

## Exam 1 Practice Problems

*Note:* These problems illustrate many (but not necessarily all) of the concepts you will need to know for the upcoming exam. You do not need to turn them in—they are solely to help you prepare for the exam. All of the homework problems will be useful in studying for the exam as well. If you have any questions, please do not hesitate to ask.

1. All of the problems in the practice problem set that is also posted online.
2. Consider the following system of equations:

$$\begin{cases} x + 2y + 4z = 1 \\ x + y + z = 0 \\ -x + \quad + z = 0 \end{cases}$$

- (a) Write this system as a matrix equation.
  - (b) Use elimination matrices to reduce this system to one of the form  $U\mathbf{x} = \mathbf{c}$ . What are  $U$  and  $\mathbf{c}$ ?
  - (c) What is the  $LU$  factorization of  $A$ ? Also, write  $L$  in terms of the elementary matrices from part (b).
3. Suppose you are given 2 unit vectors in  $\mathbb{R}^2$ , call them  $\mathbf{u}$  and  $\mathbf{v}$ . Can  $\mathbf{u} \cdot \mathbf{v} = 2$ ? Why or why not?
  4. Suppose  $A$  is an  $n \times n$  matrix with real entries.
    - (a) Define what it means for  $A$  to be invertible.
    - (b) Suppose  $A$  is invertible. Prove that the inverse of  $A$  is unique.
    - (c) If  $A$  is invertible, what is the only solution to  $A\mathbf{x} = \mathbf{b}$ ? Why is this the only solution?
    - (d) If

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix},$$

what is  $A^{-1}$ ? Please show your work.

- (e) If  $A$  and  $B$  are invertible matrices, is  $AB$  invertible? If so, please find the inverse and prove that it is indeed the inverse (you may use the fact that if  $C$  and  $D$  are square matrices with  $CD = I$ , then  $DC = I$ ). If not, please give a counter-example (the  $2 \times 2$  case will suffice).
- (f) If  $A$  is an  $n \times n$  matrix such that  $A^2 = 0$ , and if  $I$  is the  $n \times n$  identity matrix, what is  $(I + A)^{-1}$ ? Please also prove that what you found is indeed the inverse.

5. Suppose  $R$  is an  $m \times n$  matrix. Prove that  $R^T R$  is symmetric.
6. Suppose you are given a system of linear equations with 2 equations and 2 unknowns. We know that there can be no solution, infinitely many solutions, or one solution. Draw a possible row picture for each of these scenarios (there is more than one correct answer).
7. Suppose  $V$  is a vector space.
- What does it mean for  $V$  to be closed under addition?
  - Is the set of all  $2 \times 2$  invertible matrices closed under addition? If so, please prove it. If not, please provide a counter-example.
8. Let  $V$  be the space of all  $3 \times 3$  matrices with real entries. Is  $W$ , the set of all  $3 \times 3$  lower triangular matrices, a subspace of  $V$ ? Why or why not?
9. Suppose

$$A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \text{ and } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

- What is  $A\mathbf{x}$ ?
  - What is the sum of the components of  $A\mathbf{x}$ ?
  - Using your answer to part (b), does the system  $A\mathbf{x} = \mathbf{b}$  have a solution when  $\mathbf{b} = (1, 0, 0)$ ? Why or why not?
  - How many solutions does the system  $A\mathbf{x} = \mathbf{b}$  have if  $\mathbf{b} = (0, 0, 0)$ ? (Hint: Use the result you obtained in part (a)).
  - Is the matrix  $A$  invertible? Please explain, using your answer to part (d).
  - How many solutions does the system  $A\mathbf{x} = \mathbf{b}$  have if  $\mathbf{b} = (-21, 20, 1)$ ?
10. What are 2 ways of determining whether or not a given  $n \times n$  matrix  $A$  is invertible (without using the determinant)? (Hint: See Section 2.5, pages 81 and 82).
11. Suppose  $A$  is an  $m \times n$  matrix, with the form

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n \end{bmatrix}$$

where  $\mathbf{a}_i$  are column vectors in  $\mathbb{R}^m$  ( $i = 1, 2, \dots, n$ ).

- Give the definition of the column space of  $A$ , denoted  $C(A)$ .
- Prove that  $C(A)$  is a subspace of  $\mathbb{R}^m$ . (Hint: If  $\mathbf{x}$  is a vector in  $C(A)$ , how can you express  $\mathbf{x}$  in terms of the columns of  $A$ ?