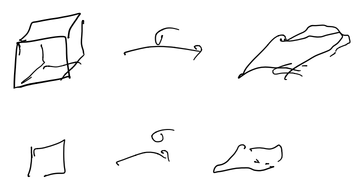
$= \{(t_1,\dots,t_k) \in \mathbb{R}^k$
$0 \le t_i \le 1\}$

U open in $\mathbb{R}^2$

or any space

Singular cube in U:

$\sigma : I^k \to U$ smooth map



Chain: linear comb of sing cubes with $\mathbb{R}$-coeffs

~~Huge Vector~~ space

$\cancel{S_k(U)} =$

Huge set $Q_k(U)$

$= \{$all sing cubes in $U\}$

$= \{$all smooth $\sigma : I^k \to U\}$

Huge Vector Space:

$\mathbb{R}$-Vectorspace with basis Q.

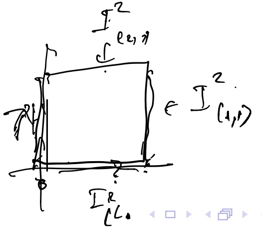
$\{ \sum_\sigma a_\sigma \sigma \,;\, a_\sigma = 0$ except for finitely many $\sigma \}$

σ.

- $\sum_{\sigma \in Q_k(U)} a_\sigma \sigma = \sum_{\sigma \in Q_k(U)} b_\sigma \sigma$
  $\Longleftrightarrow$
  $a_\sigma = b_\sigma$ for all $\sigma \in Q_k(U)$
- $\sum a_\sigma \sigma + \sum b_\sigma \sigma = \sum (a_\sigma + b_\sigma)\sigma$
- $a(\sum a_\sigma \sigma) = \sum (a a_\sigma)\sigma$.
- The elements of $S_k(U)$ are called *singular cubical chains* in U.

I^k

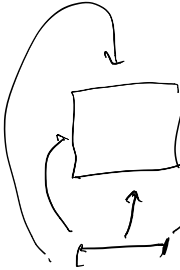
$I^k_{(i,\varepsilon)}$ $\quad I^2_{(1,0)}$



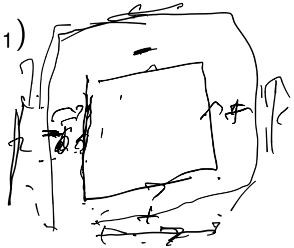
$Q_{k-1}(I^k)$

- $id : I^k \to I^k$ is an element of $Q_k(I^k)$.
- Faces of I^k come in pairs: as subsets
  $I^k_{i,\epsilon} = \{t_1, \ldots .t_{i-1}, \epsilon, t_{i+1}, \ldots .t_k) \in I^k\}$ for $\epsilon = \cancel{\pm 1}$ 0 or 1
- As singular $k-1-$ cubes in I^k the maps $\phi^k_{i,\epsilon}$

$$\phi^k_{i,\epsilon}(t_1, \ldots, t_{k-1}) = (t_1, \ldots, t_{i-1}.\epsilon, t_i, \ldots, t_{k-1})$$



I^{k-1}

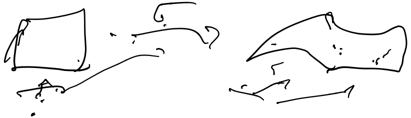


- Define $\partial(I^k)$, the boundary of I^k, to be

$$\boxed{\partial I^k = \sum (-1)^{i+\epsilon} \phi^k_{i,\epsilon} \in S_{k-1}(I^k)}$$

$\varphi^1_{1,0}(t_1) = (0, t_1)$

$$\boxed{-\varphi_{1,0} + \varphi_{1,1} + \varphi_{2,0} - \varphi_{2,1}}$$



- If $\sigma : I^k \to U$ is a singular k cube in U, define its boundary to be

$$\boxed{\partial\sigma = \sum_{i,\epsilon} (-1)^{i+\epsilon} \sigma \circ \phi^k_{i,\epsilon}}$$

- If $c = \sum_\sigma a_\sigma \sigma$ is a singular k-chain in U, define its boundary to be

$$\partial c = \partial\left(\sum a_\sigma \sigma\right) = \sum a_\sigma \partial\sigma$$

- Check $\partial^2 = 0$

- If $\alpha \in A^k(U)$ is a k-form and $c = \sum a_\sigma \sigma \in S_k(U)$ is a k-chain, define

$$\int_c \alpha = \sum_\sigma a_\sigma \int_\sigma \alpha = \sum_\sigma a_\sigma \boxed{\int_{I^k} \sigma^* \alpha}$$

- **Theorem (Stokes's Theorem)**

  For all $\alpha \in A^{k-1}(U)$ and for all $c \in S_k(U)$

$$\boxed{\int_c d\alpha = \int_{\partial c} \alpha}$$

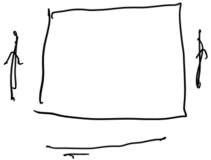
$$\int \sigma^* \alpha = \int d\sigma^* \alpha \qquad \forall \alpha \in A^k(I^k)$$

$$\int_{\partial I^k} \qquad \int_{I^k}$$

$$\forall\ \eta \in A^{k-1}(I^k)$$

$$\boxed{\int_{\partial I^k} \eta = \int_{I^k} d\eta}$$

$$\eta = \varphi_{i}\, f_i \cdot dx_1 \cdots \widehat{dx_i} \cdots dx^k$$



$k = 2$ 1-form on I^2

$$f\,dx + g\,dy$$

$$\int_{\partial I} = \sum (-1)^{i+\epsilon} \varphi^{2}_{i,\epsilon} + (\ ,$$

$\frac{\partial f}{\partial y}\, dx\, dy$

$\frac{\partial g}{\partial x}\, dx$

$i=1 \qquad I$

$\int g\, dy \qquad \int f\, dx$

$\int f\, dy \quad \cancel{f(x, y)}$

$d\alpha\,?$

$\alpha = A^1(U)$

$d\alpha\,?$

$$\int_{\partial D} \alpha = \int_{D} d\alpha$$

$$\frac{1}{\mathrm{vol}(D)}\left(\int_{\partial D} \alpha\right) \to d\alpha$$