Foundations of Analysis II
Week 11

Domingo Toledo

University of Utah

Spring 2019

Integration

- Problem: define integration over k-dimensional subsets of $\mathbb{R}^n$.
- In this context need to understand:
  - Integrands
  - Domains of integration
  - Integrals

Model: Line integrals ($k = 1$)

- Looked at
$$\int_\gamma \omega$$
where
  - $U \subset \mathbb{R}^n$ open.
  - $\gamma : [a, b] \to U$ a parametrized curve.
  - ω a one-form on U (written $\omega \in A^1(U)$)
- One-form on U means a function
$$\omega : U \times \mathbb{R}^n \to \mathbb{R} \text{ written } \ (x, v) \to \omega_x(v)$$
which is smooth in x and *linear* in v

- By definition

$$\int_\gamma \omega = \int_a^b \omega_{\gamma(t)}(\gamma'(t))dt \tag{1}$$

- There exists a unique collection of smooth functions $p_1 \dots, p_n : U \to \mathbb{R}$ such that

$$\omega = \sum_{i=1}^{n} p_i \, dx_i$$

- Writing $\gamma(t) = (\gamma_1(t), \dots, \gamma_n(t))$

$$\int_\gamma \omega = \int_a^b \left(\sum_{i=1}^{n} p_i(\gamma(t)) \, \gamma_i'(t) \right) dt$$

Pull-back

- Give the integrand in (1) a name.
- $$\gamma^*\omega = \omega_{\gamma(t)}(\gamma'(t))dt \tag{2}$$
  is called the *pull-back* of ω to $[a, b]$
- The definition of line integral now reads

$$\int_\gamma \omega = \int_a^b \gamma^*\omega$$

Independence of Parametrization

- $\gamma : [a, b] \to U$ smooth curve.
- $\phi : [c, d] :\to [a, b]$ smooth, strictly increasing and surjective.
- $\tilde{\gamma} = \gamma \circ \phi : [c, d] \to U$
- Then for all $\omega \in A^1(U)$

$$\int_{\tilde{\gamma}} \omega = \int_{\gamma} \omega$$

- Follows from the change of variables formula for integrals

Higher Dimensions

- For k-dimensional integration in an open set $U \subset \mathbb{R}^n$ we will need to define the corresponding objects:
  - Integrands: smooth k-forms $\omega \in A^k(U)$.
  - Domains of integration: smooth maps $\Phi : D \to U$, where D is a domain in $\mathbb{R}^k$.
  - Integral:

$$\int_\Phi \omega = \int_D \Phi^* \omega \tag{3}$$

- For this to make sense need

  - Definition of k forms $\omega \in A^k(U)$
  - Definition of pull-back $\Phi^* : A^k(U) \to A^k(D)$
  - Definition of the integral

$$\int_D : A^k(D) \to \mathbb{R}$$

- If $\eta \in A^k(D)$, where D is a domain in $\mathbb{R}^k$, then

$$\int_D \eta = \int_D \phi(t_1, \ldots, t_k) dt_1 \ldots dt_k$$

  an ordinary multiple integral.

- For all this to have geometric meaning, wnat *Independence from parametrization*

- This should follow from change of variable formula for mutiple integrals.

Summary

Differential forms reduce the theory of integration over k-dimensional subspaces of $\mathbb{R}^n$ to ordinary multiple integrals over domains $D \subset \mathbb{R}^k$.

line int $\to \int_a^b \cdots dt$

Surf int $\to \int\int f(\cdots)\, dt_1 dt_2$

Domains of integration

k-dim int
in $\mathbb{R}^k$

- Want to reduce to $\mathbf{I}^k$, the cartesian product of k intervals:

$$\mathbf{I}^k = \prod_{i=1}^{k} [a_i, b_i], \quad a_i, b_i \in \mathbb{R}, \quad a_i < b_i. \tag{4}$$

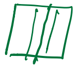
- Not open.
- If $C \subset \mathbb{R}^k$ is compact, say $f : C \to \mathbb{R}$ is smooth if it extends to a smooth function on some neighborhood of C.

- If $f : \mathbf{I}^k \to \mathbb{R}$ is continuous can define

$$\int_{\mathbf{I}^k} f(t_1, \dots, t_k)\, dt_1 \dots dt_k$$

as
  - Limit of Riemann sums
  - Iterated integral

← Rudin



$a \le s \le b$

$c \le t \le d$

$\int f(s,t)\, dt \rightsquigarrow$ fn of s

$f = h_1(t_1) h_2(t_2) \ldots h_k(t_k)$ $\int_a^b \left(\int_c^d f(s,t)\,dt \right) ds$

does this depend on order?

$\int = \prod \int_{a_i}^{b_i} h_i(t_i)\,dt_i$

Stone-Weierstrass

$\sum a_{i_1 \ldots i_k} h$

dense in $C(I \times \cdots \times I)$

- For $D \subset \mathbb{R}^k$, if $\bar{D}$ is compact, then $\exists$ sets $\mathbf{I}^k$ with $D \subset \mathbf{I}^k$.
- Given $f : D \to \mathbb{R}$ continuous, extend by 0 to $g : \mathbf{I}^k \to \mathbb{R}$, no longer continuous.
- If g is Riemann integrable, could define

$$\int_D f(t)dt_1 \ldots dt_k = \int_{\mathbf{I}^k} g(t)dt_1 \ldots dt_k$$

$f : D \to \mathbb{R}$, $D \subset I^k$

$g : I^k \to \mathbb{R}$

$g(x) = \begin{cases} f(x) & x \in D \\ 0 & x \notin D \end{cases}$

not cont.

if ∂D meas. 0, R-int.

Support of a function

- If X is a metric space and $f : X \to \mathbb{R}$ is continuous, the *support* of f is defined to be

closure

$$supp(f) = \overline{\{x \in X \ : \ f(x) \neq 0\}}$$

- If $f \in \mathcal{C}(\mathbb{R}^k)$ and $supp(f)$ is compact, can define

$$\int_{\mathbb{R}^k} f(t)dt = \int_{\mathbf{I}^k} f(t)dt$$

for any $\mathbf{I}^k$ containing $supp(f)$.

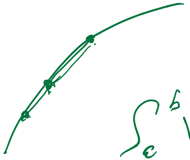
- Differntial forms should be (linear) functions on a space that contains the tangent spaces to the images of the maps σ

- Given vectors $v_1, \ldots, v_k \in \mathbb{R}^n$ linearly independent, want a way to manipulate the subspaces

$$\{t_t v_1 + \cdots + t_k v_k \;:\; 0 \le t \le 1\} \subset \mathbb{R}^n \tag{5}$$

(the parallelipiped spanned by $v_1, \ldots, v_k$)
and their volumes

$\int_a^b f(t)\,dt$

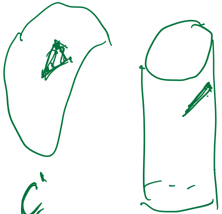


$\int_a^b \sqrt{\gamma'\cdot\gamma'}\,dt$ $\quad \gamma(t)$

Whitney

· Geometric Integration theory

$|\gamma| = \int_0^b |\gamma'(t)|\,dt$



C^1

$\sum |\gamma(t_i) - \gamma(t_{i-1})|$

$= \sum |\gamma'(\xi_i)(t_i - t_{i-1})|$

$= \sum |\gamma'(\xi_i)|(t_i - t_{i-1})$

$= \sum |\gamma'(\xi_i)|$

" Triangulation theorem"

The Grassmann Algebra

- For each k, $1 \leq k \leq n$ want a symbol

$$v_1 \wedge v_2 \wedge, \cdots \wedge v_k$$

  that represents the parallelipiped (7).
- Operations on these symbols that reflect the geometry.
- Example:

$$v_2 \wedge v_1 \wedge v_3 \cdots \wedge v_k = -v_1 \wedge v_2 \cdots \wedge v_k$$

  reflecting change of orientation.

Define The Grassmann Algebra

- Start with $\mathbb{R}^n$ with its standard inner product $x \cdot y$ and it standard ON basis $e_1, \ldots, e_n$

- For each increasing sequence I of k integers

$$I = \{i_1, \ldots, i_k\} \text{ with } 1 \leq i_1 < i_2 < \cdots < i_k \leq n$$

  define a symbol

$$e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}$$

- There are $\left(\begin{array}{c} n \\ k \end{array} \right) = \frac{n!}{k!(n-k)!}$ such symbols.

- For $k = 0, \ldots, n$ define spaces $\Lambda^k = \Lambda^k(\mathbb{R}^n)$ by
- $\Lambda^0 = \mathbb{R}$
- For $1 \leq k \leq n$,

  $\Lambda^k =$ the $\mathbb{R}$- vector space with basis $\{e_I : \text{ card } (I) = k\}$

- $\dim(\Lambda^k) = \binom{n}{k}$
- $\Lambda^1 =$ the original $\mathbb{R}^n$ with basis $e_1, \ldots, e_n$

Product

- If $I = \{i_1 < \cdots < i_k\}$ as above, write$|I| = k$
- If $|I| = k$ and $|J| = \ell$, define $e_I \wedge e_J$ by

$$e_I \wedge e_J = \begin{cases} 0 \text{ if } I \cap J \neq \emptyset, \\ \varepsilon(I, J) e_K \text{ if } I \cap J = \emptyset. \end{cases} \tag{6}$$

- K and $\varepsilon(I, J)$ defined as follows:
  - Let $I \cup J$ denote the sequence $\{i_1, \ldots, i_k, j_1, \ldots j_\ell\}$
  - K is the sequence $I \cup J$ arranged in increasing order.
  - $\varepsilon(I, J)$ is the sign of the permutation that takes $I \cup J$ to K.

- This determines a product

$$\Lambda^k \times \Lambda^l \to \Lambda^{k+l}$$

- If $a = \sum_I a_I e_I$ and $b = \sum_J b_J e_J$, then

$$a \wedge b = \sum_{I,J} a_I b_J e_I \wedge e_J$$

- This sum can be rewritten, using the definition of $e_I \wedge e_J$ above, as

$$\sum_K c_K e_K$$

  This is $a \wedge b$.

- Multiplication is associative

$$(a \wedge b) \wedge c = a \wedge (b \wedge c)$$

- Distributive law holds

$$(a + b) \wedge c = a \wedge c + b \wedge c$$

- If $a \in \Lambda^k$ and $b \in \Lambda^l$, then

$$b \wedge a = (-1)^{kl} a \wedge b$$

- $\Lambda^k(\mathbb{R}^n)$ has an inner product, with $\{e_I \;:\; |I| = k\}$ as ON basis.

- Explicitly, if $|I| = |J|$

$$(\sum_i a_i e_I) \cdot (\sum_J e_J) = \sum_I a_I b_I$$

  and 0 otherwise.

- The corresponding *norm* is

$$|a| = |\sum_I a_i e_I| = \sqrt{\sum_I a_I^2}$$

- If $v_1, \ldots, v_k \in \mathbb{R}^n$ are linearly independent, then

$$\boxed{v_1 \wedge \cdots \wedge v_k \in \Lambda^k(\mathbb{R}^n)}$$

represents the oriented parallelipiped (7)

$$\{\underbrace{t_1 v_1 + \cdots + t_k v_k}\ :\ 0 \leq t_i \leq 1\}$$

- The norm

$$|\underline{v_1 \wedge \cdots \wedge v_k}|$$

is the *k-dimensional volume* of the parallelipiped.

Vol parametrized k-dim surf in $\mathbb{R}^n$

$\int \partial \Phi$

$$\Phi : \underbrace{[a_1, b_1] \times \cdots \times [a_k, b_k]}_{D} \longrightarrow \mathbb{R}^n$$

C^1

$$\mathrm{Vol}(\Phi) = \int_D \left| \frac{\partial \Phi}{\partial t_1} \wedge \frac{\partial \Phi}{\partial t_2} \wedge \cdots \wedge \frac{\partial \Phi}{\partial t_k} \right| dt_1 \cdots dt_k$$

$k = 2$ area

$(x_1, \ldots, x_n) = \Phi(t_1, t_2)$

$x_1(t_1, t_2), \ldots, x_n(t_1, t_2)$

$$\iint \left| \frac{\partial \Phi}{\partial t_1} \wedge \frac{\partial \Phi}{\partial t_2} \right| dt_1\, dt_2$$

$$\frac{\partial \Phi}{\partial t_1} = \left(\frac{\partial x_1}{\partial t_1}, \frac{\partial x_2}{\partial t_1}, \ldots \right)$$

$$= \frac{\partial x_1}{\partial t_1} e_1 + \frac{\partial x_2}{\partial t_1} e_2 + \cdots$$

$$\left(\frac{\partial x_1}{\partial t_1} e_1 + \frac{\partial x_2}{\partial t_1} e_2 + \cdots + \frac{\partial x_n}{\partial t_1} e_n \right) \wedge \left(\frac{\partial x_1}{\partial t_2} e_1 + \cdots + \frac{\partial x_n}{\partial t_2} e_n \right)$$

$$= \sum_{i<j} \begin{vmatrix} \frac{\partial x_i}{\partial t_1} & \frac{\partial x_i}{\partial t_2} \\ \frac{\partial x_j}{\partial t_1} & \frac{\partial x_j}{\partial t_2} \end{vmatrix} e_i \wedge e_j$$

$$\sqrt{\sum \left(\left| \quad \right| \right)^2}$$

Introduction

Geometric Integration Theory

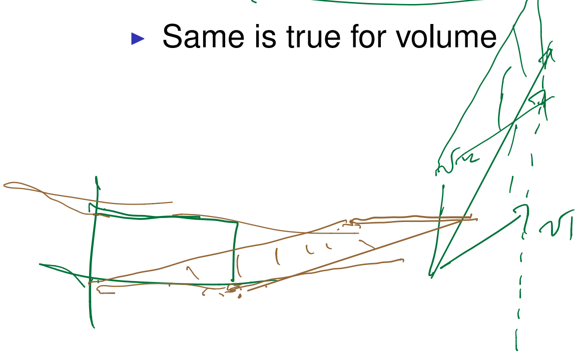
by Hassler Whitney

Reality check

- $v_1, \ldots, v_k$ are linearly independent and $w =$ linear combination of $v_2, \ldots, v_k$, then

$$v_1 \wedge v_2 \wedge \cdots \wedge v_k = (v_1 + w) \wedge v_2 \wedge \cdots \wedge v_k$$

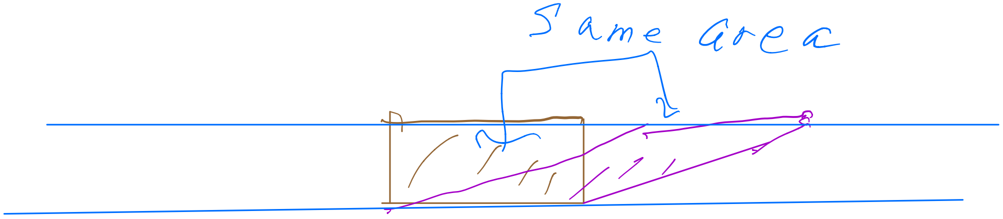
- Same is true for volume



- Example: for $k = 2$

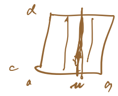
$$v_1 \wedge v_2 = (v_1 + \alpha v_2) \wedge v_2 \quad \text{for all} \quad \alpha \in \mathbb{R}$$

- Picture for area:



Rudin's pf for change of vars formula

$\mathbb{R}^2$



$(u,v) \to (u, g(u,v)) = (x,y)$

$\iint f(x,y)\,dy\,dv$

$= \iint f(u, g(u,v))\,dy\,dv \qquad y = g(u,v)$

$dy = \frac{\partial g}{\partial u}du + \frac{\partial g}{\partial v}dv$

$\cancel{dy = \left(\frac{\partial g}{\partial u}du + \frac{\partial g}{\partial v}dv\right)}$

$= \iint f(u, g(u,v)) \frac{\partial g}{\partial v}(u,v)\,(du\,dv)$

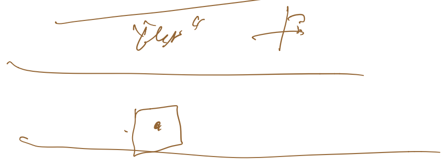
formula, change of variables

$(u, g(u,v)) = \begin{pmatrix} 1 & 0 \\ \partial g/\partial u & \frac{\partial g}{\partial v} \end{pmatrix}$

$\det = \partial g/\partial v$

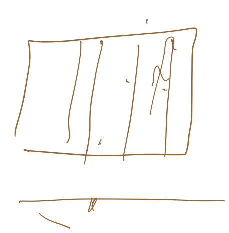
$G(u,v) = (u, g(u,v))$

$\text{Jac} \det G = \partial g/\partial v$



$\cancel{f\,dG}$

$|\det(dG)|$



- For $k = n$, if $v_i = Ae_i$ for $i = 1, \ldots, n$ then

$$v_1 \wedge \cdots \wedge v_n = \det(A) e_1 \wedge \cdots \wedge e_n$$

- Known $|\det(A)| =$ volume of parallelipiped.

- If $k = 2$, let

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \\ \cdots & \cdots \\ a_{n,1} & a_{n,2} \end{pmatrix}$$

- Let $v_1 = \sum_i a_{i,1} e_i$ and $v_2 = \sum_i a_{i,2} e_i$
- Check

$$v_1 \wedge v_2 = \sum_{i<j} \begin{vmatrix} a_{i,1} & a_{i,2} \\ a_{j,1} & a_{j,2} \end{vmatrix} e_i \wedge e_j$$

- For $k = n = 2$ get

  $v_1 \wedge v_2 = \pm$ area of parallelogram $\{t_1 v_1 + t_2 v_2 : 0 \leq t_i \leq 1\}$

- equivalently

$$v_1 \wedge v_2 = \det(A) e_1 \wedge e_2$$

- equivalently

$$|\det(A)| = \text{ area of parallelogram}$$

- For $k = 2$ and $n = 3$ get $v_1 \wedge v_2$ is the sum

$$\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} e_1 \wedge e_2 + \begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{3,1} & a_{3,2} \end{vmatrix} e_1 \wedge e_3 + \begin{vmatrix} a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{vmatrix} e_2 \wedge e_3$$

- This looks like the cross product $v_1 \times v_2$

$$\left(\begin{vmatrix} a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{vmatrix}, -\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{3,1} & a_{3,2} \end{vmatrix}, \begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} \right)$$

- In any case, the two vectors have the same magnitude:
$$|v_1 \wedge v_2| = |v_1 \times v_2|$$
- So the new formula $|v_1 \wedge v_2|$ and the old formula $|v_1 \times v_2|$ for the area of the parallelogram agree.
- Similarly one can check the case $k = n = 3$

$$|v_1 \wedge v_2 \wedge v_3| = |\det(A)| = |(v_1 \times v_2) \cdot v_3| \quad \text{etc}$$

for the volume of the parallelipiped.

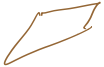
General Formula

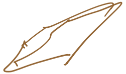
- The cases already discussed:
  - $k = 1$, n arbitrary
  - $k = 2$, n arbitrary, particularly $n = 3$,
  - $k = n$, particularly both $= 3$.

  are the most common

- General formula:
  If for $j = 1, \ldots, k$, $v_j = \sum_{i=1}^{n} a_{i,j} e_i \in \Lambda^1(\mathbb{R}^n)$,
  then $v_1 \wedge \cdots \wedge v_k$ is given by

$$\sum_{i_1 < \cdots < i_k} \begin{vmatrix} a_{i_1,1} & \ldots & a_{i_1,k} \\ \ldots & \ldots & \ldots \\ a_{i_k,1} & \ldots & a_{i_k,k} \end{vmatrix} e_{i_1} \wedge \cdots \wedge e_{i_k} \tag{7}$$

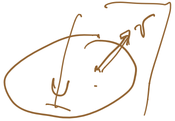
$v_1 \wedge v_2$ 

$v_1 \wedge v_2 \wedge v_3$ 

Differential k-forms in $\mathbb{R}^n$ ($k \leq n$)

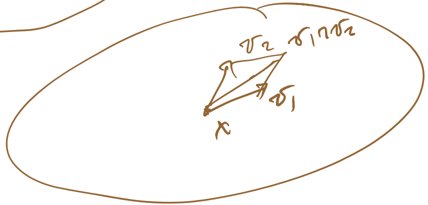
- k-dimensional integrands in $\mathbb{R}^n$ are the differential k-forms.
- $U \subset \mathbb{R}^n$ open.
- A (smooth) differential k- form on U is smooth function
$$\omega : U \times \Lambda^k(\mathbb{R}^n) \to \mathbb{R}$$
written $\omega_x(w)$ for $x \in U$ and $w \in \Lambda^k$, which is smooth in x and linear in w.
- Notation: $A^k(U) = \{\omega \ : \omega \text{ smooth } k - \text{form on } U\}$

1-form x, v : tangent vector to U at y



2-form $U \times \underbrace{\Lambda^2 \mathbb{R}}^n$

parallelogram tangent to U at x



$\Lambda^k \mathbb{R}^n$

- If $e_1, \ldots, e_n$ is an ON basis for $\mathbb{R}^n$, ω is determined by the $\binom{n}{k}$ functions

$$a_I(x) = \omega_x(e_I)$$

for all $I = \{i_1 < \cdots < i_k\}$.

- Write $e_I = e_{i_1} \wedge \cdots \wedge e_{i_k}$ for the basis elements of Λ^k
- Write $dx^I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ for the dual basis of $L(\Lambda^k, \mathbb{R})$.

$dx_1, \ldots, dx_n$ dual basis for $e_1, \ldots, e_n$

$dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ dual basis for $e_{i_1} \wedge \cdots \wedge e_{i_k}$

- Then

$$\omega = \sum_I a_I(x)dx^I$$

- Explicitly

$$\omega = \sum_{i_1<\cdots<i_k} a_{i_1,\dots,i_k}(x)dx_{i_1}\wedge\cdots\wedge dx_{i_k} \tag{8}$$

$\Lambda^k \mathbb{R}^n$ has basis $\{dx^{i_1}\wedge\cdots\wedge dx^{i_k} : i_1,\dots,i_k,\ 1\le i_1 < \cdots < i_k \le n\}$

$\binom{n}{k}$

$(dx_{i_1}\wedge\cdots\wedge dx_{i_k})(e_{j_1}\wedge\cdots\wedge e_{j_k}) =$

$$\det \begin{vmatrix} dx_{i_1}(e_{j_1}) & - & dx_{i_1}(e_{j_k}) \\ dx_{i_2}(e_{j_1}) & - & dx_{i_2}(e_{j_k}) \\ - & - & - \end{vmatrix}$$

$$dx_i(e_j) = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$$

$$\underline{(dx_{i_1} \wedge \dots \wedge dx_{i_k})}\,(e_{j_1}, \dots, e_{j_k})$$

$$\Lambda^k \mathbb{R}^n \qquad v_1 \wedge \dots \wedge v_k$$

$$\Lambda^k L(\mathbb{R}^n; \mathbb{R}) \qquad \underline{dx_{i_1} \wedge \dots \wedge dx_{i_k}}$$

- Let $A : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation ($A \in L(\mathbb{R}^m, \mathbb{R}^n)$)
- Define the associated linear transformation

$$\Lambda^k A : \Lambda^k(\mathbb{R}^m) \to \Lambda^k(\mathbb{R}^n)$$

  by

$$\Lambda^k A(e_I) = Ae_{i_1} \wedge Ae_{i_2} \wedge \cdots \wedge Ae_{i_k}$$

- Also called the *induced* linear transformation.

- Often it's easier to say that $\Lambda^k A$ is defined by

$$\Lambda^k A(v_1 \wedge \cdots \wedge v_k) = Av_1 \wedge \cdots \wedge Av_k$$

  for all $v_1, \ldots, v_k \in \mathbb{R}^m$.

- Since

$$\{v_1 \wedge \cdots \wedge v_k \ : \ v_1, \ldots, v_k \in \mathbb{R}^n\}$$

  spans $\Lambda^k(\mathbb{R}^m)$, $\Lambda^k A$ is determined by these values.

- To know that the definition makes sense, that is, $Av_1 \wedge \cdots \wedge Av_k$ depends just on $v_1 \wedge \cdots \wedge v_k$, need

$$v_1 \wedge \cdots \wedge v_k = 0 \Rightarrow Av_1 \wedge \cdots \wedge Av_k = 0$$

- This is equivalent to

  $v_1, \ldots, v_k$ linearly dependent
  $\Rightarrow$
  $Av_1, \ldots, Av_k$ linearly dependent

- Clear

Pull-back

- $V \subset \mathbb{R}^m$, $U \subset \mathbb{R}^n$ open sets
- $f : V \to U$ smooth map
- Pull-back $A^k(U) \to A^k(V)$ is defined by

$$(f^*\omega)_t(v_1 \wedge \cdots \wedge v_k) = \omega_{f(t)}(d_t f(v_1) \wedge \cdots \wedge d_t f(v_k))$$

  for all $t \in V$ and for all $v_1, \ldots, v_k \in \mathbb{R}^m$

- More concisely

$$(f^*\omega)_t = \omega_{f(t)} \circ \Lambda^k d_t f$$

  for all $t \in V$.

$$f = (f_1(t_1,\dots,t_m), \dots, f_n(t_1,\dots,t_m))$$

$$\begin{tikzcd} V \arrow[r, "f"', "\text{smooth}"] & U \end{tikzcd} \qquad V \subset \mathbb{R}^m, \quad U \subset \mathbb{R}^n$$

$$t = (t_1,\dots,t_m)$$

$$\to x = (x_1,\dots,x_n) = (f_1(t_1,\dots,t_m), \dots, f_n(t_1,\dots,t_m))$$

$$A^k(V) \xleftarrow{f^*} A^k(U)$$

$$\omega \in A^k(U)$$

$v_1,\dots,v_k \in \mathbb{R}^m$

$$(f^*\omega)_t\,(\underbrace{v_1\wedge\dots\wedge v_k}_{\in \Lambda^k V}) = \omega_{f(t)}\big(d_tf(v_1)\wedge\dots\wedge d_tf(v_k)\big)$$

$$\begin{tikzcd} V \arrow[r, "f"] & U \arrow[r, "\varphi"] & \mathbb{R} \end{tikzcd}$$

$k=0$ $\quad A^0(V) \xleftarrow{f^*} A^0(U)$

$\varphi \in A^0(U)$, $\quad \varphi : U \xrightarrow{\text{smooth}} \mathbb{R}$

$\varphi \to \varphi \circ f$

$(\varphi \circ f)(t) = \varphi(f(t))$

eg $\quad V \overset{i}{\subset} U$ inclusion of a subsp

$i^*\varphi$ = restriction of φ

$k=0$ $\quad \varphi = A^0(U)$.

$f^*\varphi(t) = \varphi(f(t))$

$k=1$

$A^k(U)$ = diff k-forms on U

$\{\omega : U \times \Lambda^k \mathbb{R}^n \to \mathbb{R} \mid \omega_x(v)$ smooth in x, linear in $v\}$

$$\boxed{\omega = \sum_I a_I(x)\, dx_I \qquad I = (i_1, \dots, i_k) \qquad 1 \le i_1 < i_2 < \dots < i_k \le n}$$

$a_I : U \to \mathbb{R}$ smooth fun

$A^0(U)$ = smooth fns

$A^1(U)$ = smooth 1-forms

$= \{a_1 dx_1 + \dots + a_n dx_n\}$

$a_1, \dots, a_n : U \to \mathbb{R}$ smooth fun

$\omega = a_1(x) dx_1 + \dots + a_n(x) dx_n$

$\cancel{f^*\omega = a_1(f(t))\,(dx_1)(d_t f(v))}$

$f^*(dx_i)(e_j) = dx_i\big((df)(e_j)\big) = dx_i\big(\sum \frac{\partial f_k}{\partial t_j} e_k\big)$

$=$

$$f^* dx^i = df^i$$

$$= \frac{\partial f_i}{\partial t_1} dt_1 + \frac{\partial f_i}{\partial t_2} dt_2 + \cdots$$

$$f^*\Big(\overbrace{a_1(x)\, dx_1 + a_2(x)\, dx_2 + \cdots + a_n(x)\, dx_n}\Big)$$

$$= f^*(a_1(x)\, dx_1) + f^*(a_2(x)\, dx_2) \cdots f^*(a_n(x)\, dx_n)$$

$$= (f^* a_1) f^*(dx_1) + (f^* a_2) f^* dx_2 + \cdots (f^* a_n) f^* dx_n$$

$$a_1(f(t))\, df^1 + a_2(f(t))\, df^2 + \cdots$$

$(t_1, \dots, t_m)$ m

$\downarrow$

$(x_1, \dots, x_n)$ n

$$= a_1(f(t)) \left(\frac{\partial f_1}{\partial t_1} dt_1 + \frac{\partial f_1}{\partial t_2} dt_2 + \cdots + \frac{\partial f_1}{\partial t_m} dt_m \right]$$

$$+ a_2(f(t)) \frac{\partial f_2}{\partial t_1} dt_1 + \frac{\partial f_2}{\partial t_2} dt_2 + \cdots \frac{\partial f_2}{\partial t_m} dt_m$$

$$\vdots$$

$$a_n(f(t) \cdots$$

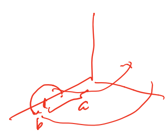
Example

(See HW 4)

$$F^*(dx \wedge dy)$$

$$F(\varphi, \theta) = ((a + b\cos\varphi)\cos\theta,\ (a + b\cos\varphi)\sin\theta,\ b\sin\varphi)$$

(x, y, z)



$$F^*(dx \wedge dy)$$

$$x = (a + b\cos\varphi)\cos\theta$$

$$F^* dx = (-b\sin\varphi\cos\theta)\, d\varphi + (a + b\cos\varphi)(-\sin\theta)\, d\theta$$

$$y = (a + b\cos\varphi)\sin\theta$$

$$F^* dy = (-b\sin\varphi\sin\theta\, d\varphi + (a + b\cos\varphi)\cos\theta\, d\theta$$

$$F^*(dx \wedge dy)$$

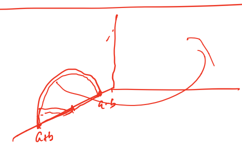
$$= -(b\sin\varphi\cos\theta)\, d\varphi + (a + b\cos\varphi)\sin\theta\, d\theta)$$

$$\wedge(-b\sin\varphi\sin\theta\, d\varphi + (a + b\cos\varphi)\cos\theta\, d\theta)$$

$F^*(dx \wedge dy)$ $dy \wedge dz$ $dx \wedge dz$

$-b(a+b\cos\varphi)\sin\varphi\,(\cos\theta\,d\varphi\wedge\theta - \sin\theta\,d\varphi \wedge d\varphi)$

$$\boxed{-b(a+b\cos\varphi)\sin\varphi\;d\varphi\wedge d\theta}$$



$(a+b\cos\varphi)$

$0 \le \varphi \le \pi$ | $\pi \le \varphi \le 2\pi$

$0 \le \theta \le 2\pi$ | $0 \le \theta \le 2\pi$

$T = T_+ \cup T_-$

$\int_{T_+} F^* dx\wedge dy = \pm$ area of proj. onto xy plane

$\int_{T_-} = \mp$!

$\pi[(a+b)^2 - (a-b)^2]$

$= 4\pi ab$

$\pi((a+b)^2 - b^2)$

$= \pi(a^2 + 2ab)$

$(a+b)^2 - (a-b)^2$

$\int_0^{2\pi}\int_0^{\pi} b(a + b\cos\varphi)\sin\varphi\,d\varphi\,d\theta$

$2\pi b\left(\int_0^{\pi} a\sin\varphi + b\cos\varphi\sin\varphi\right)$

$\frac{\sin^2\varphi}{2}\Big|_0^{\pi}$

$\int_0^{\pi}\sin\varphi\,d\varphi$

$= -\cos\varphi\Big|_0^{\pi}$

$4\pi ab$

$2\pi b\int_{\pi/2}^{3\pi/2}(a\sin\varphi + b\cos\varphi\sin\varphi)\,d\varphi$

$f^*\left(\sum a_I\, dx_I\right)$

$= \sum (f^* a_I)\,(df)_I$

$dx_{i_1}\wedge\cdots\wedge dx_{i_k}$

$x_{i_k} = f_{i_k}$

f^*

- In terms of coordinates $t = (t_1, \ldots, t_m)$ and $x = (x_1, \ldots, x_n)$
- $x = f(t) = (f_1(t_1, \ldots, t_m), \ldots, f_n(t_1, \ldots, t_m))$
- $\omega = \sum_I a_I(x) dx^I = \sum_{i_1 < \cdots < i_k} a_{i_1, \ldots, i_k}(x) dx_{i_1} \wedge \ldots dx_{i_k}$
- Then

$$f^*\omega = \sum_{i_1 < \cdots < i_k} a_{i_1, \ldots, i_k}(f(t)) f^*(dx_{i_1}) \wedge \cdots \wedge f^*(dx_{i_k}) \quad (9)$$

- Using (3), this can be rewritten as

$$f^*\omega = \sum_{i_1<\cdots<i_k} a_{i_1,\ldots,i_k}(f(t))(d_t f_{i_1}) \wedge \cdots \wedge (d_t f_{i_k}) \qquad (10)$$

- Writing $df_i = \sum_{j=1}^{m} \frac{\partial f_i}{\partial t_j} dt_j$ and expanding df^I in the same manner as (9) we get an explicit expression for $f^*\omega$ as a sum

$$\sum_J c_J(t) dt^J$$

- Perhaps more useful than an explicit but complicated formula is to observe the multiplicative properties of f^*.

- If $a : U \to \mathbb{R}$ is a smooth function, that is, $a \in A^0(U)$, let
$$f^* : A^0(U) \to A^0(V)$$
be defined by
$$(f^*a)(t) = a(f(t))$$
- Then (11) says

$$f^*\Big(\sum_{I=i_1<\cdots<i_k} a_I dx_{i_1} \wedge \cdots \wedge dx_{i_k}\Big) = \sum (f^*a_I)(f^*dx_{i_1}) \wedge \cdots \wedge (f^*dx_{i_k})$$

- Suggests the following
- There is a product

$$L(\Lambda^k, \mathbb{R}) \times L(\Lambda^l, \mathbb{R}) \to L(\Lambda^{k+l}, \mathbb{R})$$

  defined just as in (8) using the dual basis dx^I rather than e_I

- Induces a product $A^k(U) \times A^l(U) \to A^{k+l}(U)$.
- If $\omega \in A^k(U)$ and $\eta \in A^l(U)$, then $\omega \wedge \eta \in A^{k+l}(U)$.
- If $f : V \to U$ is smooth, then

$$\boxed{f^*(\omega \wedge \eta) = (f^*\omega) \wedge (f^*\eta)} \tag{11}$$

Some properties of pull-back

- $f^*(\omega \wedge \eta) = f^*(\omega) \wedge f^*(\eta)$ as above
- $f^*(\omega_1 + \omega_2) = f^*\omega_1 + f^*\omega_2$
- $f : V \to U$ and $g : W \to V$ smoooth maps of open sets. then

$$(f \circ g)^* = g^* \circ f^* : A^k(U) \to A^k(W)$$

$((f \circ g)^* a)(x)$

$= a((f \circ g)\, x)$

$= a(g(f\, x)) = g^* a(f^* x$ $\quad g^* f^*$

$$\boxed{(f\circ g)^* = g^* \circ f^*}$$

$$\int_{T^+} dx\wedge dy \qquad \underline{dx\wedge dy} \quad \Lambda^2\mathbb{R}^3$$

$\cancel{\Phi(\varphi,\theta)}$

$$\int_{T^+} dx\wedge dz \qquad F: [0,\pi]\times[0,2\pi) \quad \varphi, \theta \quad \to F(\varphi,\theta) = \ldots$$

$$\int_{T^+} dx\wedge dy = \int_{[0,\pi]\times[0,2\pi)} F^* dx\wedge dy$$

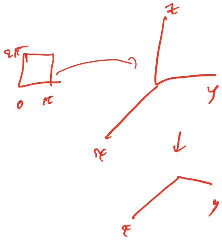
$$F: [0,\pi]\times[0,2\pi) \to \mathbb{R}^3$$

$$\searrow T^+$$

$$dx\wedge dy =$$

On $\mathbb{R}^2$ $\quad dx\wedge dy = \pm$ area

On $\mathbb{R}^3$: $\pi_{(x,y)}^*(dx\wedge dy)$



$(\pi_{(x,y)}\circ F)^* = F^*\pi^*_{(x,y)}(dx\wedge dy)$

$= \pm$ area of image

<u>Provided</u>

$\pi\circ F : [0,2\pi]\times[0,2\pi] \to$ image in $\mathbb{R}^2$

is 1-1, onto, Jacobian $\neq 0$

$[0,\pi]$

$[\pi/2, 3\pi/2]$

$\int_{\pi/2}^{3\pi/2}$

Integration over k-cells

- Let $D = \mathbf{I}^k$ be a k-*cell* as in (6)
- Let $\alpha \in A^k(D)$ be a smooth k-form.
- Then
$$\alpha = \phi(t) dt_1 \wedge \cdots \wedge dt_k$$
for some smooth $\phi : D \to \mathbb{R}$, $t = (t_1, \ldots, t_k)$
- Define
$$\int_D \alpha = \int_D \phi(t) dt_1 \ldots dt_k$$
the Riemann integral of ϕ over $D = \mathbf{I}^k$.

- If $\sigma : D \to U$ is smooth and $\omega \in A^k(U)$, define

$$\int_{\sigma} \omega = \int_{D} \sigma^*(\omega)$$

- Would like $\int_\sigma \omega$ to be *independent of parametrization*.

- This means that if E is another k-cell and

$$\Phi : E \to D$$

is smooth, bijective, $\det(d\Phi) > 0$ everywhere on E, then

$$\int_{\sigma \circ \Phi} \omega = \int_\sigma \omega$$

- This follows from the *change of variables formula*
- If $\Phi : E \to D$ and $\alpha \in A^k(D)$ as before, then

$$\int_E \Phi^* \alpha = \int_D \alpha$$

- More usual formulation:

- If $\alpha = a(t)dt_1 \wedge, \cdots \wedge dt_k$ then

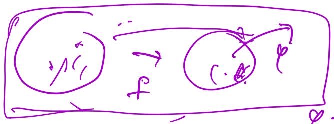
$$\int_E a(\Phi(t)) \left|\det(d_t\Phi)\right| dt_1 \ldots, dt_k = \int_D a(t)dt_1 \ldots dt_n$$

- Note how the absolute vaue $\left|\det(d\Phi)\right|$ appears, rathen than $\det(d\Phi)$. Results from orientation.

Recall Pullback

- $f : V \to U$, $x = f(t)$, where
  $x = (x_1, \ldots, x_n), t = (t_1, \ldots, t_m)$
-
$$\boxed{\omega = \sum_I a_I(x) dx_I \in A^k(U)}$$
  sum over all $i = \{i_1, \ldots, i_k\}$ with $1 \leq i_1 < \cdots < i_k \leq n$
- $a_i : U \to \mathbb{R}$ smooth functions
-
$$dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

- The *pull-back* of ω by f is defined as
- $$f^*(\omega)_t = \sum_I a_I(f(t))(d_t f)_I$$
- where
$$(d_t f)_I = d_t f_{i_1} \wedge d_t f_{i_2} \wedge \dots d_t f_{i_k}$$
- And, as usual,
$$d_t f_i = \sum_{j=1}^{m} \frac{\partial f_i}{\partial t_j}(t) dt_j$$



$\varphi \to \varphi \circ f$

$f^*(\text{function } a)_t = a(f(t))$ composition

$dx_1 \wedge dx_2$

$df_1 \wedge df_2$

- Putting all these into a single formula would be quite long.
- An easy way to remember:
  - If $a \in A^0(U)$ is a smooth function, then $f^*a \in A^0(V)$ is

$$(f^*a)(t) = a(f(t))$$

$dx_i \in A^1(U)$

  - If $x_i : U \to \mathbb{R}$ is one of the coordinate functions, then $f^*dx_i \in A^1(V)$ is

$f^*(dx_i) = df_i$

$f^* x_i = f_i$

$$(f^*(dx_i))_t = d_t f_i = \sum_{j=1}^{m} \frac{\partial f_i}{\partial t_j}(t)dt_j$$

$f^* dx_i = df_i$

  - $f^* : A(U) \to A(V)$ is *multiplicative*

$$f^*(a \wedge b) = f^*a \wedge f^*b \quad \text{for all} \quad a \in A^k(U), b \in A^\ell(U)$$

- These properties determine f^* uniquely.

Remarks

f^*, d, $+$, sum, product, ...

$A^0, A^1, A^2, \ldots A^n$ $\qquad U \subset \mathbb{R}^n$

$A^k \times A^l \to A^{k+l}$

- Used $A(U)$ for the totality of differential forms on U. Usually take this to mean direct sum

$A^0, A^1, A^2 \ldots$ c single vc sp A

$$A(U) = \oplus_0^n A^k(U)$$

$A^2 + A^3$

A^5

- $A(U)$ is then an *algebra*.
- $f^* : A(U) \to A(v)$ is an algebra homomorphism.

$A(U)$

what's geometric

+ a lot stuff

$\sum a_{i_1 \ldots i_n}(x)\, dx_{i_1} \ldots dx_{i_n}$

$f^*(a)$

$f^*(dx)$

$f^*(\alpha \wedge \beta) = f^*\alpha \wedge f^*\beta$

f^* : algebra homom

$\alpha, \beta, \gamma \ldots$
$\in A^k \in A^l \in A^d$

$\alpha \wedge \beta \in A^{k+l}$

$\alpha + \beta$? $A^k \oplus A^l$? [illegible]

$df = \sum \frac{\partial f}{\partial x_i} dx_i$

$\Phi^*(dx \wedge dy)$

$= (\Phi^* dx) \wedge (\Phi^* dy)$

$d(\Phi_1) \wedge d\Phi_2$

$\Phi_1(\varphi, \theta) = (a + b\cos\varphi)\cos\theta$

$\Phi_2(\varphi, \theta) = (a + b\cos\varphi)\sin\theta$

Form algebra $\alpha \wedge \beta = (-1)^{|\alpha||\beta|} \beta \wedge \alpha$

d

$|\alpha| = k \Leftrightarrow \alpha \in A^k$

$\mathbb{R}^4$ $(dx_1 \wedge dx_2) \wedge dx_3 = (-1)^{2 \cdot 1} dx_3 \wedge (dx_1 \wedge dx_2)$

$\| \quad \wedge \quad d$ "derivation" Leibniz Rule

$\cancel{d}$

$\wedge$

d

$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^{|\alpha|} \alpha \wedge d\beta$

$df =$ ~~usual~~ df

$\theta(\alpha)$

$d(\alpha + \beta) = d\alpha + d\beta$

$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{|\alpha|} \alpha \wedge d\beta$

d

S deg

$\cancel{df(x)}$. df

$f \in A^0(U) \Leftrightarrow$ function

$df =$ usual differential df

$= \sum \frac{\partial f}{\partial x_i} dx_i$

$$\boxed{\deg(\alpha) = k \Leftrightarrow \alpha \in A^k}$$

$$\alpha \wedge \beta = (-1)^{\deg(\alpha)\deg(\beta)} \beta \wedge \alpha$$

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg(\alpha)} \alpha \wedge d\beta$$

$$df = \sum \frac{\partial f}{\partial x_i} dx_i$$

$$d(dx_i) = 0$$

$d(a_1 dx_1 + a_2 dx_2 - a_n dx_n) = d(a_1 dx_1) + d(a_2 dx_2) \ldots d(a_n dx_n)$

$= da_1 \wedge dx_1 + (-1)^{\deg(a_1)} a_1 d(dx_1) +$

$f,\ dx_i$

$\Downarrow$

$\boxed{df = \sum \frac{\partial f}{\partial x_i} dx_i \qquad d(dx_i) = 0}$

$\sum a_{i_1 \dots i_k} dx_{i_1} \wedge \dots \wedge dx_{i_k}$

$d(dx_i) \qquad \boxed{d^2 = 0} \qquad \boxed{d(dx_i) = 0}$

$dx_i = \frac{\partial x_i}{\partial x_i}$

$= da_1 \wedge dx_1 + a\, d(dx_1)$

$d(a_1 dx_1 + a_2 dx_2 + \dots$

$= da_1 \wedge dx_1 + a\, \cancel{d^2 x_1}_{0}$

$d(a_1 dx_1 + a_2 dx_2 + \dots)$

$da_1 \wedge dx_1 + a_1 \cancel{d(dx_1)}_{0} + da_2 \wedge dx_2 + \dots$

$da_1 \wedge dx_1 + da_2 \wedge dx_2 + \dots + da_n \wedge dx_n$

$d\Big(a_{12}\, dx_1 \wedge dx_2 + a_{13}\, dx_1 \wedge dx_3 + a_{14}\, dx_1 \wedge dx_4$
$a_{23}\, dx_2 \wedge dx_3 + a_{24}\, dx_2 \wedge dx_4$
$a_{34}\, dx_3 \wedge dx_4\Big)$

$da_{12} \wedge dx_1 \wedge dx_2 + a_{12}\, d(dx_1 \wedge dx_2)$

$d(dx_i \wedge dx_j) = \cancel{d(dx_i)}_{0} \wedge dx_j - dx_i \wedge \cancel{d(dx_j)}_{0} = 0$

$d(dx_{i_1} \wedge \dots \wedge dx_{i_k}) = 0$

$d(a_{i_1 \dots i_k} dx_{i_1} \dots dx_{i_k}) = da_{i_1 \dots i_k} \wedge dx_{i_1} \dots dx_{i_k}$

$d(df) = d\left(\frac{\partial f}{\partial x_1} dx_1 + \dots \frac{\partial f}{\partial x_n} dx_n\right)$

$= d\left(\frac{\partial f}{\partial x_1}\right) dx_1 + \dots d\left(\frac{\partial f}{\partial x_n}\right) dx_n$

$\boxed{\left(\frac{\partial^2 f}{\partial x_1^2} dx_1 + \frac{\partial^2 f}{\partial x_2 \partial x_1} dx_2\right) + \left(\frac{\partial^2 f}{\partial x_1 \partial x_2} dx_1 + \frac{\partial^2 f}{\partial x_2^2} dx_2\right)} \leftarrow \begin{matrix} \wedge dx_1 \\ \wedge dx_2 \end{matrix}$

$df = \sum \frac{\partial f}{\partial x_i} dx_i$

$\underline{d(dx_i) = 0}$

definition

defining d

$\partial f \qquad dx_i \wedge dx_j$

$i < j$

$\left(\frac{\partial^2 f}{\partial x_i \partial x_j} - \frac{\partial^2 f}{\partial x_j \partial x_i}\right) dx_i \wedge dx_j \qquad \frac{\partial^2 f}{\partial x_2 \partial x_1} dx_2 \wedge dx_1 + \frac{\partial^2 f}{\partial x_1 \partial x_2} dx_1 \wedge dx_2 = \left(\frac{\partial^2 f}{\partial x_1 \partial x_2} - \frac{\partial^2 f}{\partial x_2 \partial x_1}\right) dx_1 \wedge dx_2$

$f \in C^2 \Rightarrow \boxed{d^2 f = 0}$

$d^2 = 0$ $\qquad$ $d^2 f = 0$ on fcn

$d(\sum a_{\ldots} dx \ldots)$

$d(\sum da_{\ldots} \wedge dx \ldots)$

$= \sum d^2 a \ldots + \sum da_i \, d(\ldots)$

$\boxed{d^2 = 0 \text{ always}}$

$\Updownarrow$

$\boxed{\text{algebraic topology}}$

Very long story

---

algebraic properties f^*, d

$\boxed{d(f^*\omega) = f^* d\omega}$

chain rule

k form $\to$ int over k-dim submanifolds of $\mathbb{R}^n$

$\Phi : D^k \subset \mathbb{R}^k \to \mathbb{R}^n$

($d_x\Phi$ injective)

$$\int_\Phi \omega = \int_D \Phi^* \omega$$

$A^k(\mathbb{R}^k) \leftarrow A^k(\mathbb{R}^n)$ on ...

$\int : A^k(\mathbb{R}^k) \to \mathbb{R}$

$(f^*\omega)_x (v_1 \ldots v_k) = \omega_{f(x)} (df(v_1) \ldots df(v_k))$

$v_1 \wedge v_2 \to d\Phi(v_1), d\Phi(v_2)$

Area $\qquad$ Length $\int_a^b \|\gamma'(t)\| dt$

$\|\gamma(t_1) - \gamma(t_2)\| = |\gamma'(\xi_i)(t_i - t_{i-1})|$

$\Phi : D^2 \to \mathbb{R}^3$

$$\iint_D \left\| \frac{\partial \Phi}{\partial t_1} \wedge \frac{\partial \Phi}{\partial t_2} \right\| dt_1 \, dt_2$$

area of parallelogram

$\frac{\partial \Phi}{\partial t_1}$, $\frac{\partial \Phi}{\partial t_2}$

Smooth surf

Integrals of k-forms

- $D \subset \mathbb{R}^k$ a "domain", for example $D = \mathcal{I}^k$ a product of intervals.
- If $\eta \in A^k(D)$ is a smooth k form, should know how to integrate η over D:
  - $\eta = \phi(t) dt_1 \wedge \cdots \wedge dt_k$
  - Define

$$\int_D \eta = \int_D \phi(t) dt_1 \ldots dt_k \quad \text{or, simply} \quad \int_D \phi(t) dt$$

  - RHS is an ordinary k-dimensional integral, a Riemann integral or itetared integral.
  - RHS is independent of orientation, while LHS depends on orientation.

- Suppose $\Phi : D \to U$ is a smooth map.
- May also want to assume $d_t\Phi$ is injective at each $t \in D$.
- If $\omega \in A^k(U)$ is a smooth k-form, define

$$\int_\Phi \omega = \int_D \Phi^*\omega$$

Change of Variables Formula

- D_1, D_2 domains in $\mathbb{R}^k$
- $F : D_1 \to D_2$ smooth, bijective, $d_s F$ isomorphism for all $s \in D_1$
- $\phi : D_2 \to \mathbb{R}$ smooth function.
- Then

$$\int_{D_1} \phi(F(s))|\det(d_s F)| ds_1 \dots ds_k = \int_{D_2} \phi(t) dt_1 \dots dt_k$$

- The absolute value of the Jacobian determinat $\det(dF)$ comes because of orientation: if F is orientation reversion reversing $(\det(dF) < 0)$ does not affect these integrals.
- For differential forms $\eta \in A^k(D_2)$ the theorem says

$$\int_{D_1} F^*\eta = \int_{D_2} \eta$$

Independence of Parametrization

- Finally, if $\Phi : D_2 \to U$ is a parametrized k-surface and $f : D_1 \to D_2$ is a change of variables, then

$$\int_{\Phi \circ F} \omega = \int_{\Phi} \omega$$

since

$$\int_{D_1} (\Phi \circ F)^* \omega = \int_{D_1} F^*(\Phi^* \omega) = \int_{D_2} \Phi^* \omega$$

Recall Torus

Recall we looked at the parametrization $\Phi : [0, 2\pi] \times [0, 2\pi] \to \mathbb{R}^3$ of a torus

$$\begin{aligned} x &= (a + b\ \cos\phi)\cos\theta \\ y &= (a + b\ \cos\phi)\sin\theta \\ z &= b\ \sin\phi \end{aligned}$$

$dy \wedge dz$

$dx \wedge dy$

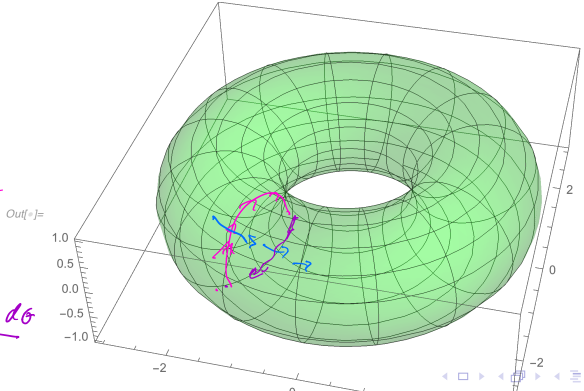
$\boxed{d((a+b\cos\varphi)\cos\theta)} \wedge d((a+b\cos\varphi)\sin\theta)$

$(-b\sin\varphi\cos\theta\, d\varphi + (a+b\cos\varphi)(-\sin\theta\, d\theta))$

$\wedge(-b\sin\varphi\sin\theta\, d\varphi + (a+b\cos\varphi)\cos\theta\, d\theta)$

$0 \le \varphi \le \pi$

$d\varphi \wedge d\theta$



$-4\pi ab$

$2\pi(-b)\underbrace{\int_0^\pi (a+b\cos\varphi)\sin\varphi\, d\varphi}$

$\int_0^\pi a\sin\varphi\, d\varphi = 2a$

- Worked out (scratchwork above)

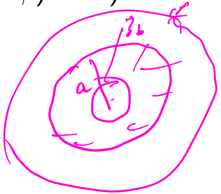
$\int_0^{\pi} b \cos\varphi \sin\varphi\, d\varphi$

$$\boxed{\Phi^*(dx \wedge dy)} = \boxed{-b(a + b\cos\phi)\sin\phi \; d\phi \wedge d\theta}$$

- Did this by calculating

$$d((a + b\ \cos\phi)\cos\theta) \wedge d((a + b\ \cos\phi)\sin\theta)$$

by following the procedures above.



$\pi(a+b)^2$

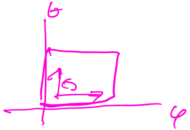
$-\pi(a-b)^2$

$=\pi(a^2+2ab+b^2)$

$-\pi(a^2-2ab+b^2)$

$=4\pi ab$

$d\varphi \wedge d\theta$



- Reality check: worked out

$$\int_{[0,\pi]\times[0,2\pi]} (-b(a+b\cos\phi)\sin\phi)\ d\phi d\theta$$

- Compared our answer with the area of the projection of the top half of the torus to the x, y-plane. This can be done by elementarty goemetry.
- If all is correct, one answer should be $\pm$ the other.
- It worked!

Formula for Area

H. A. Schwarz example

From H. A. Schwarz
Mathematische Abhandlungen
Vol II
p. 309–311

Les équations

$$x = r\cos u, \quad y = r\sin u, \quad z = v, \quad 0 \leqq u \lesseqgtr 2\pi, \quad 0 \leqq v \lesseqgtr h$$

représentent un cylindre droit dont la surface courbe a l'aire $2r\pi h$. A ce cylindre on peut inscrire un polyèdre à $4mn$ faces triangulaires qui satisfait aux conditions suivantes:

1°. — Tous les sommets de ce polyèdre sont donnés par les équations:

$$u = u' = \frac{2\mu\pi}{m}, \qquad v = v' = \frac{\nu h}{n},$$
$$(\mu = 0, 1, 2, \dots m-1, \quad \nu = 0, 1, 2, \dots n);$$
$$u = u'' = \frac{(2\mu+1)\pi}{m}, \qquad v = v'' = \frac{(2\nu+1)h}{2n},$$
$$(\mu = 0, 1, 2, \dots m-1, \quad \nu = 0, 1, 2, \dots n-1).$$

2°. Toutes les faces triangulaires sont isocèles et congrues entre elles.

3°. Les bases de tous ces triangles isocèles sont situées dans les plans $z = v'$, $z = v''$.

La base et la hauteur d'une face triangulaire ont les longueurs:

$$2r\sin\left(\frac{\pi}{m}\right), \quad \sqrt{r^2\left[1-\cos\left(\frac{\pi}{m}\right)\right]^2 + \left(\frac{h}{2n}\right)^2}.$$

Par conséquent l'aire totale du polyèdre inscrit a la valeur:

m
n

$$S' = 4mnr\sin\left(\frac{\pi}{m}\right)\sqrt{4r^2\sin^4\left(\frac{\pi}{2m}\right) + \left(\frac{h}{2n}\right)^2}.$$

$2\pi rh$

De cette formule on conclut:

1°. Si l'on fait $n = am$, on a la limite cherchée $S = 2r\pi h$ pour $m = \infty$.

2°. Si l'on fait $n = am^2$, on trouve que la limite S de l'aire S' est donnée par l'équation

$$S = 2r\pi\sqrt{a^2r^2\pi^2 + h^2}.$$

Il est évident que la valeur de cette limite dépend du nombre a et qu'elle peut surpasser une grandeur donnée.

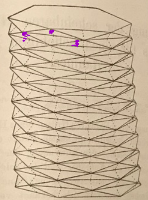
3°. Si l'on fait $n = am^3$, on trouve que l'aire S' est plus grande que

$$8r^2m^4\sin\left(\frac{\pi}{m}\right)\sin^2\left(\frac{\pi}{2m}\right).$$

Dans ce cas, il n'y a plus de limite pour la quantité S', car cette quantité surpasse, pour lim. $m = \infty$, une grandeur quelconque donnée.

De ce qui précède on conclura que la définition de l'aire d'une surface courbe donnée par M. Serret doit être modifiée par l'addition d'une condition restrictive concernant la construction du polyèdre inscrit en question.

Fig. 23.

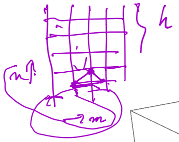


La figure 23 représente un des polyèdres inscrits dont il vient d'être question.

Similar Pictures

$(u, v) \to (\cos u, \sin u, v)$

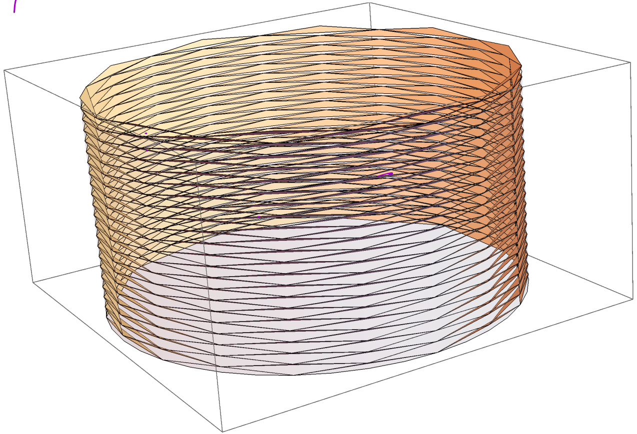
Sum area Δ's $\to \infty$



$n = m^3$

$\to \infty$

Out[ ]=



Out[•]=