The Gromov-Witten potential of the local $\mathbb{P}(1, 2)$

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Recent Progress on the Moduli Space of Curves, BIRS
Outline

1. The Crepant Resolution Conjecture.
   - Motivations and origins
   - Gromov-Witten invariants
   - Versions of the Conjecture

2. Examples
   - $\mathbb{C}^2/\mathbb{Z}_2$
   - $\mathbb{C}^2/\mathbb{Z}_3$

3. Local $\mathbb{P}(1, 2)$
   - Set-up
   - The invariants
   - Positive degree
   - The Potential
McKay Philosophy

“Geometry on a quotient orbifold \([X/G]\) (that is the \(G\)-equivariant geometry on \(X\)) is equivalent to geometry on a crepant resolution \(Y\) of \(X/G\)”

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“Propagating strings can’t tell the difference between an orbifold and its crepant resolution”
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- multilinear functions $\langle \ldots \rangle_{g,\beta}^Y$ on its cohomology $H^*(Y)$.
- involve integrals over the space of stable maps $\overline{M}_{g,n}(Y, \beta)$

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**Conjecture**

Yongbin Ruan (2002)

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There is an isomorphism between $H^*(Y)$ and $H^*_{orb}(\mathcal{X})$ that identifies the Gromov-Witten theories.
Let \( \{\beta_1, \ldots, \beta_r\} \) be a positive basis of \( H_2(Y) \). Encode genus zero Gromov-Witten invariants of \( Y \) in the Potential function

\[
F_Y(y_0, \ldots, y_a, q_1, \ldots, q_r) = \sum_{n_0,\ldots,n_a=0}^{\infty} \sum_{\beta} \langle \gamma_0^{n_0} \cdots \gamma_a^{n_a} \rangle_{\beta} \frac{y_0^{n_0}}{n_0!} \cdots \frac{y_a^{n_a}}{n_a!} q_1^{d_1} \cdots q_r^{d_r}
\]

where \( \beta = d_1 \beta_1 + \cdots + d_r \beta_r \) is summed over all non-negative \( \beta \).

Analogously for an orbifold \( X \) can define a \( F^X \).
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Conjecture
Bryan-Graber (2005)

The orbifold Gromov-Witten potential $\mathcal{F}^\mathcal{X}$ is equal to the (ordinary) Gromov-Witten potential $\mathcal{F}^\mathcal{Y}$ of a crepant resolution after a change of variables induced by:

- a linear isomorphism

$$L : H^*_{\text{orb}}(\mathcal{X}) \rightarrow H^*(\mathcal{Y});$$

- an analytic continuation of the potential to a specialization of the excess quantum parameters (on the resolution side) to roots of unity.
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The first example (Bryan-Graber)

For the pair:

\[
\begin{align*}
X &= \mathbb{C}^2 / \mathbb{Z}_2 \\
Y &= \mathcal{O}_{\mathbb{P}^1}(-2)
\end{align*}
\]

\[
F^X_{xxx} = -\frac{1}{2} \tan \left( \frac{x}{2} \right) \\
F^Y_{st} = \sum d \frac{1}{d^3} e^{dy} q^d
\]

After the change of variables:

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\begin{align*}
y &= ix \\
q &= -1
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\]

We have that

\[
F^Y_{xxx} = \frac{1}{2i} \left[ \frac{1 - e^{ix}}{1 + e^{ix}} \right]
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The first example (Bryan-Graber)

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Another example (Bryan-Graber-Pandharipande)

\[ \mathcal{X} = [\mathbb{C}^2/\mathbb{Z}_3] \quad Y = \text{two } -2 \text{ curves} \]

The third derivatives of the Potential for \( \mathcal{X} \)

Sum of \( \tan \)

The third derivatives of the Potential for \( Y \)

Sum of \( \exp \)

Change of variables:

\[ y_0 = x_0 \]
\[ y_1 = \frac{i}{\sqrt{3}} (\omega x_1 + \bar{\omega} x_2) \]
\[ y_2 = \frac{i}{\sqrt{3}} (\bar{\omega} x_1 + \omega x_2) \]
\[ q_i = \omega \]
Intermediate case

Let $Z$ be the weighted blow-up of $\mathbb{C}^2/\mathbb{Z}_3$ at the origin with weights one and two.

1. $Z \cong \mathcal{O}(K_{\mathbb{P}(1,2)})$ is Local $\mathbb{P}(1,2)$
2. $Z$ is not a global quotient
3. Possible insertions
   - The fundamental class $1$ with dual $z_0$
   - Divisorial class $H$ with dual $z_1$
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Degree zero

Given by the triple intersection in (equivariant) cohomology

\[ \langle a, b, c \rangle_0 = \int_Z a \cup b \cup c \]

which are computed by Atiyah-Bott localization formula. Except the degree zero invariant with all stacky insertions. We have

\[ \langle S^n \rangle_0 = -\int_{\text{Adm}}^{g \rightarrow 0, (t_1, \ldots, t_{2g+2})} \lambda_g \lambda_{g-1}. \]

Theorem

Faber and Pandharipande (Bryan and Pandharipande, and also Bertram, Cavalieri, –) If \( G \) is the contribution to the potential then we have that

\[ G''' = \frac{1}{2} \tan \left( \frac{Z_0}{2} \right). \]
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Reductions

Use the Euler sequence on \( \mathbb{P}(1, 2) \) to reduce from \( Z \) to the total space of a CY threefold \( Z' = \mathcal{L}_1 \oplus \mathcal{L}_2 \) (the total space of a bundle over \( \mathbb{P}(1, 2) \)).

The image of non-constant maps to \( Z' \) must lie in the zero section \( E \cong \mathbb{P}(1, 2) \). We can identify

\[
\overline{\mathcal{M}}_{0,n}(Z', d[E]) \cong \overline{\mathcal{M}}_{0,n}(\mathbb{P}(1, 2), d)
\]

BUT keep in mind the difference of virtual fundamental classes given by

\[
e(R^\bullet \pi_* f^* N_{E/Z}) = e(R^\bullet \pi_* f^* \mathcal{L}_1 \oplus \mathcal{L}_2)
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Odd $d$ case

Problem 1

The moduli spaces $\overline{M}_{0,n}(\mathbb{P}(1, 2), d)$ are very complicated.

Solution: Use localization under a lifting of a $\mathbb{C}^*$ action on $\mathbb{P}(1, 2)$.

Problem 2

The fixed loci are very complicated.

Solution: Use different lifting of the action to $\mathcal{L}_1 \oplus \mathcal{L}_2$ to reduce to a single fixed loci.
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...and the winner is...

Reduce everything to computing integrals of the type

\[
\int_{\text{Adm}} \lambda_g \lambda_{g-i} \psi^{i-1} d^{i-1}
\]

which in generating function by Cavalieri contributes

\[
\cos^2d \left( \frac{Z_2}{2} \right).
\]
The Crepant Resolution Conjecture.
Examples
Local $\mathbb{P}(1, 2)$

The Potential

**Theorem**

\[
F^z = \frac{z_0^3}{36t_1t_2} + \frac{2t_1 + t_2}{12t_1t_2} z_0^2 z_1 + \frac{(t_1 - t_2)(4t_1 - t_2)}{3t_1t_2} z_0 z_1^2 \\
+ \frac{1}{4} z_0 z_2^2 + \frac{t_2 - t_1}{4} z_1 z_2^2 + \frac{(t_1 - t_2)^3 (8t_1 + t_2)}{36t_1t_2} z_1^3 - (2t_1 + t_2) G \\
+(2t_1 + t_2) \sum_{d \text{ odd}} \frac{1}{4d^2} \frac{(d - 2)!!}{(d - 1)!!} e^{dz_1} q^d \int \cos^2 \left(\frac{Z_2}{2}\right) dz_2 \\
+(2t_1 + t_2) \sum_{d \text{ even}} \frac{1}{2d^3} e^{dz_1} q^d.
\]

where $G''' = \frac{1}{2} \tan\left(\frac{z_2}{2}\right)$. 
But the third partial derivatives...

After analytic continuation and specialization we have

\[ F_{z_1 z_1 z_2}^z = \frac{\alpha}{\sqrt{1 + \alpha^2}}, \]

where

\[ \alpha = e^{z_1} \cos^2 \left( \frac{z_2}{2} \right). \]
The Crepant Resolution Conjecture.

Examples

Local $\mathbb{P}(1, 2)$

Set-up

The invariants

Positive degree

The Potential