NOTES ON LOCALLY CONVEX TOPOLOGICAL VECTOR SPACES

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of Rudin's *Functional Analysis*.

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1. Completeness

1.1 Definition. *Let X be a set. A collection $\mathcal{F}$ of subsets of X is called a filter if the following are satisfied:*

1. $\mathcal{F} \neq \emptyset$ *and* $\emptyset \notin \mathcal{F}$*;*
2. *if $A \in \mathcal{F}$ and $B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$;*
3. *if $A \in \mathcal{F}$ and $A \subset C \subset X$ then $C \in \mathcal{F}$;*

1.2 Definition. *Let X be an set. A collection $\mathcal{F}$ of subsets of X is called a filter base if the following are satisfied:*

1. $\mathcal{F} \neq \emptyset$ *and* $\emptyset \notin \mathcal{F}$*;*
2. *if $A \in \mathcal{F}$ and $B \in \mathcal{F}$ then there exists $C \in \mathcal{F}$ such that $C \subset A \cap B$;*

Note that every filter base $\mathcal{A}$ is contained in a unique filter $\mathcal{F}$ with the property that each element of $\mathcal{F}$ contains an element of $\mathcal{A}$. In fact, $\mathcal{F}$ is the set of all subsets of X which contain an element of $\mathcal{A}$.

1.3 Examples.

1. *If $x \in X$ and $\mathcal{F} = \{A \subset X : x \in A\}$ then $\mathcal{F}$ is a filter;*
2. *if X is a topological space, $x \in X$ and $\mathcal{A}$ is a nbhd base for the topology at x, then $\mathcal{A}$ is a filter base:*
3. *if X is a topological space and $\{x_n\}$ a sequence or net converging to $x \in X$ then the collection $\mathcal{A}$ of all sets of the form $\{x_n : n > N\}$ for some N is a filter base while $\mathcal{F} = \{A \subset X : \exists N \text{ with } x_n \in A\ \forall n > N\}$ is the corresponding filter;*
4. *if X is a topological space, Y a subspace of X, $x \in \bar{Y} \subset X$ and $\mathcal{F} = \{A \in Y : \exists \text{ nbhd } V \text{ of } x \text{ with } Y \cap V \subset A\}$ then $\mathcal{F}$ is a filter in Y but only a filter base in X.*

Note that if $Y \subset X$ then a filter base in Y is also a filter base in X. However, a filter in Y is not a filter in X if $Y \neq X$ although it is a filter base in X.

1.4 Definition. *If X is a topological space, then a filter or filter base $\mathcal{F}$ converges to $x \in X$ if for each nbhd V of x there is an $A \in \mathcal{F}$ with $A \subset V$.*

Note that a filter base converges to a point if and only if its corresponding filter does.

1.5 Definition. *If X is a topological vector space, then a filter base is Cauchy if for each 0-nbhd V there is an $A \in \mathcal{F}$ with $A - A \subset V$. If E is a subset of X then a filter base in E is called Cauchy if it is Cauchy as a filter base in X.*

Note that a filter base is Cauchy if and only if its corresponding filter is Cauchy. Note also that if E is a subset of Y and Y is a linear subspace of X then the notion of Cauchy for filter bases in E is independent of whether E is considered a subset of Y or of X.

1.6 Proposition. *If X is a Hausdorff topological space and a filter base $\mathcal{F}$ in X converges to x and to y then $x = y$.*

Proof. If $x \neq y$ let V be a nbhd of X and W a nbhd of y so that $V \cap W = \emptyset$. If $\mathcal{F}$ converges to both x and y then there exist sets $A \in \mathcal{F}$ and $B \in \mathcal{F}$ so that $A \subset V$ and $B \subset W$. But this implies that $A \cap B = \emptyset$ which contradicts the fact that $\mathcal{F}$ is a filter base.

1.7 Proposition. *If X is a topological space and $E \subset X$, then $x \in \bar{E}$ if and there is a filter base in E which converges to x.*

Proof. If $s \in \bar{E}$ set $\mathcal{A} = \{V \cap E : V \text{ a nbhd of } x\}$. Clearly $\mathcal{A}$ is a filter base in E which converges to x.

On the other hand, if $\mathcal{A}$ is a filter base in E converging to x and V is any nbhd of x, then there exists $A \in \mathcal{A}$ such that $\mathcal{A} \subset V$. This implies $V \cap E \neq \emptyset$. Hence, $x \in \bar{E}$.

1.8 Definition. *A subset E of a t.v.s. is called* **complete** *if every Cauchy filter base in E converges to a point of E.*

1.9 Proposition. *If a filter base $\mathcal{A}$ converges, then it is Cauchy.*

Proof. If W is a 0-nbhd in X and V is a symmetric 0-nbhd with $V + V \subset W$ then there is a $A \in \mathcal{A}$ such that $A \subset V$. Then

$$A - A \subset V - V = V + V \subset W$$

We conclude that $\mathcal{A}$ is Cauchy, as claimed.

Clearly a closed subset of a complete space is complete. On the other hand, we have:

1.10 Proposition. *If E is a subset of the topological vector space X and if E is complete in the relative topology it inherits from X, then E is closed in X.*

Proof. If $x \in \bar{E}$ then there is a filter base $\mathcal{A}$ in E which converges to x. By the previous proposition this is a Cauchy filter base. Since E is complete, it conveerges to some $y \in E$. Then $x = y$ by Prop. 1.6. Hence, $x \in E$ and E is closed.

1.11 Proposition. *If X is a t.v.s. then X may be embedded as a dense subspace of a complete topological vector space. This embedding is unique up to topological isomorphism.*

We leave the proof as a problem for the reader (Problem 1.1).

We end this section with a characterization of compactness for subsets of a t.v.s.

1.12 Definition. *A subset E of a t.v.s. X is called* **totally bounded** *if every 0-nbhd V in X there is a finite set $F \subset E$ so that $E \subset F + V$*

1.13 Theorem. *A subset E of a t.v.s. X is compact if and only if it is complete and totally bounded.*

Proof. If E is compact then E is clearly totally bounded. Also, if $\mathcal{F}$ is a Cauchy filter base in E then so is $\bar{\mathcal{F}} = \{\bar{A} : A \in \mathcal{F}\}$. However, a filter base of closed sets in a compact set has non-empty intersection. It follows from the fact that $\bar{\mathcal{F}}$ is Cauchy that the intersection is a single point and $\mathcal{F}$ converges to this point. Thus, E is complete if it is compact.

Now suppose that E is totally bounded and complete. We will show that E is compact by showing that every filter base of closed sets in E has a non-empty intersection (i. e. that every collection of closed sets with the finite intersection property has non-empty intersection). Thus, let $\mathcal{F}$ be a filter base of closed sets. Clearly the union of a totally

ordered family of filter bases is also a filter base. Thus, by Zorn's lemma there exists a maximal filter base $\mathcal{G}$ containing $\mathcal{F}$.

Let W be any 0-nbhd and let V be a 0-nbhd with $\bar{V} - \bar{V} \subset W$. Since E is totally bounded, there is a finite set $F \subset E$ such that $E \subset F + V$. It follows that there is at least one point $x \in F$ such that $(x + \bar{V}) \cap A \neq \emptyset$ for all $A \in \mathcal{G}$. But by the maximality of $\mathcal{G}$, this implies that $x + \mathcal{V} \in \mathcal{G}$. This, in turn, implies that $(x + \bar{V}) - (x + \bar{V}) \subset \bar{V} - \bar{V} \subset W$. hence, $\mathcal{G}$ is Cauchy. Since E is complete, $\mathcal{G}$ converges to a point of E. Since the sets in $\mathcal{G}$ are closed, this point must belong to each of them, and, hence, it belongs to $\cap\mathcal{F}$. This proves that E is compact and completes the proof.

The above result suggests that the following class of topological vector spaces may be important:

1.14 Definition. *A t.v.s. X is called* **quasi-complete** *if every closed bounded subset of X is complete.*

Since it is clear that a totally bounded subset of a t.v.s. is bounded, we have:

1.15 Corollary. *A subset of a quasi-complete topological vector space is compact if and only if it is closed and totally bounded.*

Exercises

(1) Prove Proposition 1.11 using equivalence classes of filters to construct the completion.

2. Projective and Inductive Limits

Given a family $\{\phi_\alpha : X \to X_\alpha\}$ of linear maps from a vector space X to topological vector spaces X_α, the **Projective topology** induced on X by the family is the weakest topology on X which makes each of the maps ϕ_α continuous. A basis for this topology at the origin is given by all finite intersections of sets of the form $\phi_\alpha^{-1}(V)$, where V is a 0-neighborhood in X_α for some index α.

The projective topology induced by a family of maps is characterized as follows:

2.1 Proposition. *The projective topology induced on X by a family of linear maps, as above, is the unique t.v.s. topology $\mathcal{T}$ on X with property (A): a linear map ψ from a t.v.s. Y to $(X, \mathcal{T})$ is continuous if and only if $\phi_\alpha \circ \psi : Y \to X_\alpha$ is continuous for every α.*

Proof. If $\mathcal{T}$ is the projective topology Y is an l.c.s and $\psi : Y \to X$ is a linear map then $\phi_\alpha \circ \psi$ is certainly continuous for each α if ψ is continuous. On the other hand, if $\phi_\alpha \circ \psi$ is continuous for each α then $\psi^{-1}(U)$ is open in Y whenever U is a 0-neighborhood in X of the form $\phi_\alpha^{-1}(V)$ for V a 0-neighborhood in X_α. Since sets of this form are a sub-basis for $\mathcal{T}$ at 0, ψ is continuous.

Now if $\mathcal{T}$ is any t.v.s. topology on X with property (A) and if $\mathcal{T}'$ is the projective topology, then both $\mathcal{T}$ and $\mathcal{T}'$ have property (A) and it follows that the identity maps $(X, \mathcal{T}) \to (X, \mathcal{T}')$ and $(X, \mathcal{T}') \to (X, \mathcal{T})$ are both continuous. This completes the proof.

If $\{X_\alpha\}$ is any family of topological vector spaces, then the Cartesian product $\prod X_\alpha$ (with the Cartesian product topology) is an example of a t.v.s. which has the projective topology induced by a family of linear maps. In this case, the maps are the projections $\prod X_\alpha \to X_\alpha$ onto the coordinate spaces.

2.2 Definition. *A family $\{X_\alpha, \phi_{\alpha\beta}\}$, where α and β belong to a directed set $\mathcal{A}$, X_α is a t.v.s. for each $\alpha \in \mathcal{A}$, $\{\phi_{\alpha\beta} : X_\beta \to X_\alpha\}$ is a continuous linear map for each pair $\alpha, \beta \in \mathcal{A}$ with $\alpha < \beta$ and $\phi_{\alpha\beta} \circ \phi_{\beta\gamma} = \phi_{\alpha\gamma}$ whenever $\alpha < \beta < \gamma$ is called an* **inverse directed system** *of t.v.s's. The projective limit $\varprojlim X_\alpha$ of such a system is the subspace of the Cartesian product $\prod X_\alpha$ consisting of elements $\{x_\alpha\}$ which satisfy*

$$\phi_{\alpha\beta}(x_\beta) = x_\alpha \text{ for } \alpha < \beta.$$

Note that the projective limit $\varprojlim X_\alpha$ is a closed subspace of $\prod X_\alpha$ and has the projective topology induced by the family of maps $\{\phi_\alpha : \varprojlim X_\alpha \to X_\alpha\}$ where ϕ_α is the inclusion $\varprojlim X_\alpha \to \prod X_\alpha$ followed by the projection on X_α.

If Y is a t.v.s., we say that a system of continuous linear maps $\psi_\alpha : Y \to X_\alpha$ is **compatible** with the inverse directed system $\{X_\alpha, \phi_{\alpha\beta}\}$ if $\psi_\alpha = \phi_{\alpha\beta} \circ \psi_\beta$ for all $\alpha < \beta$. Note that the system of maps $\phi_\alpha : \varprojlim X_\alpha \to X_\alpha$ is compatible with $\{X_\alpha, \phi_{\alpha\beta}\}$. It follows easily from the definitions and Prop. 2.1 that:

2.3 Proposition. *If Y is a t.v.s. and $\{\psi_\alpha\}$ is a system of continuous linear maps compatible with an inverse directed system $\{X_\alpha, \phi_{\alpha\beta}\}$ then there is a unique continuous linear map $\psi : Y \to X$ such that $\psi_\alpha = \phi_\alpha \circ \psi$ for each α.*

Proof. The system $\{\psi_\alpha\}$ determines a continuous linear map of Y into $\prod X_\alpha$ by Proposition 2.1. The compatibility condition ensures that the image of this map lies in $\varprojlim X_\alpha$.

2.4 Proposition. *The projective limit of a system of complete t.v.s's is complete.*

Proof. This will follow immediately if we can first show that the Cartesian product of a family of complete t.v.s.'s is complete since the projective limit is a closed subspace of the Cartesian product. However, a filter base in a Cartesian product is clearly Cauchy if and only if it is Cauchy in each coordinate and is convergent if and only if it is convergent in each coordinate. The proposition follows.

2.5 Proposition. *Every l.c.s. (locally convex t.v.s.) is topologically isomorphic to a dense subspace of the projective limit of an inverse directed system of Banach spaces. An l.c.s. is complete if and only if it is topologically isomorphic to the projective limit of such a system.*

Proof. If X is an l.c.s. and p is a continuous seminorm on X, then p determines a continuous linear map $\phi_p : X \to X_p$ to a Banach space as follows: Let $N_p = \{x \in X : p(x) = 0\}$. If $x, y \in N_p$ and a and b are scalars, then $0 \le p(ax + by) \le |a|p(x) + |b|p(y) = 0$ and, hence, $ax + by \in N_p$. Thus, N_p is a subspace of X. If x and y are two elements of X with $x-y \in N_p$ then $p(x) \le p(y)+p(x-y) = p(y)$. By reversing the roles of x and y we also get the reverse inequality and so $p(x) = p(y)$. This implies that p determines a well defined seminorm $\bar{p}$ on the vector space X/N_p. In fact, from the definition of N_p it is clear that this is actually a norm on X/N_p. If we complete the resulting normed space we obtain a Banach space X_p. The composition $X \to X/N_p \to X_p$ is a linear map $\phi_p : X \to X_p$ which is continuous since the inverse image of the open unit ball is just $\{x \in X : p(x) < 1\}$.

If p and q are continuous seminorms with $p \le q$ then $N_q \subset N_p$ and so there is a well defined linear map $X/N_q \to X/N_p$. This map is clearly norm decreasing and induces a norm decreasing linear map $\phi_{pq} : X_q \to X_p$. Thus, if we let p run through any system of continuous seminorms on X which is directed and is rich enough to determine the topology on X, then the system $\{X_p, \phi_{pq}\}$ will be an inverse directed system. Furthermore, the system of maps $\phi_p : X \to X_p$ is compatible with this directed system and therefore defines a continuous linear map $X \to \varprojlim X_p$. This map is injective since the continuous seminorms on X separate points and the projective topology X inherits from $\varprojlim X_p$ is the topology determined by the family of continuous seminorms on X and, hence, is the original topology of X. The proof will be complete if we can show that X is dense in $\varprojlim X_p$. However, this follows from the fact that X has dense image in each X_p and each 0-neighborhood in $\varprojlim X_p$ contains the inverse image under ϕ_p of a 0-neighborhood in X_p for some p.

In the previous proposition p was only required to run through a directed set of continuous semimorms rich enough to define the topology of X. If X is metrizable then there is a countable such set. Thus:

2.6 Proposition. *Every metrizable l.c.s. is topologically isomorphic to a dense subspace of the projective limit of a sequence of Banach spaces. Every Frechet space is topologically isomorphic to the projective limit of a sequence of Banach spaces.*

Example: Let U be an open set in $\mathbb{C}$ and let $H(U)$ denote the space of holomorphic functions on U. We give $H(U)$ the topology of uniform convergence on compact subsets of U. This is the topology determined by the family of seminorms $\{||\ ||_K : K \subset U, K \text{compact}\}$, where $||f||_K = \sup\{|f(z)| : z \in K\}$. It is clear from Proposition 2.5 that $H(U) = \lim\limits_{\leftarrow}\{\bar{H}(K) : K \subset U, K\text{compact}\}$, where $\bar{H}(K)$ is the closure in $C(K)$ of the subspace consisting of restrictions to K of functions in $H(U)$. Note that if K_n is an increasing sequence of compact subsets of U with the property that every compact subset of U is contained in some K_n then every seminorm $||\ ||_K$ is dominated by some $||\ ||_{K_n}$ and so the family $\{||\ ||_{K_n}\}$ is rich enough to determine the topology of $H(U)$. In this case, $H(U) = \lim\limits_{\leftarrow} \bar{H}(K_n)$ expresses $H(U)$ as the limit of a sequence of Banach spaces. In particular, this shows that $H(U)$ is a Frechet space. For example, if U is a disc, then $\{K_n\}$ may be chosen to be an increasing sequence of closed discs with union U. In this case, the space $\bar{H}(K_n)$ is the space of functions which are continuous on K_n and holomorphic on the interior of K_n. An analogous construction represents $C(U)$ as $\lim\limits_{\leftarrow}\{C(K) : K \subset U, K\text{compact}\}$. Here, U may be any locally compact space. In case U is the union of an increasing sequence $\{K_n\}$, as above, then $C(U)$ is the Frechet space $\lim\limits_{\leftarrow}\{C(K_n)\}$.

The **locally convex direct sum** $\bigoplus X_\alpha$ of a family $\{X_\alpha\}$ of locally convex spaces is the algebraic direct sum (the subspace of the direct product consisting of finitely non-zero elements) with the topology determined by the 0 neighborhood base consisting of those convex balanced sets whose intersections with each coordinate space are zero neighborhoods. We will typically consider each X_α to be a subspace of $\bigoplus X_\alpha$ through identification with its image under the obvious inclusion $X_\alpha \to \bigoplus X_\alpha$.

A **directed system** of t.v.s.'s is defined in the same way as an inverse directed system but with the arrows reversed.

2.7 Definition. *Given a directed system of locally convex spaces $\{X_\alpha\}$ and continuous linear maps $\{\phi_{\beta\alpha} : X_\alpha \to X_\beta$ for $\alpha < \beta\}$, the* **inductive limit** $\lim\limits_{\rightarrow} X_\alpha$ *is the quotient of the locally convex direct sum $\bigoplus X_\alpha$ by the linear subspace spanned by*

$$\{x_\alpha - \phi_{\beta\alpha}(x_\alpha) : \alpha < \beta,\ x_\alpha \in X_\alpha\}$$

provided this subspace is closed. *If this subspace is not closed then the inductive limit does not exist (or is not Hausdorff if one allows non-Hausdorff t.v.s's)*

Note that the inclusion $X_\alpha \to \bigoplus X_\alpha$ composed with the quotient map $\bigoplus X_\alpha \to \lim\limits_{\rightarrow} X_\alpha$ determines a continuous linear map $\phi_\alpha : X_\alpha \to \lim\limits_{\rightarrow} X_\alpha$ for each α.

Note also that the topology on $\bigoplus X_\alpha$ is the strongest locally convex topology that makes each of the inclusions continuous. That is, every convex balanced set that possibly could be a 0-neighborhood and have all the inclusions be continuous is a 0-neighborhood. For the same reason, the topology on $\lim\limits_{\rightarrow} X_\alpha$ is the strongest locally convex topology for which all the maps $\phi_\alpha : X_\alpha \to \lim\limits_{\rightarrow} X_\alpha$ are continuous.

If Y is an l.c.s., then a family of continuous linear maps $\{\psi_\alpha : X_\alpha \to Y\}$ is said to be compatible with the directed system $\{X_\alpha, \phi_{\beta\alpha}\}$ if $\psi_\alpha = \psi_\beta \circ \phi_{\beta\alpha}$ whenever $\alpha < \beta$.

The following is proved in a manner similar to Proposition 2.3:

2.8 Proposition. *If Y is an l.c.s. and $\psi : X_\alpha \to Y$ is a family of continuous linear maps compatible with the directed system $\{X_\alpha, \phi_{\beta\alpha}\}$, then there is a unique continuous linear map $\psi : \varinjlim X_\alpha \to Y$ such that $\psi_\alpha = \psi \circ \phi_\alpha$ for each α. This property characterizes $\varinjlim X_\alpha$.*

Example: Let $H(U)$ denote the space of holomorphic functions on a domain $U \subset \mathbb{C}$. We give $H(U)$ the topology of uniform convergence on compact subsets of U. For a fixed $z_0 \in \mathbb{C}$ we consider the directed set $\mathcal{U}_{z_0}$ consisting of all domains containing z_0 with order relation given by reverse containment ($V > U$ iff $V \subset U$). Given domains $V \subset U$ we let $r_{VU} : H(U) \to H(V)$ be the restriction map. Then $\{H(U), r_{VU}\}$ for $U, V \in \mathcal{U}_{z_0}$ is a directed limit system. The resulting inductive limit $H_{z_0} = \varinjlim\{H(U) : z_0 \in U\}$ is the space of **germs** of holomorphic functions at z_0. It is a Hausdorff l.c.s. because there are enough continuous linear functionals to separate points. In fact, for each non-negative integer n, the n^{th} derivative map $f \to f^{(n)}(z_0)$ is a continuous linear functional on $H(U)$ for each $U \in \mathcal{U}_{z_0}$ which is compatible with restriction and, hence, defines a continuous linear functional on H_{z_0} by Prop. 2.8.

Note that if one repalces $H(U)$ by $C(U)$ or $C^\infty(U)$ in the above example, the resulting inductive limit does not exist (is not Hausdorff) by Problem 2.7

2.9 Definition. *If, in a directed system of locally convex spaces, each $\phi_{\alpha\beta}$ is a topological isomorphism of X_α onto a subspace of X_β, then the system is called a* **strict inductive limit system***.*

The notion of inductive limit for locally convex spaces is not well behaved in general. However, we shall show that the strict inductive limit of an increasing sequence of locally convex spaces is quite well behaved. We first need the following lemma:

2.10 Lemma. *If X is an l.c.s. and $Y \subset X$ a linear subspace then each convex balanced 0-neighborhood V in Y is the intersection with Y of a convex balance 0 neighborhood U in X. Furthermore, if $x_0 \in X$ is a point not in the closure of Y then U may be chosen so that it does not contain x_0.*

Proof. Let W be a convex, balanced 0-neighborhood in X such that $W \cap Y \subset V$ and let U be the convex, balanced hull of $W \cup V$. Clearly $V \subset U \cap Y$. On the other hand, if $u \in U \cap Y$ then $u = \lambda x + \mu y$ with $x \in W, y \in V$ and $|\lambda| + |\mu| \le 1$. This implies that $\lambda x = u - \mu y \in Y$ which implies that $\lambda = 0$ or $x \in Y$. In either case, $u \in V$. Thus, $U \cap Y = V$.

Because W is open in X it is clear that each point $u = \lambda x + \mu y$ in U with $\lambda \neq 0$ is an interior point of U. It remains to show that points of V are interior points of U. However, if $u \in V$ we may choose t with $0 < t < 1$ small enough so that $x = 2tu \in W$ and $y = 2(1-t)u \in V$. Then $u = \frac{1}{2}(x + y)$ and, as before, u is an interior point of U. Thus, U is open.

Finally, if $x_0 \in X$ is not in the closure of Y, then the neighborhood W in the above argument may be chosen so that $(x_0 + W) \cap Y = \emptyset$. It follows that $x_0 \notin U$ for if $x_0 = \lambda x + \mu y \in U$, as above, then $x_0 - \lambda x = \mu y \in Y$ and $-\lambda x \in W$, contradicting the choice of W.

2.11 Proposition. *A sequence $\{X_n, \phi_{nm}\}$ which is a strict inductive limit system yields a Hausdorff inductive limit X and each induced map $\phi_n : X_n \to X$ is a topological isomorphism onto a subspace. the resulting space X is called the strict inductive limit of the system. If the image of X_n in X_{n+1} is closed for each n then the image of X_n in X is closed for each n.*

Proof. We identify each X_n with its image under ϕ_n so that the spaces X_n form an increasing sequence of subspaces of X, each of which has the relative topology it inherits from being a subspace of the next. Let V be a convex balanced 0-neighborhood in X_n. Using the previous lemma and induction, we may construct a sequence $\{V_m\}_{m \geq n}$ where V_m is a 0-neighborhood in X_m, $V_{m+1} \cap X_m = V_m$, and $V_n = V$. Then $U = \bigcup V_m$ is a convex, balanced subset of X with the property that $U \cap X_m = V_m$ for $m \geq n$. This implies that $U \cap V_m$ is a 0 neighborhood in V_m for all m and, hence, that U is a 0 neighborhood in X. It also implies that $V = U \cap X_n$. I follows that $\phi_n : X_n \to X$ is an open map and, hence, is a topological isomorphism. This also implies that X is Hausdorff, since each non-zero element of X belongs to some X_n and there is a 0-neighborhood in X_n which does not contain it.

If each X_n is closed in X_{n+1} then each X_n is closed in each X_m for $m \geq n$. If $x \in X$ and $x \notin X_n$ then $x \in X_m$ for some $m > n$. Since X_m is Hausdorff, it follows that there is a 0-neighborhood V in X_m for which $(x + V) \cap X_n = \emptyset$. Since $X_m \to X$ is a topological isomorphism by the above paragraph, it follows that there is a 0-neighborhood U in X with $U \cap X_m = V$. Then, $(x + U) \cap X_n = \emptyset$. Hence, X_n is closed in X.

2.12 Proposition. *If X is the strict inductive limit of an sequence of spaces $\{X_n\}$ with X_n closed in X_{n+1} for each n, then*

(1) *A sequence $\{x_i\} \subset X$ converges in X if and only there is an n so that $\{x_i\} \subset X_n$ and $\{x_i\}$ converges in X_n;*
(2) *a set $B \subset X$ is bounded if and only if there is an n so that $B \subset X_n$ and B is bounded in X_n.*

Proof. If $\{x_i\} \subset X_m$ and $\{x_i\}$ converges in X then its limit x belongs to X_n for some $n \geq m$. Then $\{x_i\}$ is a sequence contained in X_n which converges in X_n. This proves (1) in one direction. To prove the converse, it suffices to prove that a sequence which is not contained in any X_m cannot converge to 0 since, by subtracting the limit from each term of any convergent sequence we obtain a sequence converging to 0. Thus, let $\{x_i\}$ be a sequence not contained in any X_m. Then by replacing $\{x_i\}$ by a subsequence if necessary, we may assume that $x_i \notin X_i$ for each i. It then follows from Lemma 2.10 that there is a sequence of convex balanced 0-neighborhoods V_i, constructed as in the preceding Proposition, with $V_i = V_{i+1} \cap X_i$ and with $x_i \notin V_i$. If $U = \bigcup V_i$, then U is a 0-neighborhood in X which contains none of the points x_i. Thus, the sequence cannot converge to 0 in X. This proves (1).

A set which is a bounded subset of some X_n is clearly bounded in X. To prove the converse, suppose that B is a bounded set in X. If B is not contained in any X_n then it contains a sequence $\{x_n\}$ such that $x_n \notin X_n$ for each n. Since $\{x_n\}$ is bounded, $\{\dfrac{x_n}{n}\}$ converges to 0. this contradicts part (1). Hence, B is contained in some X_n. Clearly it is bounded in X_n since X_n has the relative topology it inherits from X. This proves (2).

Example: If X is any vector space, consider the directed system consisting of the finite dimensional subspaces of X, directed by inclusion. Each finite dimensional subspace has a unique t.v.s. topology and the inclusions are all topological isomorphisms. Thus, this is a strict inductive limit system. The inductive limit is X as a vector space and the inductive topology on X is called the strongest l.c.s. topology (since it is the strongest possible locally convex topology on X. If X is of countable dimension then X is the union of a sequence $\{X_n\}$ of finite dimensional subspaces. Every finite dimensional subspace of X is then contained in some X_n and it follows that $X = \lim\limits_{\rightarrow} X_n$ and the strongest l.c.s. topology on X, as described above, is the inductive topology induced by this sequence.

Other important examples of strict inductive limits will occur naturally in the study of distributions.

There is an analogue for inductive limits of the proposition (Proposition 2.5) which expresses any complete l.c.s. as a projective limit of Banach spaces. Let X be any l.c.s. and let B be a closed, convex, balanced, bounded subset of X and let $X_B = \bigcup\{tB : t > 0\}$. Then, since B is a convex, balanced absorbing subset of X_B, its Minkowski functional p_K is a seminorm on X_B with $\{x \in X_B : p_B(x) < 1\} \subset B \subset \{x \in X_B : p_B(x) \leq 1\}$. In fact, p_B is a norm on X_B since $p_K(x) = 0$ implies that $x \in tB$ for all $t > 0$ and, since B is bounded, this implies that $x \in U$ for every 0-neighborhood U which implies that $x = 0$. If we give X_B the topology induced by the norm p_B, then the inclusion map $X_B \to X$ is continuous. This is due to the fact that a sequence $\{x_n\}$ converging to 0 in X_B must eventually lie in tB for each $t > 0$. As above, this implies that it eventually belongs to U for each 0-neighborhood U. It is easy to see that X_B is complete and, hence, is a Banach space if B is a complete subset of X. This is true, in particular, in the case where X is quasi-complete.

If B and C are bounded sets with $B \subset C$ then clearly $X_B \subset X_C$ and the inclusion $i_{CB} : X_B \to X_C$ is norm decreasing. Thus, we have a direct limit system $\{X_B, i_{CB}\}$. The inclusions $X_B \to X$ form a family of continuous linear maps which are compatible with this system and, hence, they determine a continuous linear map $\lim\limits_{\rightarrow} X_B \to X$. This map is onto since its image contains every bounded subset of X and it is also clearly one to one. It will be a topological isomorphism if it is an open map. For this to be true we need that every set which is open in the inductive topology is open in X, that is, we need that every convex, balanced set in X which intersects each X_B in a set which contains a non-zero multiple of B is a 0-neighborhood in X. This will be true exactly when X has the property expressed in the following definition:

2.13 Proposition. *An l.c.s. is called* **bornological** *if every convex balanced set V in X which absorbs every bounded set B (i. e, $B \subset tV$, for some $t > 0$) is a 0-neighborhood.*

Every normed space is bornological since any set which absorbs the unit ball must contain the ball of some positive radius. Also, it is easy to see that the inductive limit (if it exists) of a family of bornological spaces is bornological (Problem 2.9). Thus, we have proved the following analogue of Proposition 2.5:

2.14 Proposition. *Each l.c.s is the image under a continuous bijection of the inductive limit $\lim\limits_{\rightarrow} X_B$ of normed spaces determined by the closed convex balanced bounded subsets*

B of X. This bijection is a topological isomorphism if and only if X is bornological. The spaces X_B are Banach spaces if X is quasicomplete.

Exercises

(1) If X has the weak topology induced by a family $\{\phi_\alpha : X \to X\alpha\}$ of linear maps, then a set $B \subset X$ is bounded in X if and only if its image under each ϕ_α is bounded in X_α.
(2) Prove that the Cartesian product of a family of quasi-complete t.v.s.'s is quasi-complete.
(3) Prove the the projective limit of an inverse system of quasi-complete t.v.s's is quasi-complete.
(4) Prove that if a subset B of the locally convex direct sum of a family of l.c.s.'s is bounded then it is contained in the direct sum of finitely many of the subspaces.
(5) Prove that the locally convex direct sum of a family of complete (quasi-complete) l.c.s.'s is complete (quasi-complete).
(6) Prove that the strict inductive limit of a sequence of complete (quasi-complete) l.c.s.'s is complete (quasi-complete).
(7) Let x_0 be a point on the real line and let $C(U)$ denote the space of continuous functions on $U \subset \mathbb{R}$ with the topology of uniform convergence on compact subsets of U. Prove that $\varinjlim\{C(U) : x_0 \in U\}$ has no non-zero continuous linear functionals and, hence, is not Hausdorff.
(8) Prove Proposition 2.8.
(9) Prove that the inductive limit of a family of bornological spaces is bornological and any Hausdorff quotient of a bornological space is bornological.

3. Tensor Products

In this section we will make considerable use of the notion of a continuous bilinear map $X \times Y \to Z$ where X, Y and Z are topological vector spaces. In particular, we will make use of the following:

3.1 Proposition. *If X, Y and Z are topological vector spaces then any bilinear map $\phi : X \times Y \to Z$ is continuous if it is continuous at* $(0, 0)$.

Proof. Let (x_0, y_0) be a point of $X \times Y$ and let W be a 0-neighborhood in Z. We choose a 0-neighborhood W_1 with $W_1 + W_1 + W_1 \subset W$. Since ϕ is continuous at $(0, 0)$, there are 0-neighborhoods $U \subset X$ and $V \subset Y$ such that $\phi(U \times V) \subset W_1$. If we choose $s, t \in (0, 1)$ so that $x_0 \in sU$, $y_0 \in tV$, $sU \subset U$ and $tV \subset V$, then for $x \in x_0 + sU$ and $y \in y_0 + tV$ we have

$$\phi(x, y) - \phi(x_0, y_0) = \phi(x - x_0, y_0) + \phi(x - x_0, y - y_0) + \phi(x_0, y - y_0) \subset W_1 + W_1 + W_1 \subset W.$$

This proves that ϕ is continuous at (x_0, y_0).

The algebraic tensor product $X \otimes Y$ of two vector spaces X and Y is the vector space generated by the set of symbols $\{x \otimes y : x \in X, y \in Y\}$ subject to the relations

$$(ax_1 + bx_2) \otimes y = a(x_1 \otimes y) + b(x_2 \otimes y),\ x_1, x_2 \in X, y \in Y, a, b \in \mathbb{C}$$

$$x \otimes (ay_1 + by_2) = a(x \otimes y_1) + b(x \otimes y_2),\ x \in X, y_1, y_2 \in Y, a, b \in \mathbb{C}.$$

It follows that every element $u \in X \otimes Y$ may be written as a finite sum $u = \sum_{i=1}^{n} x_i \otimes y_i$. The minimal number n of terms required in a representation of u as above is called the **rank** of u. If u is expressed as above using a minimal number of terms, that is, so that the number of terms n is equal to the rank of u, then it turns out that the sets $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ must both be linearly independent (problem 3.6). Of course, this representation of u is far from being unique.

One easily shows that the tensor product is characterized by the following universal property:

3.2 Proposition. *The map $\theta : X \times Y \to X \otimes Y$, defined by $\theta(x, y) = x \otimes y$, is a bilinear map with the property that any bilinear map $X \times Y \to Z$ to a vector space Z is the composition of θ with a unique linear map $\psi : X \otimes Y \to Z$.*

If X and Y are locally convex topological vector spaces, there are at least two interesting and useful ways of giving $X \otimes Y$ a corresponding locally convex topology. The most natural of these is the projective tensor product topology, which we describe below.

If p and q are continuous seminorms on X and Y, respectively, we define the tensor product seminorm $p \otimes q$ on $X \otimes Y$ as follows:

$$(p \otimes q)(u) = \inf\{\sum p(x_i)q(y_i) : u = \sum x_i \otimes y_i\}$$

It follows easily that $p \otimes q$ is, indeed, a seminorm. Furthermore, we have:

3.3 Lemma. *For seminorms p and q on X and Y,*

(1) $(p \otimes q)(x \otimes y) = p(x)q(y)$ *for all* $x \in X, y \in Y$;
(2) *if* $U = \{x \in X : p(x) < 1\}$ *and* $V = \{y \in Y : q(y) < 1\}$, *then*

$$co(\theta(U \times V)) = \{u \in X \otimes Y : (p \otimes q)(u) < 1\}.$$

Proof. From the definition, it is clear that

$$(p \otimes q)(x \otimes y) \le p(x)q(y) \text{ for all } x \in X, y \in Y$$

On the other hand, for a fixed $(x, y) \in X \times Y$, using the Hahn-Banach theorem we may choose linear functionals f on X and g on Y such that $f(x) = p(x)$, $g(y) = q(y)$ and $|f(x')| \le p(x')$, $|g(y')| \le q(y')$ for all $(x', y') \in X \times Y$. Then if $x \otimes y = \sum x_i \otimes y_i$ is any representation of $x \otimes y$ as a sum of rank one tensors, we have

$$p(x)q(y) = f(x)g(y) = \sum f(x_i)g(y_i) \le \sum p(x_i)q(y_i)$$

Since $(p \otimes q)(x \otimes y)$ is the inf of the expressions on the right side of this inequality we have $p(x)q(y) \le (p \otimes q)(x \otimes y)$. This proves (1).

Certainly $co(\theta(U \times V)) \subset \{u \in X \otimes Y : (p \otimes q)(u) < 1\}$ since the latter is a convex set containing $\theta(U \times V)$. To prove the reverse containment, let u be an element of $X \otimes Y$ with $(p \otimes q)(u) < 1$. Then we can represent u as $u = \sum x_i \otimes y_i$ with

$$\sum p(x_i)q(y_i) = r^2 < 1$$

If we set $x_i' = rp(x_i)^{-1}x_i$ and $y_i' = rq(y_i)^{-1}y_i$, then $p(x_i') = r = q(y_i')$. Thus, $x_i' \in U$ and $y_i' \in V$. Furthermore, if $t_i = r^{-2}p(x_i)q(y_i)$, then

$$u = \sum t_i(x_i' \otimes y_i') \quad \text{and} \quad \sum t_i = 1$$

Thus, $u \in co(\theta(U \times V))$ and the proof of (2) is complete.

3.4 Definition. *The topology on $X \otimes Y$ determined by the family of seminorms $p \otimes q$, as above, will be called the* **projective tensor product topology**. *We will denote $X \otimes Y$, endowed with this topology, by $X \otimes_\pi Y$.*

If $f \in X^*$ and $g \in Y^*$ then we may define a linear functional $f \otimes g$ on $X \otimes Y$ by

$$(f \otimes g)(\sum x_i \otimes y_i) = \sum f(x_i)g(y_i)$$

One easily checks that this is well defined and linear.

3.5 Proposition. *The projective tensor product topology is a Hausfdorff locally convex topology on $X \otimes Y$ with the following properties:*

(1) *the bilinear map $\theta : X \times Y \to X \otimes_\pi Y$ is continuous;*
(2) *$f \otimes g \in (X \otimes_\pi Y)^*$ for each $f \in X^*$ and $g \in Y^*$;*
(3) *A neighborhood base for the topology at 0 in $X \otimes_\pi Y$ consists of sets of the form $co(\theta(U \times V))$ where U is a 0-neigborhood in X and V is a 0 neighborhood in Y.*
(4) *any continuous bilinear map $X \times Y \to Z$ to a locally convex space Z factors as the composition of θ with a unique continuous linear map $X \otimes_\pi Y \to Z$;*

Proof. Lemma 3.3 (1) implies that each $(p \otimes q) \circ \theta$ is continuous at $(0,0)$ and this implies that θ is continuous at $(0,0)$ and, hence, is continuous everywhere by Proposition 3.1.

The continuity of $f \otimes g$ for $f \in X^*$ and $g \in Y^*$ follows from the fact that $|f|$ and $|g|$ are continuous seminorms on X and Y and $|(f \otimes g)(\sum x_i \otimes y_i)| \le \sum |f(x_i)||g(y_i)| \le |f| \otimes |g|(\sum x_i \otimes y_i)$. This proves (2).

The fact that the projective topology is Hausdorff follows from (2). In fact, if $u \in X \otimes Y$ then we may write $u = \sum x_1 \otimes y_i$, where the set $\{x_i\}$ is linearly independent. Then we may choose $f \in X^*$ such that $f(x_i) \neq 0$ if and only if $i = 1$ and we may choose $g \in Y^*$ such that $g(x_1) \neq 0$. Then the element $f \otimes g \in (X \otimes Y)^*$ has the non-zero value $f(x_1)g(x_1)$ at u. Thus, $U = \{v \in X \otimes Y : |(f \otimes g)(v)| < f(x_1)g(x_1)\}$ is an open set containing 0 but not containing u.

Part (3) is an immediate consequence of Lemma 3.3(2)

If $\phi : X \times Y \to Z$ is a continuous bilinear map, then $\phi = \psi \circ \theta$ for a unique linear map $\psi : X \otimes_\pi Y \to Z$ by Proposition 3.2. To prove (4) we must show that ψ is continuous. Let W be a convex 0-neighborhood in Z. Since ϕ is continuous, there exist 0-neighborhoods U and V in X and Y, respectively, such that $\phi(U \times V) \subset W$. Then the convex hull of $\theta(U \times V)$ is a 0-neighborhood in $X \otimes Y$ by (3) and it clearly maps into W under ψ. Thus ψ is continuous.

Note that (4) of the above proposition says that projective tensor product topology is the strongest locally convex topology on $X \otimes Y$ for which the bilinear map $\theta : X \times Y \to X \otimes Y$ is continuous.

Note also that if X and Y are normed spaces then the tensor product of the two norms is a norm on $X \otimes Y$ which determines its topology. Also, if X and Y are metrizable then so is $X \otimes Y$.

If X, Y, and Z are locally convex spaces and $\alpha : X \to Y$ is a continuous linear map then the composition

$$X \times Z \to Y \times Z \to Y \otimes_\pi Z$$

is a continuous bilinear map $X \times Z \to Y \otimes_\pi Z$ and, by Propostion 3.5(4), it factors through a unique continuous linear map $\alpha \otimes id : X \otimes_\pi Z \to Y \otimes_\pi Z$. This shows that, for a fixed l.c.s. Z, $(\cdot) \otimes_\pi Z$ is a functor from the category of locally convex spaces to itself. Similarly, the projective tensor product is also a functor in its second argument for each fixed l.c.s. appearing in its first argument.

3.6 Proposition. *If $\alpha : X \to Y$ is a continuous linear open map, then so is $\alpha \otimes id : X \times Z \to Y \otimes Z$.*

Proof. To show that $\alpha \otimes id$ is open we must show that each 0-neighborhood in $X \otimes Z$ maps to a 0-neighborhood in $Y \otimes Z$. However, this follows immediately from Proposition 3.5(3) and the hypothesis that α is an open map.

The space $X \otimes_\pi Y$ is generally not complete. It is usually useful to complete it.

3.7 Definition. *The completion of $X \otimes_\pi Y$ will be denoted $X \hat{\otimes}_\pi Y$ and will be called the completed projective tensor product of X and Y.*

Note that if $\alpha : X \to Y$ is a continuous linear map, the map $\alpha \otimes id$ extends by continuity to a continuous linear map $\alpha \otimes id : X \hat{\otimes}_\pi Z \to Y \hat{\otimes}_\pi Z$. Even under the hypotheses of Proposition 3.6 this map is not generally a surjection. However, we do have:

3.8 Proposition. *If X, X and Z are Frechet spaces and $\alpha : X \to Y$ is a surjective continuous linear map, then $\alpha \otimes id : X \hat{\otimes}_\pi Z \to Y \hat{\otimes}_\pi Z$ is a surjection.*

Proof. By the open mapping theorem, the map α is open. Then $\alpha \otimes id$ is open by Proposition 3.6. Since, the topologies of X, Y and Z have countable bases at 0 the same is true of $X \otimes_\pi Z$ and $Y \otimes_\pi Z$. However, an open map between metrizable t.v.s's has the property that every Cauchy sequence in the range has a subsequence which is the image of a Cauchy sequence in the domain (Exercise 3.1). Since every point in the completion $X \hat{\otimes}_\pi Z$ is the limit of a Cauchy sequence in $Y \otimes_\pi Z$, the result follows.

Obviously, the analogues of Proposition 3.6 and 3.8 with the roles of the left and right arguments reversed are also true.

There are other hypotheses under which the conclusion of the above Proposition is true and we will return to this question when we have developed the tools to prove such results.

In the case where X and Y are Frechet spaces, elements of the completed projective tensor product $X \hat{\otimes}_\pi Y$ may be represented in a particularly useful form:

3.9 Proposition. *If X and Y are Frechet spaces then each element $u \in X \hat{\otimes}_\pi Y$ may be represented as the sum of a convergent series*

$$u = \sum_{i=1}^{\infty} \lambda_i \, x_i \otimes y_i$$

where $\sum_{i=1}^{\infty} |\lambda_i| < \infty$ and $\{x_i\}$ and $\{y_i\}$ are sequences converging to 0 in X and Y, respectively.

Proof. Let $\{p_n\}$ and $\{q_n\}$ denote increasing sequences of seminorms generating the topologies of X and Y, respectively. Let r_n denote the extension to $X \hat{\otimes}_\pi Y$ of the product seminorm $p_n \otimes q_n$. If $\{u_n\}$ is a sequence in $X \otimes Y$ converging to u in the projective topology, then we may, by replacing $\{u_n\}$ by an appropriately chosen subsequence, assume that the sequence $\{v_n\}$, where $v_1 = u_1$ and $v_n = u_n - u_{n-1}$ for $n > 1$, satisfies

$$r_n(v_n) < n^{-2} 2^{-n} \quad \text{and} \quad \sum v_n = u$$

It follows that we may write each v_n as a finite sum

$$v_n = \sum_i x'_{ni} \otimes y'_{ni} \quad \text{with} \quad \sum_i p_n(x'_i) q_n(y'_i) < n^{-2} 2^{-n}$$

If we set

$$x_{ni} = n^{-1} p_n(x'_{ni})^{-1} x'_{ni}, \quad y_{ni} = n^{-1} p_n(y'_{ni})^{-1} y'_{ni}, \quad \lambda_{ni} = n^2 p_n(x_{ni}) q_n(y_{ni})$$

then

$$p_n(x_{ni}) = q_n(y_{ni}) = n^{-1}, \quad \sum_i |\lambda_{ni}| < 2^{-n} \text{and} \quad v_n = \sum_i \lambda_{ni}\, x_i \otimes y_i$$

and so

$$u = \sum_{n,i} \lambda_{ni}\, x_i \otimes y_i, \quad \sum_{n,i} |\lambda_{ni}| < \infty \quad \text{and} \quad \lim_n p_m(x_{ni}) = \lim_n p_m(x_{ni}) = 0 \quad \forall\, m$$

The proof is complete if we reindex $\{x_{ni}\}, \{y_{ni}\}$, and $\{\lambda_{ni}\}$ to form singly indexed sequences.

3.10 Definition. *Let $\mathcal{B}(X, Y)$ denote the space of continuous bilinear forms (continuous bilinear maps to $\mathbb{C}$) on $X \times Y$. Let $B(X, Y)$ denote the space of bilinear forms on $X \times Y$ which are continuous in each variable separately.*

3.11 Proposition. *The dual space of $X \hat{\otimes}_\pi Y$ is naturally isomorphic to $\mathcal{B}(X, Y)$.*

Proof. If $\theta : X \times Y \to X \hat{\otimes}_\pi Y$ is the bilinear map defined by $\theta(x, y) = x \otimes y$ then $f \to f \circ \theta$ is clearly a linear map of $X \hat{\otimes}_\pi Y$ to $\mathcal{B}(X, Y)$. It is an isomorphism by Prop. 3.5(4).

We now proceed to the second important way of topologizing the tensor product of two locally convex spaces.

Let X^*_σ denote X^* with its weak-* tolology - that is, with the weak topology that X induces on X^*. With a similar meaning for Y^*, we note that $X \otimes Y$ is naturally embedded in $B(X^*_\sigma, Y^*_\sigma)$. In fact, if $u = \sum x_i \otimes y_i \in X \otimes Y$ the map

$$(f, g) \to \phi_u(f, g) = \sum f(x_i) g(y_i) : X^* \times Y^* \to \mathbb{C}$$

is a separately continuous bilinear form on $X^*_\sigma \times Y^*_\sigma$. If we choose the set $\{x_i\}$ to be linearly independent, then it is easy to see that $\phi_u \neq 0$ if $u \neq 0$. Thus, $u \to \phi_u$ is an embedding of $X \otimes Y$ into $B(X^*_\sigma, Y^*_\sigma)$.

3.12 Definition. *The topology of bi-equicontinuous convergence on $B(X^*_\sigma, Y^*_\sigma)$ is the topology of uniform convergence on sets of the form $A \times B$ where A and B are equicontinous subsets of X^* and Y^*, respectively. We denote by $B_e(X^*_\sigma, Y^*_\sigma)$ the space $B(X^*_\sigma, Y^*_\sigma)$ with this topology. We denote by $X \otimes_e Y$ the space $X \otimes Y$ with the topology it inherits from its natural embedding in $B_e(X^*_\sigma, Y^*_\sigma)$. Finally, we denote by $X \hat{\otimes}_e Y$ the completion of $X \otimes_e Y$.*

Note that a family of seminorms determining the topology on $B_e(X^*_\sigma, Y^*_\sigma)$ consists of the seminorms of the form $p_{A,B}$, where A and B are equicontinuous subsets of X^* and Y^*, respectively and:

$$p_{A,B}(\phi) = \sup\{|\phi(f, g)| : (f, g) \in A \times B\}.$$

A typical 0-neighborhood in this topology has the form

$$V_{A,B} = \{\phi \in B(X_\sigma^*, Y_\sigma^*) : |\phi(f,g)| < 1 \; \forall \; f \in A, \; g \in B\}$$

For this to make sense, we need to know that each $\phi \in B(X_\sigma^*, Y_\sigma^*)$ is bounded on each set of the form $A \times B$ with A and B equicontinuous. However, note that every equicontinuous subset of X^* is contained in one of the form V° for V a 0-neighborhood and sets of the form V° are compact in X_σ^* by the Banach-Alaoglu theorem. thus, we may assume that A and B are compact and convex. Since an element $\phi \in B(X_\sigma^*, Y_\sigma^*)$ is separately continuous, it is bounded on $\{x\} \times B$ for each $x \in A$ and on $A \times \{y\}$ for each $y \in B$. It follows from the Banach-Steinhaus Theorem for compact convex sets that ϕ is bounded on $A \times B$. Thus, the seminorm $p_{A,B}$ is defined on all of $B(X_\sigma^*, Y_\sigma^*)$. Clearly, the topology generated by the family of such seminorms makes $B_e(X_\sigma^*, Y_\sigma^*)$ into an l.c.s.

If $\alpha : X \to Y$ is a continuous linear map between l.c.s.'s and Z is any l.c.s, then the adjoint map $\alpha^* : Y_\sigma^* \to X_\sigma^*$ is also continuous and maps equicontinuous sets to equicontinuous sets. It follows that α^* induces a continuous linear map

$$(\alpha^* \times id)' : B_e(X_\sigma^*, Z_\sigma^*) \to B_e(Y_\sigma^*, Z_\sigma^*)$$

where

$$(\alpha^* \times id)'\phi(f,g) = \phi(\alpha^*(f), g)$$

Restricted to the image of $X \otimes Z$ in $B_e(X_\sigma^*, Z_\sigma^*)$, this map is just $\alpha \otimes id : X \otimes Z \to Y \otimes Z$. Thus, it follows that $\alpha \otimes id$ is a continuous linear map from $X \otimes_e Z$ to $Y \otimes_e Z$ and extends to a continuous linear map $\alpha \otimes id : X \hat{\otimes}_e Z \to Y \hat{\otimes}_e Z$ between their completions. Thus, for a fixed l.c.s Z, $(\cdot) \otimes_e Z$ and $(\cdot)\hat{\otimes}_e Z$ are functors from the category of locally convex spaces to itself.

3.13 Lemma. *If X and Y are l.c.s.'s and $\alpha : X \to Y$ is a topological isomorphism onto its range. Then under $\alpha^* : Y^* \to X^*$ each equicontinuous set in X^* is the image of an equicontinuous set in Y^*.*

Proof. Since α is a topological isomorphism onto its image, we may identify X with its image and consider it a subspace of Y. Then α is the inclusion and α^* the restriction map from functionals on Y to functionals on X. Thus, we must show that each equicontinuous set in X^* is the restriction of an equicontinuous set in Y^*. Since the class of equicontinuous sets is closed under passing to subsets, it is enough to show that each equicontinuous set A in X^* is contained in the restriction of an equicontinuous set B in Y^*. Furthermore, we may assume that $A = U^\circ$ for a convex, balance 0-neighborhood $U \subset X$ since every equicontinuous set is contained in one of this form. However, by Lemma 2.10, for each such neighborhood U there is a convex, balanced 0 neighborhood $V \subset Y$ such that $V \cap X = U$. The proof will be complete if we can establish that U° is the image of V° under the restriction map. To this end, we let p_V be the Minkowski functional of V. Since $V \cap X = U$, the restriction of p_V to X is the Minkoski functional p_U of U. Now if $f \in X^*$, it is easy to see that $f \in U^\circ$ if and only if $|f| \le p_U$ (Problem 3.5). Thus, if $f \in U^\circ$ then $|f| \le p_V$. Now the Hahn-Banach theorem implies that f has an extention g to Y which satisfies $|g| \le p_V$. This, in turn, implies that $g \in V^\circ$. Thus, U° is the restriction to X of V° and the proof is complete.

3.14 Proposition. *If X, Y and Z are l.c.s.'s and $\alpha : X \to Y$ is a topological isomorphism onto its image, then the induced maps $\alpha \otimes_e id : X \otimes Z \to Y \otimes_e Z$ and $\alpha \otimes id : X \hat{\otimes}_e Z \to Y \hat{\otimes}_e Z$ are as well.*

Proof. This will certainly follow if we can show that the above map

$$(\alpha^* \times id)' : B_e(X^*_\sigma, Z^*_\sigma) \to B_e(Y^*_\sigma, Z^*_\sigma)$$

is a topological isomorphism onto its range. However, the fact that α is a topological isomorphism onto its range implies that every equicontinuous set in X is the image of an equicontinuous set in Y under α^* by the previous lemma. This, in turn, implies that the 0-neighbohood in $B_e(X^*_\sigma, Z^*_\sigma)$ consisting of elements bounded by one on $A \times C$ for a pair of equicontinuous sets $A \subset X^*$ and $C \subset Z$, will be the inverse image under $(\alpha^* \times id)'$ of the 0-neighborhood, defined analogously, by $B \times C$, where B is an equicontinuous set in Y^* with $\alpha^*(B) = A$. This gives the result for the uncompleted tensor product. The result for the completed tensor products follows from Problem 3.4.

How are the projective and bi-equicontinuous convergence tensor product topologies related? The projective is stronger. That is,

3.15 Proposition. *The identity map $X \otimes_\pi Y \to X \otimes_e Y$ is continuous.*

Proof. A typical 0-neighborhood in $X \otimes_e Y$ has the form

$$W = \{u \in X \otimes Y : |(f \otimes g)(u)| \le 1 \; \forall \; f \in A, g \in B\}$$

where A and B are equicontinuous subsets of X^* and Y^*, respectively. But if $U = A^\circ$ and $V = B^\circ$, then U and V are 0-neighborhoods in X and Y, respectively, and $\theta(U \times V) \subset W$. Since W is convex, we also have $co(\theta(U \times V) \subset W$. Since $co(\theta(U \times V)$ is a 0-neighborhood in $X \times_\pi Y$, the proof is complete.

Exercises

(1) Prove that a continuous linear open map $\phi : X \to Y$ between two metrizable spaces t.v.s.'s has the property that each Cauchy sequence in Y is has a subsequence which is the image under ϕ of a Cauchy sequence in X.

(2) Prove that a continuous linear open map between two metrizable l.c.s.'s induces a surjection between their completions.

(3) Let μ be a positive countably additive measure on a measurable space Ω and let E be a Banach space. We define $L^1(\mu, E)$ to be the completion of the space of simple (finite valued) measurable functions with values in E in the norm defined by $||f|| = \int |f| \, d\mu$. Prove that $L^1(\mu, E)$ is isometrically isomorphic to $L_1(\mu) \hat{\otimes}_\pi E$.

(4) Prove that if $\alpha : X \to Y$ is a tolological isomorphism onto its image then the same is true of the induced map on completions $\bar{\alpha} : \bar{X} \to \bar{Y}$

(5) Prove that if U is a convex, balanced open subset of an l.c.s. X and p_U is its Minkowski functional, then $f \in U^\circ$ if and only if $f \in X^*$ and $|f| \le p_U$.

(6) Prove that if $u \in X \otimes Y$ is expressed as $u = \sum_{i=1}^{n} x_i \otimes y_i$ with $n = \text{rank } u$, then the sets $\{x_i\}_{i=1}^{n}$ and $\{y_i\}_{i=1}^{n}$ are both linearly independent.

(7) Prove that if Δ is a compact Hausdorff space, E is any banach space, and $C(\Delta, E)$ is the space of continuous E-valued functions on Δ, then $C(\Delta, E)$ is isomorphically isometric to $C(\Delta)\hat{\otimes}_\pi E$.

4. Duality

The key to most of the results in topological vector space theory is to exploit duality – the relationship between an l.c.s. X and its dual X^*. The results of this section, particularly, show how this works.

We will need to work with a variety of topologies on an l.c.s. X and its dual X^*. Below we give a list of some of the more important topologies on X and X^*.

4.1 Definition. *Given an l.c.s. X and its dual X^* we define the following spaces, each of which is X or X^* with the indicated topology:*

(1) X_σ – *(the weak or $\sigma(X, X^*)$ topology) the topology of uniform convergence on finite subsets of X^* ;*
(2) X^*_σ – *(the weak-* or $\sigma(X^*, X)$ topology) the topology of uniform convergence on finite subsets of X ;*
(3) X_τ – *(the Mackey topology of X) the topology of uniform convergence on compact convex subsets of X^*_σ ;*
(4) X^*_τ –*(the Mackey topology of X^*) the topology of uniform convergence on compact convex subsets of X_σ;*
(5) X^*_β – *(the strong or $\beta(X^*, X)$ topology) the topology of uniform convergence on bounded subsets of X ;*
(6) X – *(the original topology on X) the topology of uniform convergence on equicontinuous subsets of X^*_σ.*

The topology of uniform convergence on equicontinuous sets in X^* is the original topology of X due to the fact that a set is equicontinuous in X^* if and only if it is contained in V° for V a 0-neighborhood in X and the bipolar theorem, which implies that $V^{\circ\circ} = \bar{V}$ if V is convex and balanced.

Each topology in 4.1 is the topology of uniform convergence on sets belonging to a class $\mathcal{S}$ of subsets of a space in a dual pair relation with the space we are topologizing. Let us consider a dual pair of vector spaces (X, Y) and ask what properties of a family $\mathcal{S}$ of subsets of Y are needed in order that the topology of uniform convergence on members of $\mathcal{S}$ determines an l.c.s. topology on X. The topology of uniform converence on members of $\mathcal{S}$ is the topology generated by the family of seminorms of the form p_A for $A \in \mathcal{S}$ where

$$q_A(x) = \sup\{|x(y)| = | < x, y > | : y \in A\}$$

Obviously this supremum will not be finite for all $x \in X$ unless each function $x(y)$ for $x \in X$ is bounded on the set A. Thus, we need the sets in $\mathcal{S}$ to be bounded for the $\sigma(Y, X)$ topology. If every $y \in Y$ belongs to some $A \in \mathcal{S}$ then each of the functions $y(x) =< x, y >$ will be dominated by one of the seminorms q_A and, hence, each $y \in Y$ will determine a continuous linear functional on X. Since the collection of such functions separates points, this will imply that X is Hausdorff in the topology generated by the seminorms q_A. If we want the family of seminorms to be directed, then for each finite set $\{A_i\} \subset \mathcal{S}$ there should exist $B \in \mathcal{S}$ such that $\bigcup A_i \subset B$. Under these conditions, the sets of the form

$$\{x \in X : q_A(x) < \epsilon\} \quad A \in A, \quad \epsilon > 0$$

form a neighborhood base at 0 for a Hausdorff l.c.s. topology on X with the property that the elements of Y are all continuous as linear functionals on X. If we also insist that $tA \in \mathcal{S}$ whenever $A \in \mathcal{S}$ and $t > 0$ then the polars A° of sets $A \in \mathcal{S}$ form a neighborhood base at 0 for this topology. This discussion is summarized in the following proposition:

4.2 Proposition. *Let (X, Y) be a dual pair and let $\mathcal{S}$ be a family of subsets of Y with the following properties:*

(1) *$\mathcal{S}$ is a family of $\sigma(Y, X)$-bounded subsets of Y;*
(2) *$\bigcup\{A : A \in \mathcal{S}\} = Y$;*
(3) *the union of the members of any finite subset of $\mathcal{S}$ is contained in a member of $\mathcal{S}$;*
(4) *if $A \in \mathcal{S}$ and $t > 0$ then $tA \subset \mathcal{S}$.*

Then the topology of uniform convergence on members of $\mathcal{S}$ is a Hausdorff l.c.s topology on X relative to which every $y \in Y$ determines a continuous linear functional. The polars in X of members of $\mathcal{S}$ form a neighborhood base at 0 for this topology.

We will call a family $\mathcal{S}$ of subsets of Y that satisfies (1) - (4) of Prop. 4.2 a **basic family**. We will call the topology of uniform convergence on members of a basic family $\mathcal{S}$ the $\mathcal{S}$-topology. Clearly the families determining the topologies described in Definition 4.1 are basic families.

By the bipolar theorem, a set $A \subset Y$ and its closed, convex, balanced hull have the same polar in X. Therefore, a basic family $\mathcal{S}$ will determine the same topology as the basic family obtained by replacing each set in $\mathcal{S}$ by its $\sigma(Y, X)$-closed, convex, balanced hull. Thus, we could insist that the families $\mathcal{S}$ we use to define topologies consist of closed, convex, balanced sets. The reason we don't do this is that it is easier to say "unifom convergence on all finite sets" than to say "uniform convergence on all convex balanced hulls of finite sets". Never the less, whenever it is convenient, we shall assume that the sets in a basic family $\mathcal{S}$ are closed, convex and balanced.

4.3 Definition. *Given a dual pair (X, Y), an l.c.s topology on X is said to be* **consistent** *with the pairing (X, Y) provided the dual of X with this topology is equal to Y as a vector space.*

Note this means that each function $y(x) = < x, y >$, for $y \in Y$ is continuous in the given topology on X and that every continuous linear functional has this form.

4.4 Theorem(Mackey-Arens). *A topology on X is consistent with the pairing (X, Y) if and only if it is the $\mathcal{S}$-topology for a basic family of $\sigma(Y, X)$ compact, convex, balanced subsets of Y.*

Proof. Let X be given a topology consistent with the pairing, so that $Y = X^*$, then the polar of each 0-neighborhood in X is a convex, balanced and $\sigma(X^*, X)$-compact subset of X^* by the Banach-Aloaglu Theorem. The bipolar theorem implies that the given topology on X is the $\mathcal{S}$-topology for the basic family $\mathcal{S}$ consisting of all polars of 0-neighborhoods in X.

On the other hand, suppose $\mathcal{S}$ is a basic family of convex, balanced, $\sigma(X, Y)$ compact subsets of Y and give X the $\mathcal{S}$-topology. Then for each $y \in Y$ the function $y(x) = < x, y >$ is a continuous linear functional on X. We must show that every continuous linear functional

on X arises in this way. Thus, suppose $f \in X^*$. Then $\{x \in X : |f(x| < 1\}$ is a 0-neighborhood in X since f is continuous. It follows that this neighborhood must contain a neighborhood of the form $V = A^\circ$ for a set $A \in \mathcal{S}$. Now $f \in V^\circ = A^{\circ\circ}$ where the bipolar is taken in X^* and A is considered a subset of X^* through the embedding $Y \to X^*$. Since A is $\sigma(Y, X)$ compact and $Y \to X^*$ is continuous from the $\sigma(Y, X)$ topology to the $\sigma(X^*, X)$ topology, A is closed in X^*. Now the bipolar theorem implies that $V^\circ = A$ and, hence, that $f \in A$. In particular, $f \in Y$. This completes the proof.

4.5 Corollary. *Given a dual pair (X, Y) there is a strongest locally convex topology on X consistent with the pairing. It is the $\tau(X, Y)$-topology where $\tau(X, Y)$ is the basic family consisting of all convex, balanced $\sigma(Y, X)$-compact subsets of Y. This is called the Mackey topology on X for the pairing (X, Y).*

When the pairing is (X, X^*) for an l.c.s X, the space X with the Mackey topology is the space X_τ of Definition 4.1. When the pairing is (X^*, X) the space X^* with the Mackey topology is the space X^*_τ of Definition 4.1.

Note that the $\sigma(X, Y)$ topology is the weakest locally convex topology consistent with the pairing (X, Y).

4.6 Definition. *An l.c.s. X for which the original topology is the Mackey topology $\tau(X, X^*)$ is called a* **Mackey space***.*

4.7 Definition. *If X is an l.c.s. then a* **barrel** *in X is a closed, convex, balanced absorbing set. If every barrel in X is a 0-neighborhood, then X is said to be* **barreled***.*

Note Problem 4.1 which says that every Frechet space is barreled, Problem 4.2 which says that every quasi-complete bornological space is barreled and Problem 4.3 which says that every bornological space is a Mackey space and every barreled space is a Mackey space.

4.8 Proposition. *An l.c.s. X is barreled if and only if every $\sigma(X^*, X)$-bounded subset of X^* is equicontinuous.*

Proof. A set $B \subset X^*$ is $\sigma(X^*, X)$-bounded if and only if each $x \in X$ is bounded on B when considered a linear functional on X^*. This is the case if and only if each x belongs to $tB_\circ$ for some $t > 0$, that is, if and only if $B_\circ$ is absorbing. In other words, the $\sigma(X^*, X)$-bounded subsets of X^* are those whose polars are absorbing. On the other hand, the equicontinuous subsets of X^* are those whose polars are 0-neighborhoods. Obviously then, every barrel in X is a 0-neighborhood if and only if every $\sigma(X^*, X)$-bounded subset of X^* is equicontinuous.

Note that the pairing (X^*, X) expresses X as a space of linear functionals on X^* and these are clearly continuous in any $\mathcal{S}$-topology for a basic family $\mathcal{S}$ of bounded subsets of X. This is true, in particular, for the strong topology on X^*. Thus, there is a canonical linear map $X \to (X^*_\beta)^*$. If we also give $(X^*_\beta)^*$ its strong topology, then we have a canonical map $X \to (X^*_\beta)^*_\beta$. This is not, in general, even continuous. However, we have:

4.9 Proposition. *The canonical map $X \to (X^*_\beta)^*_\beta$ is an open map onto its range. It is a topological isomorphism onto its range if X is barreled.*

Proof. A typical 0-neighborhood in $(X^*_\beta)^*_\beta$ is of the form $U = B^\circ$ for a bounded set $B \subset X^*_\beta$. also, $V = U \cap X$ is the polar, taken in X, of B. This will be a 0-neighborhood in X exactly when B is equicontinuous. Equicontinuous sets in X^* are strongly bounded and, so the map $X \to (X^*_\beta)^*_\beta$ is always open onto its range. If X is barreled then $\sigma(X^*, X)$-bounded sets of X^* are equicontinuous and, hence, strongly bounded sets are equicontinuous. It follows that $X \to (X^*_\beta)^*_\beta$ is continuous if X is barreled. This completes the proof.

4.10 Definition. *An l.c.s. X is called* **reflexive** *if the natural map $X \to (X^*_\beta)^*_\beta$ is a topological isomorphism.*

4.11 Proposition. *An l.c.s X is reflexive if and only if it is barreled and every bounded subset of X has weakly compact weak closure.*

Proof. If X is barreled then $X \to (X^*_\beta)^*_\beta$ is a topological isomorphism onto its range. If every bounded subset of X has weakly compact weak closure then every closed, convex, balanced, bounded set is weakly compact and it follows that the strong and Mackey topologies agree on X^*. Thus, by the Mackey-Arens theorem, the dual of X^*_β is X. In other words, $X \to (X^*_\beta)^*_\beta$ is surjective.

To prove the converse, note first that if X is reflexive then the $\sigma(X^*, X)$ topology on X^* is the weak topology of the l.c.s X^*_β, since this space has dual equal to X. Thus, $\sigma(X^*, X)$ bounded sets in X^* are weakly bounded and, hence, strongly bounded subsets of X^*_β. Also by reflexivity, a set in X^*_β is bounded if and only if its polar is a 0-neighborhood in X. This implies that bounded sets in X^*_β are equicontinuous and, hence, that $\sigma(X^*, X)$ bounded sets are equicontinuous. By Prop. 4.8, X is barreled. Now if X is reflexive then X^*_β is also (problem 4.9) and its strong dual is X. Thus, we also conclude that bounded subsets of X are equicontinuous as sets of functionals on X^*_β. The Banach-Aloaglu theorem then implies that bounded sets have $\sigma(X, X^*)$ compact closures. This completes the proof.

4.12 Corollary. *If X is reflexive then both X and X^*_β are barreled.*

4.13 Corollary. *If X is reflexive then both X and X^*_β are quasi-complete.*

Proof. If X is reflexive and $B \subset X$ is closed and bounded then the weak closure of B is weakly compact. A Cauchy filter base $\mathcal{F}$ in B in the original topology will be Cauchy in the weak topology as well and, hence, will converge weakly to a point x in the weak closure of B. But since $\mathcal{F}$ is Cauchy in the original topology, it must converge to x in the original topology. Thus, $x \in B$ and B is complete. This proves that X is quasi-complete and now the same argument applies to X^* since it is also reflexive.

The method used in Proposition 4.8 can be used to give a very general version of the Banach-Steinhaus Theorem.

4.14 Theorem. *If X and Y are l.c.s.'s and X is barreled then*

(1) *if $\mathcal{F}$ is a set of continuous linear maps from X to Y such that $\{\phi(x) : \phi \in \mathcal{F}\}$ is bounded for each $x \in X$, then $\mathcal{F}$ is equicontinuous;*
(2) *if a filter base of continuous linear maps from X to Y converges pointwise, then the limit is also a continuous linear map from X to Y.*

Proof. If V is a closed convex balanced 0-neighborhood in Y, then $U = \cap\{\phi^{-1}(V) : \phi \in \mathcal{F}\}$ is a closed, convex, balanced subset of X. For each $x \in X$, since $\{\phi(x) : \phi \in \mathcal{F}\}$ is bounded it is contained in tV for some $t > 0$. Then, $x \in tU$. Thus, U is absorbing and is, therefore a barrel. Since X is barreled, U is a 0-neighborhood. This implies that $\mathcal{F}$ is equicontinuous and completes the proof of (1). Part (2) follows immediately from part (1).

We next take up the problem of characterizing completeness in terms of properties of the dual space.

4.15 Proposition. *Let X be an l.c.s and $\mathcal{S}$ a basic family of bounded subsets of X. The space X^* is complete in the $\mathcal{S}$-topology if and only if every linear functional on X which is continuous on each member of $\mathcal{S}$ is also continuous on X.*

Proof. Let $\mathcal{F}$ be a Cauchy filter base in X^* with the $\mathcal{S}$ topology. Then $\mathcal{F}$ will converge pointwise to a linear functional f. Since the convergence is uniform on each member of $\mathcal{S}$, the functional f is continuous on each member of $\mathcal{S}$. If the condition of the Proposition is satisfied then f is continuous and, hence, f is an element of X^*. Clearly $\mathcal{F}$ converges uniformly to f on each member of $\mathcal{S}$. Thus, X^* is complete in the $\mathcal{S}$-topology.

To prove the converse, we assume that X^* is complete in the $\mathcal{S}$ topology and f is a linear functional on X which is continuous on each member of $\mathcal{S}$. Then we claim that f can be uniformly approximated on each member of $\mathcal{S}$ by elements of X^*. If we can show this then it will follow that f is the uniform limit on members of $\mathcal{S}$ of a filter in X^* and, hence, that $f \in X^*$ since X^* is complete in the $\mathcal{S}$ topology. To establish the claim, let $S \in \mathcal{S}$. We may assume that S is convex, balanced and closed (otherwise we just replace it by its closed balance convex hull). If f is continuous on S then for each $\epsilon > 0$ there exists a closed, convex, balanced 0-neighborhood $U \subset X$ so that $|f(x)| < \epsilon$ for all $x \in U \cap S$. Thus, $\epsilon^{-1} f \in (U \cap S)^\circ$, where the polar is taken in the space X' of all (not neccessarily continuous) linear functionals on X. The pair (X, X') is a dual pair and the sets U and S are closed for the $\sigma(X, X')$ topology since it is stronger than the $\sigma(X, X^*)$ topology. It follows from the bipolar theorem that $(U \cap S)^\circ$ is the $\sigma(X', X)$ closed, convex, balanced hull of $U^\circ \bigcup S^\circ$ and, hence, is contained in the $\sigma(X', X)$ closure of $U^\circ + S^\circ$. However, U° is $\sigma(X', X)$ compact by the Aloaglu theorem and S° is $\sigma(X', X)$ closed, so $U^\circ + S^\circ$ is already $\sigma(X', X)$ closed. It follows that $f = g + h$ with $g \in \epsilon U^\circ$ and $h \in \epsilon S^\circ$. However, every element of the (X', X) polar of a 0-neighborhood in X is neccessarily continuous and, hence, in X^*. Thus $g \in X^*$ and $|f - g| = |h| < \epsilon$ on S.

4.16 Corollary. *If X is any l.c.s. then the completion of X may be identified with the space of all linear functionals on X^* which are $\sigma(X^*, X)$ continuous on each equicontinuous subset of X^*, provided this space is endowed with the topology of uniform convergence on equicontinuous sets.*

Proof. Let X_1 denote the space of linear functionals on X^* which are $\sigma(X^*, X)$-continuous on each equicontinuous set. Since each equicontinuous set in X^* is contained in one which is the polar of a 0-neighborhood in X and each of these is $\sigma(X^*, X)$ compact by the Banach-Aloaglu Theorem, each element of X_1 is bounded on equicontinuous sets. Thus, the topology of uniform convergence on equicontinuous sets is defined on X_1 and makes it into an l.c.s. which is clearly complete. We have $X \subset X_1$ and the topology X_1 induces

on X is the original topology. Then the closure of X in X_1 is complete. It follows from Proposition 4.15, with X^*_σ here playing the role of the space X of Prop. 4.15, that this closure is all of X_1. Hence, X_1 is the completion of X.

4.17 Corollary. *An l.c.s. X is complete if and only a hyperplane H in X^* is $\sigma(X^*, X)$ closed whenever $H \cap A$ is $\sigma(X^*, X)$ closed in A for each equicontinuous set A.*

Proof. Since a linear functional is continuous if and only if the hyperplane which is its kernel is closed, this is just Prop. 4.15 with X^*_σ here playing the role of the space X of the proposition.

The above corollary provides some motivation for the following definition:

4.18 Definition. *We say that an l.c.s X is* **B-complete** *if a linear subspace $Y \subset X^*$ is $\sigma(X^*, X)$ closed whenever $Y \cap U^\circ$ is closed for each 0-neighborhood $U \subset X$.*

By corollary 4.17, every B-complete space is complete. For metrizable spaces the two notions agree. To prove this, we must first prove a couple of major theorems concerning duality:

4.19 Theorem(Banach-Dieudonne). *If X is a Frechet space then the Mackey topology of X^* is the $\mathcal{S}$-topology where $\mathcal{S}$ is the family consisting on the ranges of all null sequences in X and this is the strongest topology on X^* which agrees with $\sigma(X^*, X)$ on each equicontinuous subset of X^*.*

Proof. Since the closed convex hull of a compact set is compact in a Frechet space, the Mackey topology on X^* agrees with the topology of uniform convergence on compact subsets of X. Clearly this topology is stronger than the $\mathcal{S}$-topology.

Now given any compact set $K \subset X$ and any 0-neighborhood U in X there is a finite set $F \subset X$ so that $2K \subset F + U$. Then

$$F^\circ \cap U^\circ \subset (\frac{1}{2}F + \frac{1}{2}U)^\circ \subset K^\circ$$

It follows from this that the topology of uniform convergence on compact sets agrees with the topology of uniform convergence on finite sets (the $\sigma(X^*, X)$ topology on each set of the form U° for U a 0-neighborhood and, hence, on each equicontinuous set.

To complete the proof, we will show that any topology which agrees with $\sigma(X^*, X)$ on each equicontinuous set is weaker than the $\mathcal{S}$ topology. That is, we will show that if $V \subset X^*$ has the property that its intersection with each polar U° of a 0-neighborhood is $\sigma(X^*, X)$ open in U° then V is open in the $\mathcal{S}$-topology. That is, we will show that for each point $g \in V$ there is a $\mathcal{S}$ neighborhood of g contained in V. Without loss of generality we may assume that V contains 0 and $g = 0$. Let $\{U_n\}$ be a decreasing sequence of closed, convex sets forming a neighborhood base for the topology of X. We construct by induction a sequence $\{F_n\}$ of non-empty finite subsets of X such that

$$F_n \subset U_n, \quad H^\circ_{n-1} \cap U^\circ_n \subset V \quad \text{where} \quad H_n = \bigcup_{k=1}^{n} F_k$$

Suppose the F_n have been chosen for $n < m$. If it is not possible to choose F_m then $F^\circ \cap H_{m-1}^\circ \cap U_{m+1}^\circ = (F \cup H_{m-1})^\circ \cap U_{m+1}^\circ$ fails to be contained in V for all finite non-empty sets $F \subset U_m$. Then, if $\tilde{V}$ is the complement of V, the sets $F^\circ \cap H_{n-1}^\circ \cap U_{m+1}^\circ \cap \tilde{V}$ for F a finite non-empty subset of U_m form a collection of $\sigma(X^*, X)$ closed subsets with the finite intersection property of the $\sigma(X^*, X)$-compact set $U_{m+1}^\circ \cap \tilde{V}$. Thus, there is a point f in the intersection. Since $U_m^\circ = \cap\{F^\circ : F \subset U_m, F \text{ finite}\}$, f must be a point of $U_m^\circ \cap H_{m-1}^\circ \cap U_{m+1}^\circ \cap \tilde{V}$ which is impossible since $U_m^\circ \cap H_{m-1} \subset V$. The resulting contradiction completes the induction step. To get the induction started, we choose $U_0 = X$ and $H_{-1} = \emptyset$. Then the condition $H_{-1}^\circ \cap U_0^\circ \subset V$ is just the condition that V contain 0.

Having constructed the sets F_n as above, we set $S = \bigcup_n F_n$ and note that $S^\circ \cap U_n^\circ \subset V$ for all n. Since $\bigcup_n U_n^\circ = X^*$, this implies that $S^\circ \subset V$. Since $S \in \mathcal{S}$, this proves that V contains a 0-neighborhood in the $\mathcal{S}$ topology and completes the proof of the theorem.

4.20 Theorem(Krein-Smulian). *A metrizable space X is complete if and only if a convex set in X^* is $\sigma(X^*, X)$ closed whenever its intersection with U° is $\sigma(X^*, X)$ closed for each 0-neighborhood U in X. In particular, every Frechet space is B-complete.*

Proof. By 4.17, we need only prove that a metrizable complete space (a Frechet space) has the above property. Let X be a Frechet space and let B be a convex set for which $B \cap A$ is $\sigma(X^*, X)$ closed for every equicontinuous set $A \subset X^*$. Then the complement of B in X^* is a set whose intersection with each equicontinuous A set is $\sigma(X^*, X)$ open in A and, hence, by the previous theorem, it is open in the Mackey topology. Thus, B is closed in the Mackey topology. Since the Mackey topology is a topology consistent with the pairing (X^*, X), the dual of X_τ^* is X. It follows that B is also $\sigma(X^*, X)$ closed since it is convex. This completes the proof.

Finally, we are able to prove the general open mapping and close graph theorems.

4.21 Theorem(Ptak). *If $\phi : X \to Y$ is a surjective continuous linear map from a dense subspace X_0 of a B-complete space X to a barreled space Y, and if ϕ has a closed graph in $X \times Y$ then ϕ is an open map.*

Proof. We first prove this under the assumption that ϕ is injective. Let U_0 be any convex, balanced neighborhood of 0 in X_0. Then $\phi(U_0)$ is convex, balanced and absorbing in Y since ϕ is surjective. Hence, the closure of $\phi(U_0)$ is a barrel and is, thus, a 0-neighborhood in Y since Y is barreled. Thus, we have proved that the image under ϕ of each 0-neighborhood in X_0 is dense in a 0-neighborhood in Y.

Let $\phi^* : Y^* \to X^* = X_0^*$ be the adjoint of ϕ. That is, $\phi^*(f)(x) = f(\phi(x))$ for $f \in Y^*$ and $x \in X_0$. Let U be any 0-neighborhood in X and set $V = \phi(U \cap X_0)$. Note that $f \in V^\circ$ if and only if $\phi^*(f) \in (U \cap X_0)^\circ = U^\circ$. Thus,

$$(\phi^*)^{-1}(U^\circ) = V^\circ$$

and V° is closed and equicontinuous and, hence, compact in Y_σ^*. Since ϕ^* is continuous from Y_σ^* to X_σ^*, we conclude that $\phi^*(V^\circ)$ is compact in $X_0^* = X^*$ with the $\sigma(X^*, X_0)$ topology. Note that, $\phi^*(V^\circ) = U^\circ \cap \phi^*(Y^*)$.

We claim that the topologies $\sigma(X^*, X)$ and $\sigma(X^*, X_0)$ agree on the set $\phi^*(V^\circ)$. To prove this, we will show that for each $x \in X$ the function $f \to x(f) =< x, f >$ is continuous

on $\phi^*(V^\circ) = U^\circ \cap \phi^*(Y)$. Since X^0 is dense in X, given $x \in X$ and $\epsilon > 0$ there exists $x_0 \in X_0$ such that $x - x_0 \in \epsilon U$. Then $|x(f) - x_0(f)| < \epsilon$ for $f \in U^\circ$. This means that, as a function on $\phi^*(V^\circ)$, x is the uniform limit of functions $x_0 \in X_0$. Since these are all continuous, x is continuous on $\phi^*(V^\circ)$ as well. This establishes the claim and proves that $\phi^*(V^\circ) = U^\circ \cap \phi^*(Y^*)$ is compact and, hence, closed in X^*_σ. Since this is true for every 0 -neighborhood U, the B-completeness of X implies that $\phi^*(Y^*)$ is closed in X^*_σ.

Next we show that $\phi^*(Y^*) = X^*$. In fact, if u is a continuous linear functional on X^*_σ which vanishes on $\phi^*(Y^*)$, then u is an element of X, since $(X^*_\sigma)^* = X$, and $\phi^{**}(u) = 0$. However, $\phi^{**} : X \to Y$ is continuous and agrees with ϕ on X_0. Since, X_0 is dense in X, it follows that $(u, \phi^{**}(u)) \in X \times Y$ is in the closure of the graph of ϕ. Since ϕ has closed graph, it follows that $u \in X_0$. Then $\phi(u) = \phi^{**}(u) = 0$ and this implies $u = 0$ since we assumed that ϕ was injective. This proves that $\phi^*(Y^*) = X^*$ since $\phi^*(Y^*)$ is closed in X^*_σ and the only continuous linear functional on X^*_σ which vanishes on $\phi^*(Y^*)$ is 0.

The fact that $\phi^*(Y^*) = X^*$ now implies that every finite subset of X^* is the image of a finite subset of Y^*. On taking polars, this implies that $\phi : (X_0)_\sigma \to Y_\sigma$ is an open map. In other words, $\phi^{-1} : Y_\sigma \to (X_0)_\sigma$ is continuous. This, in turn, implies that if U_0 is any closed convex 0-neighborhood in X_0 then $V = \phi(U)$ is a weakly closed convex subset of Y and, hence, is closed in the original topology of Y. Since ϕ is surjective, V is absorbing and, hence, is a barrel. Since Y is barreled, V is a 0-neighborhood in Y. this completes the proof that ϕ is open in the case where ϕ is injective.

We now reduce the general case to the case where ϕ is injective. Let $N = \ker(\phi)$ then N is a closed subspace of X since it is the inverse image of the graph of ϕ under the continuous map $x \to (x, 0) : X \to X \times Y$. Thus, X/N is Hausdorff and is a B-complete space by Problem 4.8. It contains X_0/N as a dense subspace. The quotient map $X \to X/N$ is continuous and open in both the weak and original topologies and ϕ is the composition of $X_0 \to X_0/N$ with a continuous isomorphism $\psi : X_0/N \to Y$. Furthermore, the graph of ψ is the image of the graph G of ϕ under the quotient map $X \times Y \to (X \times Y)/(N \times \{0\})$. This is an open map and the image of G is the complement of the image of the complement of G because $N \times \{0\} \subset G$. Thus, the graph of ψ is also closed. We now have that ψ is an injective map which satisfies the hypotheses of the Theorem and, hence, ψ is an open map. It follows that ϕ is open also since the composition of open maps is open. This completes the proof.

4.22 Corollary(Open Mapping Theorem). *If X is B-complete and Y is barreled then every continuous surjective linear map from X to Y is open.*

Proof. A continuous linear map which is defined everywhere on X has a closed graph in $X \times Y$. Hence, this follows directly from Ptak's Theorem.

4.23 Corollary(Closed Graph Theorem). *If X is B-complete and Y is barreled then every linear map from $\psi : Y \to X$ which has a closed graph is continuous.*

Proof. If ψ is injective and has dense image in X, then $\phi = \psi^{-1}$ satisfies the hypotheses of Theorem 4.21 and, hence, is an open map. This implies that ψ is continuous. Thus, we need only show that the general case can be reduced to the case where ψ is injective with dense image. To this end, let ψ be a any map satisfying the hypotheses of the corollary. We then replace Y by $Y_1 = Y/N$, where $N = \ker\psi$, X by the closure X_1 of $\psi(Y)$ in X

and ψ by the map $\psi_1 : X_1 \to Y_1$ induced by ψ. By Problem 7, the quotient of a barreled space by a closed subspace is barreled and by Problem 8 a closed subspace of a B-complete space is B-complete. Furthermore, as in the proof of Theorem 4.21, we conclude that ψ_1 has closed graph since ψ does. Thus, ψ_1 satisfies the hypotheses of the corollary and is injective with dense image. We conclude that ψ_1 is continuous. Since ψ is just ψ_1 preceded by the quotient map $Y \to Y/N = Y_1$ and followed by the inclusion $X_1 \to X$, it is also continuous.

Exercises

(1) Prove that every Frechet space is barreled.
(2) Prove that every quasi-complete bornological space is barreled.
(3) Prove that bornological spaces and barreled spaces are Mackey spaces.
(4) Prove that the Mackey dual X^*_τ of a Frechet space is B-complete and, hence, that the strong dual of a reflexive Frechet space is B-complete.
(5) Prove that if $\phi : X \to Y$ is a continuous linear map between l.c.s's and $\phi^* : Y^* \to X^*$ is the adjoint, then $\phi^* : Y^*_\sigma \to X^*_\sigma$ is continuous. Prove ϕ is injective if and only if ϕ^* has dense image in X^*_σ,
(6) If X is an l.c.s. and $N \subset X$ is a closed subspace, then the adjoint of the projection $X \to X/N$ identifies $(X/N)^*$ with $\{f \in X^* : f(x) = 0 \; \forall \; x \in N\} = N^\circ$.
(7) Prove that every Hausdorff quotient of a barreled space is barreled.
(8) Prove that every closed subspace and every Hausdorff quotient of a B-complete space is B-complete.
(9) Prove that if X is a reflexive l.c.s. then so is X^*_β.

5. Nuclear Spaces

In this section we introduce and study a class of l.c.s.'s with striking properties. These are the Nuclear spaces introduced by Grothedieck.

Let X and Y be l.c.s.'s. The space $X^*\otimes Y$ is linearly isomorphic to a subspace of $L(X,Y)$ - the space of continuous linear maps from X to Y - as follows: if $u = \sum f_i \otimes y_i \in Y \otimes X^*$, we assign to u the linear map $\ell_u \in L(X,Y)$ defined by

$$\ell_u(x) = \sum f_i(x) y_i$$

If the y_i are chosen to be linearly independent (as they may), then the only way this sum can be zero is if $f_i(x) = 0$ for every i. If this happens for every x, then the f_i are all zero and so is u. Thus, the map $u \to \ell_u$ is injective as well as (obviously) linear. Thus, we may identify $X^* \otimes Y$ with a linear subspace of $L(X,Y)$. In fact, it is obviously the linear subspace consisting of finite rank continuous linear maps from X to Y.

Now suppose that X and Y are Banach spaces. Then X^* is also a Banach space, as is the completed projective tensor product $X^*\hat{\otimes}_\pi Y$. The map $u \to \ell_u : X^* \otimes Y \to L(X,Y)$ above is norm decreasing if $L(X,Y)$ is given the operator norm and $X^* \otimes Y$ the tensor product norm. In fact,

$$||\ell_u|| = \sup\{||\ell_u(x)|| : ||x|| \le 1\} \le \sum ||f_i|| ||y_i|| ||x|| \text{ if } u = \sum f_i \otimes y_i$$

Thus

$$||\ell_u|| \le \inf\{\sum ||f_i|| ||y_i|| : u = \sum f_i \otimes y_i\} = ||u||$$

It follows that $u \to \ell_u$ extends to a norm decreasing linear map $X^*\hat{\otimes}_\pi Y \to L(X,Y)$.

5.1 Definition. *If X and Y are Banach spaces then a* **nuclear map** *from X to Y is an element of $L(X,Y)$ of the form ℓ_u for some $u \in X^*\hat{\otimes}_\pi Y$. If X and Y are arbitrary l.c.s.'s then a nuclear map from X to Y is a map $\phi \in L(X,Y)$ which factors as $\mu \circ \psi \circ \nu$ where E and F are Banach spaces, $\nu \in L(X,E)$, $\mu \in L(F,Y)$, and $\psi : E \to F$ is nuclear.*

5.2 Proposition. *If X and Y are l.c.s.'s then a linear map $\phi \in L(X,Y)$ is nuclear if and only if it has the form*

$$\phi(x) = \sum_{n=1}^{\infty} \lambda_n f_n(x) y_n$$

where $\{f_n\}$ is a sequence in X^* which converges uniformly to 0 on some 0-neighborhood $V \subset X$, $\{y_n\}$ is a sequence which converges to 0 in the space Y_B for some balanced, convex, bounded subset $B \subset Y$ for which Y_B is complete and $\sum |\lambda_n| < \infty$.

Proof. If ϕ has the above form, we let X_V denote the Banach space constructed, as in Prop. 2.5, from X and the Minkowski functional p_V of V. We let $\nu : X \to X_V$ be the natural map. Clearly, from the choice of V, the sequence $\{f_n\}$ determines a null sequence $\{g_n\} \subset X_V^*$ such that $f_n = g_n \circ \nu$. Now let Y_B be the Banach space and $Y_B \to Y$ the

embedding determined by the bounded, convex, balanced set $B \subset Y$ as in Prop. 2.14. Then $\{y_n\}$ may be regarded as a null sequence in Y_B. The fact that $\sum |\lambda_n| < \infty$ implies that the series $\sum \lambda_n \, g_n \otimes y_n$ converges in the tensor product norm to an element $u \in X_V^* \hat{\otimes}_\pi Y_B$. If $\psi = \ell_u$ then, clearly, $\phi = \mu \circ \psi \circ \nu$. Thus, ϕ is nuclear if it has the above form.

On the other hand, if $\phi = \mu \circ \psi \circ \nu$ is nuclear, with E and F Banach spaces and $\nu : X \to E$, $\mu : F \to Y$ and $\psi : E \to F$ as in Definition 5.1, then, since ψ is nuclear it has the form ℓ_u for an element $u \in E^* \hat{\otimes} F$. It then follows from Proposition 3.9 that ϕ has the above form.

5.3 Definition. *If X and Y are l.c.s.'s and $\phi \in L(X, Y)$, then ϕ is called* **compact** *if there exists a 0-neighborhood U in X such that $\phi(U)$ has compact closure in Y.*

5.4 Proposition. *Every nuclear map between l.c.s.'s is compact.*

Proof. Let $\phi : X \to Y$ be a nuclear map between two Banach spaces and represent it as in Proposition 3.9. By modifying the f_i, y_i, and λ_i by appropriate constant factors, we may assume that $||f_i|| \le 1$ for all i and $\sum \lambda_i = 1$. Then the image of the unit ball in X under ϕ will be contained in the closed convex hull of the null sequence $\{y_i\}$. Since Y is a Banach space, and a null sequence is a set with compact closure, it follows that the closed convex hull of the set $\{y_i\}$ is compact in Y. This proves the proposition in the case of a nuclear map between Banach spaces. The general case follows from this and the fact that the composition of a compact linear map with a continuous linear map (on either side) is compact (Problem 5.1).

Note that every finite rank operator is nuclear and, hence, compact.

The following proposition is obvious from the definition of nuclear map.

5.5 Proposition. *The composition of a nuclear map with a continuous linear map (on either side) is nuclear.*

5.6 Proposition. *If $\phi : X \to Y$ is a nuclear map, then ϕ has a unique extension to a nuclear map $\bar{\phi} : \bar{X} \to Y$ from the completion of X to Y.*

Proof. Let $\phi = \mu \circ \psi \circ \nu$, as in Definition 5.1, then $\nu : X \to E$, is a continuous linear map of X into a Banach space E and, hence, it has a unique extension $\bar{\nu} : \bar{X} \to E$ to the completion of X. Then $\bar{\phi} = \mu \circ \psi \circ \bar{\nu}$ is the require extension of ϕ. It is obviously nuclear and unique.

5.7 Proposition. *If $\phi : X \to Y$ is a nuclear map then its adjoint $\phi^* : Y_\beta \to X_\beta$ is also nuclear.*

Proof. If X and Y are Banach spaces and ϕ is the nuclear map determined by $u \in X^* \bar{\otimes}_\pi Y$ then we write $u == \sum \lambda_i f_i \otimes y_i$ as in Proposition 3.9. so that

$$\phi(x) = \sum f_i(x) y_i$$

It follows that, for $g \in Y^*$,

$$\phi^*(g) = \sum g(y_i) f_i \in X^*$$

Thus, ϕ^* is determined by u^t where $u' = \sum \lambda_i y_i \otimes f_i \in Y^{**}\bar{\otimes}_p iX^*$. It follows that ϕ^* is also nuclear. This proves the proposition when X and Y are Banach spaces. The general case follows immediately, since if $\phi = \mu \circ \psi \circ \nu$ with ψ a nuclear map beween Banach spaces, then $\phi^* = \nu^* \circ \psi^* \circ \mu^*$.

Recall that for each closed, convex, balanced 0-neighborhood U in an l.c.s. X, the Minkowski functional p_U induces a norm on X/N_U, where $N_U = \{x \in X : p_U(x) = 0\}$. If X_U is the completion of X/N_U with this norm then we have a continuous linear map $\phi_U : X \to X_U$ so that U is the inverse image of the closed unit ball in X_U under ϕ_U. Furthermore, if $V \subset U$ are two such 0-neighborhoods in X, then there is a norm decreasing linear map $\phi_{UV} : X_U \to X_V$ such that $\phi_U = \phi_{UV} \circ \phi_V$. The system $\{X_U, \phi_{UV}\}$ is then an inverse limit system which has an inverse limit which contains X as a dense subspace.

5.8 Definition. *A* **nuclear space** *is an l.c.s. X with a basis $\mathcal{U}$ of convex, balanced 0-nieghborhoods such that the map $\phi_U : X \to X_U$ is a nuclear map for each $U \in \mathcal{U}$.*

5.9 Proposition. *For an l.c.s. X the following statements are equivalent:*

(1) *X is a nuclear space;*
(2) *for every every convex, balanced 0-neighborhood U there is a convex, balanced 0-neighborhood $V \subset U$ such that the map $\phi_{UV} : X_V \to X_U$ is nuclear;*
(3) *every continuous linear map from X to a Banach space is nuclear.*
(4) *the map $\phi_U : X \to X_U$ is nuclear for every convex, balanced 0-neighborhood U.*

Proof. If $\alpha : X \to E$ is a continuous linear map from X to a Banach space E and if α has dense range, then there is a 0-neighborhood $V \subset X$ such that E and $\alpha : X \to E$ may be identified with X_V and $\phi_\alpha : X \to X_V$. In fact, if $V = \alpha^{-1}(B)$ where B is the unit ball in E then

$$\ker \alpha = N_V = \ker p_V = \{x \in X : x \in tV \ \forall t > 0\}$$

and the resulting map $X/N_V \to E$ induced by α is norm preserving. Since E is complete and $\alpha(X)$ is dense, this extends to an isometry of X_V onto E. Clearly if X_V and E are identified via this isometry then α will be identified with ϕ_α.

Now suppose X is a nuclear space and U is a convex, balanced 0-neighborhood in X. There exits a convex, balanced 0-neighborhood $W \subset U$ such that the map $\phi_U : X \to X_W$ is nuclear. This means that it factors as $\psi \circ \alpha$ where $\alpha : X \to E$ is a continuous linear map of X to a Banach space and $\psi : E \to X_W$ is a nuclear map between Banach spaces. We may assume without loss of generality that $\alpha(X)$ is dense in E since, otherwise, we may simply replace E by the closure of $\alpha(X)$ in E. Then, as noted in the first paragraph, there is a 0-neighborhood $V \subset X$ such that E may be identified with X_V, $\alpha : X \to E$ with $\phi_V : X \to X_V$ and $\psi : E \to X_W$ with the map $\phi_{WV} : X_V \to X_W$. Then ϕ_{WV} is nuclear because ψ is nuclear. However, $\phi_{UV} = \phi_{UW} \circ \phi_{WV}$ and, hence, ϕ_{UV} is also nuclear. This proves that (1) implies (2).

Now suppose that (2) holds and that $\psi : X \to E$ is any continuous linear map to a Banach space E. Again we may assume without loss of generality that α has dense range. This means, by the first paragraph, that we may choose a convex, balanced 0-neighborhood U such that E may be identified with X_U and $\psi : X \to E$ with $\phi_U : X \to X_U$. Applying (2), then gives us a convex, balanced 0-neighborhood $V \subset U$ such that $\phi_{UV} : X_V \to X_U$

is nuclear. But $\psi = \phi_U$ factors as $\phi_{UV} \circ \phi_V$ and so ψ is also nuclear. This proves that (2) implies (3). That (3) implies (4) and (4) implies (1) are trivial.

The following is an immediate consequence of the above proposition.

5.10 Corollary. *The completion of a nuclear space is nuclear and every complete nuclear space is the projective limit of an inverse system consisting of Banach spaces and nuclear maps.*

5.11 Corollary. *If X is a nuclear space and $\bar{X}$ is its completion, then every bounded subset B of X has compact closure in $\bar{X}$.*

Proof. We may identify $\bar{X}$ with $\varprojlim X_U$ where U runs through any basis of convex, balanced 0-neighborhoods by Proposition 2.5. Since each map $\phi_U : X \to X_U$ is nuclear we have that $\phi_U(B)$ has compact closure K_U in X_U for each U. Since $\varprojlim X_U$ is a closed subspace of $\prod X_U$ the image of B under $X \to \bar{X} = \varprojlim X_U$ is a subset of $\prod K_U$. The latter is compact by Tychonoff's Theorem and, hence, B has compact closure in $\bar{X}$.

5.12 Proposition. *Every barreled, quasi-complete nuclear space is reflexive. In particular, every nuclear Frechet space is reflexive.*

Proof. A space is reflexive if and only if it is barreled and every weakly closed bounded subset is weakly complact. If the space is nuclear then every bounded subset has compact closure in the completion. If the space is quasicomplete, then every weakly closed bounded subset is closed in the completion. The result follows.

A **Montel space** is a reflexive l.c.s in which every closed bounded set is compact. By 5.11 and 5.12 every nuclear Frechet space is a Montel space.

5.13 Proposition. *Every nuclear map between two Banach spaces factors through a Hilbert space. If X is a nuclear space, then each convex, balanced 0-neighborhood U contains a convex, balanced 0-neighborhood V such that X_V is a Hilbert space.*

Proof. If $\phi : X \to Y$ is a nuclear map between two Banach spaces then, by Proposition 5.2, it has the form

$$\phi(x) = \sum \lambda_i f_i(x) y_i$$

with $\{f_i\}$ a bounded sequence in X^*, $\{y_i\}$ a bounded sequence in Y and $\lambda_i \geq 0$ with $\sum \lambda_i < \infty$. We may assume without loss of generality that $||y_i|| = 1$ for all i. We define a map $\alpha : X \to \ell^2$ by

$$\alpha(x) = \{\sqrt{\lambda_i} f_i(x)\}$$

that this is in ℓ^2 follows from the summability of $\{\lambda_i\}$ and the boundedness of $\{f_i\}$, which implies $\{f_i(x)\}$ is a uniformly bounded sequence for x in some 0-neighborhood of X. This also implies that that α is continuous. Furthermore, we have

$$||\phi(x)|| = ||\sum \lambda_i f_i(x) y_i|| \leq \sum \lambda_i |f_i(x)| = \sum \left(\sqrt{\lambda_i}\right) \left(\sqrt{\lambda_i} f_i(x)\right)$$

and, on applying Holder's inequality to the last expression we have

$$||\phi(x)|| \leq \left(\sum \lambda_i\right)^{\frac{1}{2}} \left(\sum \lambda_i |f_i(x)|^2\right)^{\frac{1}{2}} = ||\alpha(x)||_2$$

It follows from this that $\alpha(x) \to \phi(x)$ is a well defined, norm decreasing map from the image of α to Y. Hence, it extends to a bounded linear map $\beta : H \to Y$ where H is the closure of the image of α. Clearly, H is a Hilbert space and $\phi = \beta \circ \alpha$. This completes the proof of the first statement.

To prove the second statement, let X be nuclear and choose $W \subset U$ so that the natural map $\phi : X_W \to X_U$ is nuclear. Then it factors as $\beta \circ \alpha$ with $\alpha : X_W \to H$ and $\beta : H \to X_U$ and H a Hilbert space. If V is the inverse image of the unit ball in H under the composition of $X \to X_W$ with α, then X_V is isomorphic to the closure of the image of α in H and is, hence, a Hilbert space.

5.14 Corollary. *Every complete nuclear space is the projective limit of an inverse system of Hilbert spaces and nuclear maps.*

5.15 Proposition. *The product of any family of nuclear spaces is nuclear and the direct sum of a countable family of nuclear spaces is nuclear.*

Proof. Let $X = \oplus_{i=1}^{\infty} X_i$ by the direct sum of a sequence of nuclear spaces X_i and suppose $\psi : X \to Y$ is a continuous linear map of X to a Banach space. Then the composition of ψ with the inclusion $X_i \to X$ is a Nuclear map $\psi_i : X_i \to Y$. Hence, it has the form

$$\psi_i(x) = \sum_{n=1}^{\infty} \lambda_{in} f_{in}(x) y_{in}$$

where we may assume that $||y_{in}|| \leq 1$, $\sum\limits_{n=1}^{\infty} |\lambda_{in}| \leq i^{-2}$, and that for each i the sequence $\{f_{in}\}_n$ is equicontinuous on X_i.

We claim that, if each f_{in} is extended to X by defining it to be zero on the subspaces X_j for $j \neq i$ then the entire family $\{f_{in}\}_{i,n}$ is equicontinuous. In fact, if

$$U_i = \{x \in X_i : |f_{ni}| \leq 1, \forall n\}$$

then U_i is a 0-neighborhood in X_i by the equicontinuity of $\{f_{in}\}_n$ and if U is the convex hull of the union of the images of the U_i in X then U is a 0-neighborhood in $X = \oplus_{i=1}^{\infty} X_i$ in the direct sum topology. Since $|f_{in}| \leq 1$ on U for each i, n the double sequence $\{f_{in}\}_{i,n}$ is equicontinuous. Since the map ψ can be written as:

$$\psi(x) = \sum_{i,n} \lambda_{in} f_{in}(x) y_{in}$$

and $\sum\limits_{i,n} |\lambda_{in}| < \infty$ we have that ψ is a nuclear map. This proves that X is a nuclear space, since ψ was an arbitrary continuous linear map to a Banach space.

Now suppose that $X = \prod X_\alpha$ is the product of an arbitrary family of nuclear spaces and let $\psi : X \to Y$ be a continuous linear map to a Banach space Y. If B is the unit ball in Y, then $\psi^{-1}(B)$ is a 0-neighborhood in X and, hence, contains a neighborhood of the form

$$V = \{\{x_\alpha\} : x_{\alpha_i} \in V_i,\ i = 1, \cdots, n\}$$

for some finite set $\{\alpha_1, \cdots, \alpha_n\}$ of indices and a corresponding set of 0-neighborhoods $V_i \subset X_{\alpha_i}$. It follows that ψ must vanish on the subspace $N = \prod\limits_{\alpha \neq \alpha_i} X_\alpha$ since N is contained in V but is closed under multiplication by scalars. Thus, ψ factors through $X/N = \oplus_{i=1}^n X_{\alpha_i}$. This is a nuclear space by the result of the previous paragraph and so ψ is the composition of a nuclear map with the quotient map $X \to X/N$ and is, therefore, also nuclear. This proves that X is nuclear.

5.16 Proposition. *Every subspace of a nuclear space is nuclear and every Hausdorff quotient of a nuclear space is nuclear.*

Proof. Let X be a nuclear space and $Y \subset X$ a subspace. We claim that every continuous linear map ϕ from Y to a Banach space E has an extension which is a continuous linear map from X to E. If we can establish this claim then we may conclude that every such ψ is nuclear since its extension to X must be nuclear. From this we conclude that Y is nuclear. To establish the claim, note that X has a basis of convex, balanced 0-neighborhoods U with the property that X_V is a Hilbert space by Prop. 5.13. If U is such a neighborhood and $V = Y \cap U$, then Y_V is also a Hilbert space and, in fact, is the closure in X_V of the image of Y under the natural map $X \to X_V$. Since neighborhoods of this form comprise a basis of 0-neighborhoods in Y and since E is a Banach space, the map $\phi : Y \to E$ factors through Y_V for some V of this form. That is, ϕ is the composition of the natural map $Y \to Y_V$ with a continuous linear map $Y_V \to E$. However, we may extend the latter map to a continuous linear map $X_U \to E$ by defining it to be 0 on the orthogonal complement of Y_V in X_U. The compostion of this map with the natural map $X \to X_U$ is the desired extension of ϕ. This completes the proof that a subspace of a nuclear map is nuclear.

Now suppose Y is a closed subspace of a nuclear space X. We wish to prove that the quotient X/Y is also nuclear. If $\phi : X/Y \to E$ is a continuous linear map of X/Y to a Banach space, then the composition ψ of this map with the quotient map $X \to X/Y$ is nuclear since X is nuclear. Hence, there is a 0-neighborhood $U \subset X$ such that X_U is a Hilbert space and ψ is the composition of the natural map $X \to X_U$ with a nuclear map $\alpha : X_U \to E$. Since α must vanish on the closure K of the image of Y in X_U, it defines a continuous linear map $\beta : X_U/K \to E$ and $\phi : X/Y \to E$ is the composition of the natural map $X/Y \to X_U/K$ with β. Furthermore, β is nuclear, since it may be represented as the composition of α with the embedding of X_U/K as the orthogonal complement of K in X_U. Thus, we conclude that ϕ is nuclear, which competes the proof.

5.17 Corollary. *The inverse limit of any inverse system of nuclear spaces is nuclear and the inductive limit of any sequence of nuclear spaces is nuclear provided it is Hausdorff.*

Note that it is also true that the projective tensor product of two nuclear spaces is nuclear by Problem 5.3. In fact, if one of the spaces is nuclear, then tensor product is an especially nice operation. The remainder of this section is devoted to proving this.

Recall that the space $B_e(X^*_\sigma, Y^*_\sigma)$ is the space of separately continuous bilinear forms on $E \times F$ with the topology of uniform convergence on products of equicontinuous sets. The tensor product $X \otimes Y$ embedds in $B_e(X^*_\sigma, Y^*_\sigma)$ and the topology it inherits from this embedding is the topology of $X \otimes_e Y$.

5.18 Theorem. *If X is a nuclear space and Y any l.c.s then $X \otimes_e Y$ and $X \otimes_\pi Y$ are topologically isomorphic and the canonical embedding of this space in $B_e(X^*_\sigma, Y^*_\sigma)$ has dense range.*

Proof. We first prove that $X \otimes Y$ is dense in $B_e(X^*_\sigma, Y^*_\sigma)$. A typical 0-neighborhood in $B_e(X^*_\sigma, Y^*_\sigma)$ has the form

$$\Omega_{AB} = \{u \in B_e(X^*_\sigma, Y^*_\sigma) : |u(f,g)| < 1 \; \forall f \in A, g \in B\}$$

where A is an equicontinuous set in X^*_σ and B is an equicontinuous set in Y^*_σ. We may assume that A and B are closed, convex and balanced. Let $U = A^\circ$. Then U is a 0-neighborhood in X and we may choose a 0-neighborhood $W \subset U$ such that $\phi_{UW} : X_W \to X_U$ is nuclear. Then

$$\phi_{UW} = \sum \lambda_i f_i \otimes \bar{x}_i$$

for bounded sequences $\bar{x}_i \in X_U$ and $f_i \in X^*_W$ and a sequence $\{\lambda_i\}$ of positive numbers with $\sum \lambda_i = 1$. The dual of ϕ_{UW} is the natural map $\phi_{CA} : X^*_A \to X^*_C$ where $C = W^\circ$. This is also nuclear and has the form

$$\phi_{CA} = \sum \lambda_i \bar{x}_i \otimes f_i$$

Now fix $u \in B_e(X^*_\sigma, Y^*_\sigma)$. Then for $g \in B$ the restriction of $u(\,\cdot\,, g)$ to X^*_C is continuous in the norm on X^*_C. It follows that if $f \in A \subset X^*_A$ then $u(f,g) = u(\phi_{CA}(f), g)$. In other words,

$$u(f,g) = \sum \lambda_i f(\bar{x}_i) u(f_i, g)$$

We choose N so that

$$\sum_{i=N+1}^{\infty} |\lambda_i| ||\bar{x}_i|| |u(f_i,g)| < 1/2$$

for all $g \in B$ and we choose $x_i \in X$ for $i \le N$ so that

$$||\phi_U(x_i) - \bar{x}_i|| |u(f_i,g)| < 1/2$$

for all $g \in B$. For each $i < N$ the functional $g \to u(f_i, g)$ is $\sigma(X^*, X)$ continuous and, hence, is given by an element $y_i \in Y$. Then we set $u_0 = \sum\limits_{i=1}^{N} x_i \otimes y_i \in X \otimes Y$. When u_0 is interpreted as an element in $B_e(X^*_\sigma, Y^*_\sigma)$ we have

$$u_0(f,g) = \sum_{i=1}^{N} f(x_i) g(y_i) = \sum_{i=1}^{N} f(x_i) u(f_i, g)$$

Also, if $f \in A$ and $g \in B$ we have

$$|(u-u_0)(f,g)| \leq |\sum_{i=1}^{N} \lambda_i f(\bar{x}_i - \phi_U(x_i))u(f_i,g)| + |\sum_{i=N+1}^{\infty} \lambda_i f(\bar{x}_i)u(f_i,g)| < 1$$

by our choices of N and $\{x_i\}$. Thus, $u - u_0$ is in the given 0- neighbohood Ω_{AB} and we have proved that $X \otimes Y$ is dense in $B_e(X^*\sigma, Y^*_\sigma)$.

Next, we prove that $X \otimes_\pi Y \to B_e(X^*_\sigma, Y^*_\sigma)$ is a topological isomorphism onto its image (which is $X \otimes_e Y$). We already know this map is continuous and has dense image, so we only need to show it is an open map onto its image. Since the adjoint map is injective, we may consider $B_e(X^*_\sigma, Y^*_\sigma)^*$ as a subspace of $(X \otimes_\pi Y)^* = \mathcal{B}(X,Y)$. The map $X \otimes_\pi Y \to B_e(X^*_\sigma, Y^*_\sigma)$ will be open onto its image if and only if for each closed convex balanced 0-neighborhood $U \subset X \otimes_\pi Y$ there is a 0-neighborhood $V \subset B_e(X^*_\sigma, Y^*_\sigma)$ such that $V \cap (X \otimes_\pi Y) \subset U$. Suppose $U^\circ \subset (X \otimes_\pi Y)^*$ is contained in $B_e(X^*_\sigma, Y^*_\sigma)^*$ and equicontinuous there. Then, by the bipolar theorem, the polar of U° in $X \otimes_\pi Y$ is U while the polar, V, of U° in $B_e(X^*_\sigma, Y^*_\sigma)$ is a 0-neighborhood in $B_e(X^*_\sigma, Y^*_\sigma)$. Necessarily, $V \cap (X \otimes_\pi Y) = U$. Thus, to prove that $X \otimes_\pi Y \to B_e(X^*_\sigma, Y^*_\sigma)$ is open it is enough to prove that every equicontinuous subset Q of $(X \otimes_\pi Y)^* = \mathcal{B}(X,Y)$ is contained in and equicontinuous in $B_e(X^*_\sigma, Y^*_\sigma)^*$.

Now each equicontinuous subset of $\mathcal{B}(X,Y)$ is contained in one of the form

$$A = \{u \in \mathcal{B}(X,Y) : |u(x,y)| \leq 1 \ \forall (x,y) \in U \times V\}$$

where U and V are 0-neighborhoods in X and Y. Since X is nuclear, the map $\phi_U : X \to X_U$ is nuclear. Thus,

$$\phi_U = \sum \lambda_i f_i \otimes x_i$$

for a bounded sequence $x_i \in X_U$ and an equicontinuous sequence $f_i \in X^*$ and a sequence $\{\lambda_i\}$ of positive numbers with $\sum \lambda_i = 1$.

Since each $u \in A$ is bounded on $U \times V$ it defines a continuous bilinear functional v on $X_U \times Y$ so that $u(x,y) = v(\phi_U(x), y)$ where $\phi_U : X \to X_U$ is the natural map. Clearly $|v(x,y)| \leq 1$ for all $(x,y) \in B \times V$, where B is the unit ball in X_U. Thus, we have

$$u(x,y) = \sum_{i=1}^{\infty} \lambda_i f_i(x) v(x_i, y) = \sum_{i=1}^{\infty} \lambda_i f_i(x) g_i(y)$$

where $g_i(y) = v(x_i, y)$. Note that the sequence $\{g_i\}$ is equicontinuous by the joint continuity of v and, since $\{f_i\}$ is also equicontinuous, we have that u lies in the closed convex hull of the product of an equicontinuous set in $E \subset X^*$ and an equicontinuous set in $F \subset Y^*$. Thus, A is contained in the closured convex hull of $E \times F$). It follows that A not only consists of continuous linear functionals on $B_e(X^*_\sigma, Y^*_\sigma)$, it lies in the polar of the zero neighborhood determined in $B_e(X^*_\sigma, Y^*_\sigma)$ by the pair of equicontinuous sets E and F. Thus, it is equicontinuous in $B_e(X^*_\sigma, Y^*_\sigma)^*$. This completes the proof.

The next corollary is sometimes called the abstract Kernel Theorem. Here $\mathcal{L}_e(X^*_\tau, Y)$ is the space of continuous linear maps from X^*_τ to Y with the topology of uniform convergence on equicontinuous subsets of $X^*\tau$.

5.19 Corollary. *If X is a complete nuclear space and Y is any complete l.c.s then*

$$X\hat{\otimes}_\pi Y \simeq X\hat{\otimes}_e Y \simeq B_e(X^*_\sigma, Y^*_\sigma) \simeq \mathcal{L}_e(X^*_\tau, Y).$$

Proof. The first two equivalences follow immediately from the preceding theorem provided we can show that $B_e(X^*_\sigma, Y^*_\sigma)^*$ is complete whenever X and Y are complete. However, a Cauchy filter in $B_e(X^*_\sigma, Y^*_\sigma)^*$ will converge uniformly on each product of equicontinuous sets $E \times F$. The limit will be bilinear and continuous on each equicontinuous set in X for each fixed y and continuous on each equicontinuous set in Y for each fixed value of x. By Corollary 4.16, the limit will actually be separately continuous and, hence, in $B_e(X^*_\sigma, Y^*_\sigma)^*$. Thus, $B_e(X^*_\sigma, Y^*_\sigma)^*$ is complete.

To prove the third equivalence, we note that if $u \in B_e(X^*_\sigma, Y^*_\sigma)$ and $f \in X^*$, then $u(f, \cdot)$ is a continuous linear functional on Y^*_σ and, hence, determines an element of Y. Thus, each $u \in B_e(X^*_\sigma, Y^*_\sigma)$ determines a linear operator $\tilde{u} : X^* \to Y$. This is clearly continuous from X^*_σ to Y_σ. In fact, it is easy to see that

$$B_e(X^*_\sigma, Y^*_\sigma) \simeq \mathcal{L}_e(X^*_\sigma, Y_\sigma)$$

where the latter is the space of continuous linear functionals from X^*_σ to Y_σ with the topology of uniform convergence on equicontinuous sets. However,

$$\mathcal{L}_e(X^*_\sigma, Y_\sigma) \simeq \mathcal{L}_e(X^*_\tau, Y).$$

To see this, note that each $u \in \mathcal{L}_e(X^*_\tau, Y)$ is clearly weakly continuous. Conversely, if $u \in \mathcal{L}_e(X^*_\sigma, Y_\sigma)$ then the adjoint of u exists and maps weakly compact convex balanced subsets of Y^* into weakly compact convex balanced subsets of X. It follows that $\tilde{u}$ is continuous from X^*_τ to Y_τ and, hence, to Y. This completes the proof.

5.20 Theorem. *The strong dual of a nuclear Frechet space is nuclear and is a Montel space.*

Proof. Let X be a nuclear Frechet space and let $\phi : X^*_\beta \to Y$ be a continuous linear map to a Banach space Y. Since X is reflexive $\phi : X^*_\sigma \to Y_\sigma$ is continuous. Then

$$u(f, g) =< \phi(f), g >$$

for $(f, g) \in X^*_\sigma \times Y^*_\sigma$ defines a separately continuous bilinear map, that is, an element of $B_e(X^*_\sigma, Y^*_\sigma)$. But by the previous Corollary each such element is given by an element of $X\hat{\otimes}_\pi Y$ and such an element has the form

$$\sum \lambda_i x_i \otimes y_i$$

for bounded sequences $\{x_i\} \subset X$ and $\{y_i\} \subset Y$ and a summable sequence $\{\lambda_i\}$. Then

$$u(f, g) = \sum \lambda_i f(x_i) \otimes g(y_i)$$

and so
$$\phi(f) = \sum \lambda_i f(x_i) \otimes y_i$$
which implies that ϕ is nuclear. Thus, X^*_β is nuclear.

Since the strong dual of a Frechet space is also complete (in fact, B-complete by Problem 4.4) and reflexive, we have that the strong dual of a Frechet space is a Montel space by Proposition 5.11.

We know quite a lot about Frechet spaces but not so much about their duals. The next proposition states some of the more important properties of strong duals of Frechet spaces.

5.21 Proposition. *Consider the following properties that may be possessed by an l.c.s. X:*

(1) *X contains an increasing sequence of bounded sets $\{B_n\}$ such that every bounded set is contained in some B_n;*
(2) *if $\{V_n\}$ is sequence of convex, balanced 0-neighborhoods such that $V = \cap V_n$ absorbs every bounded subset of X, then V is a 0-neighborhood;*
(3) *if the union of a countable family of equicontinuous subsets of X^*_β is bounded then it is equicontinuous;*
(4) *the strong dual of X is a Frechet space;*
(5) *if X is barreled (which it will be if it is reflexive) then it is bornological.*

The dual of every Frechet space has properties (1) and (2). If any l.c.s. has properties (1) and (2), then it also has properties (3), (4), and (5).

Proof. Suppose X is the strong dual of a Frechet space E. If $\{U_n\}$ is a decreasing sequence of 0-neighborhoods forming a base for the topology of E and $B_n = U_n^\circ$, then $\{B_n\}$ is an increasing sequence of equicontinuous sets in $X = E^*_\beta$ and every equicontinuous set is contained in some B_n. However, equicontinuous subsets of a strong dual are always bounded and since E is Frechet, bounded subsets of $\{E^*\}$ are equicontinuous by the Banach-Steinhaus Theorem. Thus, the family $\{B_n\}$ has the properties required in (1).

Now given $\{V_n\}$ and V as in (2) we choose a basic sequence $\{B_n\}$ of bounded sets as in (1). For each n there is a $t_n > 0$ such that $2t_nB_n \subset V$ since V absorbs every bounded set. Note that the B_n may be chosen to be polars of neighborhoods in E and, hence, $\sigma(E^*, E)$ compact, convex and balanced. The same is true of the convex hull C_n of $\bigcup_{k \le n} t_kB_k$. Since each V_n is a 0-neighborhood in the $\beta(E^*, E)$ topology, it contains a neighborhood of the form $2U_n$ where U_n is the polar of a bounded set in E. Then U_n is convex, balanced and $\sigma(E^*, E)$ closed. In other words, it is a barrel in E^*_σ. If $W_n = U_n + C_n$, then $W_n \subset V_n$. Furthermore, $t_kB_k \subset C_n \subset W_n$ for all $n \ge k$, and there is an s so that $sB_k \subset U_n \subset W_m$ for all $m \le k$ from which it follows that $W = \cap W_n$ is a subset of V which absorbs every B_k and, hence, every bounded set. Also, Each W_n is convex, balanced and $\sigma(E^*, E)$ closed. Thus, the same things are true of W. We conclude, from the bipolar theorem, that W is the polar of its polar in E and, from the fact that W is absorbing, that its polar in E is bounded. It follows that W is a 0-neighborhood in $X = E^*_\beta$. Since $W \subset V$, we conclude that V is a 0-neighborhood. This concludes the proof that the strong dual of a Frechet space has properties (1) and (2).

Now suppose that X is any l.c.s. with properties (1) and (2). Then property (3) follows immediately from property (2) on taking polars.

That X^*_β is metrizable follows from the fact that X has a basic family of bounded subsets which is countable (Property (1)). That $X^*\beta$ is complete follows from Property (3) since it implies, in particular, that every bounded sequence in $X^*\beta$ is equicontinuous and this implies that the pointwise limit of a Cauchy sequence of functionals in X^*_β will neccessarily be continuous. Thus X has property (4).

Now suppose X is barreled. We choose a basic sequence of bounded sets $\{B_n\}$ which are $\sigma(E^*, E)$ compact, convex and balanced, as in (1). If A is a convex, balanced set which absorbs every bounded set then we may choose $t_n > 0$ for each n so that $2t_nB_n \subset A$. As above, we let C_n be the convex hull C_n of $\bigcup_{k\leq n} t_kB_k$ and note that C_n is also $\sigma(E^*, E)$ compact, convex and balanced. We set $C = \bigcup C_n$ and note that $2C \subset A$. We will show that the closure $\bar{C}$ of C is contained in $2C$. Suppose $x \neq 2C$. Then for each n there is a 0-neighborhood V_n in X such that $(x + V_n) \cap 2C_n = \emptyset$. If $W_n = V_n + C_n$, then each W_n is a 0-neighborhood. If we set $W = \cap W_n$ then, as in (2) above, W absorbs each B_n and, hence, each bounded set and thus, by (2) it is a 0-neighborhood. If x were in the closure of C there would be a point $x' \in (x + W) \cap C$. Then $x' \in (x + W) \subset (x + V_n + C_n)$ for all n while $x' \in C_n$ for some n. For this n we would have that $(x + V_n) \cap 2C_n \neq \emptyset$ which contradicts our choice of V_n. Thus, $x \notin \bar{C}$ and we have proved that $\bar{C} \subset 2C \subset A$. Then $\bar{C}$ is a barrel, hence a 0-neighborhood, contained in A. It follows that A is a 0-neighborhood and X is bornological. Thus, X has property (5) and the proof is complete.

Grothendieck called a space with properties (1) and (2) of the above proposition a **DF space**. These are spaces which have the essential properties of strong duals of Frechet spaces. However, there are DF spaces which are not strong duals of Frechet spaces. By the above proposition, the strong dual of every DF space is a Frechet and, of course, every reflexive DF space is the strong dual of a Frechet space.

The remainder of this section concerns spaces which are either nuclear Frechet spaces **NF spaces** or strong duals of nuclear Frechet spaces **DNF spaces**. By Prop. 5.20, DNF spaces are also nuclear. By Prob. 5.7 a complete nuclear DF space is a DNF space. Also, since $\hat{\otimes}_\pi$ and $\hat{\otimes}_e$ agree if one of the spaces involved is nuclear, we shall just use the notation $\hat{\otimes}$ in this situation.

5.22 Lemma. *If X is an NF space and Y is a Frechet space or if X is a DNF space and Y is a DF space, then every bounded subset $B \subset X\hat{\otimes}Y$ is contained in the closed convex hull of the product of a bounded set in X and a bounded set in Y.*

Proof. By assumption X is complete. We may assume without loss of generality that Y is also complete since if the conclusion is true with Y replaced by its complettion then it is also true for Y. Hence, we may identify $X\hat{\otimes}Y$ with $B_e(X^*_\sigma, Y^*_\sigma)$ and with $\mathcal{L}_e(X^*_\tau, Y)$. The latter space is $\mathcal{L}_e(X^*_\beta, Y)$ since X is reflexive. The element of $\mathcal{L}_e(X^*_\beta, Y)$ corresponding to $u \in X\hat{\otimes}y$ will be denoted $\tilde{u}$. The proof has two parts, the first of which depends on whether the spaces are F or DF spaces while the second does not. Part one of the proof is to establish the following:

Claim. *For any bounded subset $B \subset X\hat{\otimes}Y$ there exists an open set $W \subset X^*_\beta$ and a bounded set $H \subset Y$ so that $\tilde{u}(W) \subset H$ for all $u \in B$.*

In the case where both X and Y are DF spaces, we may interpret each $u \in X \hat{\otimes} Y$ as a seperately continuous bilinear form on $X^*_\beta \times Y^*_\beta$. However, both X^*_β and X^*_β are Frechet spaces (Prop. 5.21) and so separately continuous bilinear forms are jointly continuous. Since the polar of a bounded set in Y is a 0-neighborhood in Y^*_β, the claim is equivalent to the joint equicontinuity of the set B as a set of bilinear forms on $X^*_\beta \times Y^*_\beta$ or, in other words, to the joint equicontinuity of B as a set of linear functionals on $X^*_\beta \hat{\otimes} Y^*_\beta$. But $X^*_\beta \hat{\otimes} Y^*_\beta$ is a Frechet space and B is a pointwise bounded family of continuous functionals on it. Therefore, by the Banach-Steinhaus Theorem B is equicontinuous. This establishes the claim in the case where both spaces are DF spaces.

In the case where X and Y are Frechet spaces, let $\{U_n\}$ and $\{V_n\}$ be decreasing sequences forming neighborhood bases in X and Y, respectively.

Since a set is bounded in an $\mathcal{S}$ topology if and only if its members are uniformly bounded on each set in $\mathcal{S}$, we know that B is a uniformly bounded family of functions on arbitrary products of equicontinuous sets and, hence, on $U_n^\circ \times V_m^\circ$ for each n and m. Thus, for each n the set

$$F_n = \{\tilde{u}(f) : u \in B, f \in U_n^\circ\} = \{u(f, \cdot) : u \in B, f \in U_n^\circ\}$$

is a family of continuous linear functionals on Y^*_σ which is uniformly bounded on each V_m°. Taking polars this means that F_n is a subset of Y which is absorbed by every 0-neighborhood – that is, F_n is a bounded subset of Y for each n.

It follows that for each n we may choose a number $t_n > 0$ such that $t_n F_n \subset V_n$. Since $V_m \subset V_n$ if $m > n$, it follows that for each n we have $t_m F_m \subset V_n$ for all $m > n$. This clearly implies that $\bigcup_m t_m F$ is bounded in Y. We let H be the closed, convex, balanced hull of this set in Y.

For each $f \in X^*$ there is some $s > 0$ so that $f \in sU_n^\circ$ and so $\{\tilde{u}(f) : u \in B\}$ is absorbed by H. It follows that

$$W = \cap_{u \in B} \tilde{u}^{-1}(H)$$

is an absorbing subset of X^*_β. It is also closed, convex, and balanced and, hence, is a barrel. Since X^*_β is reflexive, it is barreled and, hence, W is a 0-neighborhood. By construction, each $\tilde{u}$ for $u \in B$ maps W into H. Thus the claim is true in both cases.

The remainder of the proof does not depend on whether the spaces are F or DF spaces. Suppose B is a bounded subset of $X \hat{\otimes} Y$ and suppose W and H have been chosen as in the claim, Clearly each $u \in B$ may be interpreted as a linear map from X^*_β to Y_H which factors as the natural map $X^*_\beta \to (X^*)_W$ followed by a norm decreasing linear map $\alpha_u : (X^*)_W \to Y_H$. Since X^*_β is nuclear and reflexive, the natural map $X^*_\beta \to (X^*)_W$ has the form

$$\sum \lambda_i x_i \otimes f_i$$

with $\{x_i\}$ a bounded sequence in X, $\{f_i\}$ a bounded sequence in X^*_W and $\{\lambda_i\}$ a summable sequence which we can assume is positive and sums to 1. But then each $u \in B$ can be written as

$$u(f, g) = \tilde{u}(f)(g) = \sum \lambda_i f(x_i) g(\alpha_u(f_i))$$

In other words, B is contained in the closed convex hull of the product $A \times C$ where $A = \{x_i\} \subset X$ and $C = \bigcup\{\alpha_u(f_i) : u \in B, i = 1, 2, \cdots\}$. The latter set is bounded since

$\{f_i\}$ is absorbed by W and every $u \in B$ maps W into the bounded set H. This completes the proof.

5.23 Theorem. *If X is an NF space and Y is any Frechet space or if X is a DNF space and Y is a DF space, then $(X\hat{\otimes}Y)^*_\beta$ is topologically isomorphic to $X^*_\beta\hat{\otimes}Y^*_\beta$.*

Proof. By Corollary 5.19 $X^*_\beta\hat{\otimes}Y^*_\beta$ is topologically isomorphic to $B_e(X^{**}_\sigma, Y^{**}_\sigma)$ which is $B_e(X_\sigma, Y^{**}_\sigma)$ since X is reflexive. Now the equicontinuous sets in $Y^{**} = (Y^*_\beta)^*$ are subsets of sets of the form U° where U is a 0-neighborhood in Y^*_β. But a basis for the 0-neighborhoods in Y^*_β consists of sets of the form $U = B^\circ$ where B is a bounded subset of Y. It then follows from the bipolar theorem that the correspondence which assigns to each weakly closed convex, balanced set in Y its $\sigma(Y^{**}, Y^*)$ closure $\bar{B}$ in Y^{**} is a one to one correspondence between $\sigma(Y, Y^*)$ closed convex balanced bounded subsets of Y and $\sigma(Y^{**}, Y^*)$ closed, convex, balanced equicontinuous subsets of Y^{**}. Note that a corollary of this fact is the fact that Y is dense in Y^{**}_σ. It follows that $B_e(X_\sigma, Y^{**}_\sigma)$ may be identified with $B_e(X_\sigma, Y_\sigma) = B_e(X, Y)$ with the topology of bi-bounded convergence, that is, the $\mathcal{S}$-topology for the family $\mathcal{S}$ of sets of the form $A \times B$ where A and B are bounded subsets of X and Y.

Suppose that all separately continuous linear forms on $X \times Y$ are continuous. Then $B(X, Y) = \mathcal{B}(X, Y) = (X\hat{\otimes}Y)^*$. This shows that $(X\hat{\otimes}Y)^*_\beta$ and $X^*_\beta\hat{\otimes}Y^*_\beta$ are isomorphic as vector spaces. To show that they are topologically isomorphic we must show that the family of bounded subsets of $X\hat{\otimes}Y$ generates the same topology on the dual as the family of subsets of the form $A \times B$ with A and B bounded in X and Y. This follows from the previous lemma.

In the case where X and Y are both Frechet spaces, we know all separately continuous bilinear forms are continuous and the proof is complete. Thus, suppose X is a DNF space and Y is a DF space. Then $X\hat{\otimes}Y = E^*_\beta\hat{\otimes}Y$ where $E = X^*_\beta$. By the first paragraph, $B_e(X, Y)$ may be identified with $X^*_\beta\hat{\otimes}Y^*_\beta$. This is a projective tensor product of two Frechet spaces because Y^*_β is a Frechet space by Prop. 21. Thus, each element $u \in B_e(X, Y)$ has the form

$$u(x, y) = \sum \lambda_i g_i(x) f_i(y)$$

with $\{g_i\}$ and $\{f_i\}$ bounded sequences in E and Y^*_β and $\{\lambda_i\}$ a summable sequence. Now bounded sets in $E = X^*_\beta$ are equicontinuous since E and X are reflexive. Thus, $\{g_i\}$ is equicontinuous. However, $\{f_i\}$ is also equicontinuous since a countable union of equicontinuous subsets of the dual of a DF space is equicontinuous by Prop.21. It follows that for each $\epsilon > 0$ there are open sets $U \subset X$ and $V \subset Y$ such that $|u(x, y)| < \epsilon$ for all $(x, y) \in U \times V$. This proves that u is jointly continuous and completes the proof of the theorem.

We will say that a sequence

$$0 \longrightarrow Y_1 \xrightarrow{\alpha} Y_2 \xrightarrow{\beta} Y_3 \longrightarrow 0$$

of topological vector spaces and continuous linear maps is **topologically exact** if β an open map onto Y_3 and α is a topological isomorphism onto the kernel of β. It follows from

the open mapping theorem that an exact sequence is topologically exact if the spaces Y_i are Frechet spaces or if they are nuclear DF spaces (which makes them both barreled and B-complete).

5.24 Theorem. *Let X be a l.c.s. and*

$$0 \longrightarrow Y_1 \xrightarrow{\alpha} Y_2 \xrightarrow{\beta} Y_3 \longrightarrow 0$$

a topologically exact sequence of l.c.s.'s. Assume that either X or Y_2 is nuclear and that X and Y_2 are both Frechet spaces or both DF spaces. Then the sequence

$$0 \longrightarrow X\hat{\otimes}Y_1 \xrightarrow{1\otimes\alpha} X\hat{\otimes}Y_2 \xrightarrow{1\otimes\beta} X\hat{\otimes}Y_3 \longrightarrow 0$$

is exact.

Proof. If Y_2 is nuclear, then so are Y_1 and Y_3. Thus, the hypothesis that X or Y_2 is nuclear implies that $X\hat{\otimes}_\pi Y_i = X\hat{\otimes}_e Y_i = B_e(X_\sigma, Y_\sigma)$ for each i by Prop. 5.19. Thus, the completed tensor product being used above is unambiguous.

If X and Y_2 are Frechet, then Y_1 and Y_3 are also Frechet. Then $1\otimes\beta$ is surjective and, in fact, an open map by Theorem 3.8. If X and Y_2 are DF spaces the surjectivity of $1\otimes\beta$ follows from Lemma 5.22 and the fact that every bounded subset in Y_3 is the image of a bounded subset in Y_2 (problem 5.4).

It follows from Proposition 3.14 that $1\otimes\alpha$ is an open map onto its image. It follows from this that $\mathrm{im}(1\otimes\alpha)$ is closed in $X\hat{\otimes}Y_2$. If it is not equal to $\ker 1\otimes\beta$ then there is a continuous bilinear form $u \in \mathcal{B}(X, Y_2) = (X\hat{\otimes}Y_2)^*$ which vanishes on $\mathrm{im}(1\otimes\alpha)$ but is not identically zero on $\ker(1\otimes\beta)$. But then $u(x,y) = 0$ for every $y \in \mathrm{im}\alpha = \ker\beta$, from which it follows that $u = v\circ(1\times\beta)$ for some $v \in \mathcal{B}(X, Y_3) = (X\hat{\otimes}Y_3)^*$. This means that, as a linear functional on $X\otimes Y_2$, $u = v\circ(1\otimes\beta)$ and this is a contradiction since it implies that u vanishes identically on $\ker(1\otimes\beta)$. Hence, $\mathrm{im}(1\otimes\alpha) = \ker(1\otimes\beta)$ and the proof is complete.

The next Theorem is a summary of some of the nicer properties of NF and DNF spaces. Each statement is either a restatement of or an immediate consequence of the preceding results.

5.25 Theorem. *Let $\mathcal{NF}$ be the category of NF spaces and $\mathcal{DNF}$ the caterory of DNF spaces. Then*

(1) *If $X \in \mathcal{NF}$ or $X \in \mathcal{DNF}$ then X is Montel, Mackey, reflexive, barreled, B-complete and bornological;*
(2) *each of $\mathcal{NF}$ and $\mathcal{DNF}$ is closed under completed topological tensor product and this operation is exact in each of its arguments;*
(3) *strong duality is a contravariant equivalence between the category $\mathcal{NF}$ and the category $\mathcal{DNF}$ which takes completed tensor products to completed tensor products.*
(4) *(Kernel Theorem) if X and Y are both in $\mathcal{NF}$ or both in $\mathcal{DNF}$, then*

$$\mathcal{L}(X, Y^*) \simeq X^*\hat{\otimes}Y^* \simeq (X\hat{\otimes}Y)^* \simeq \mathcal{B}(X, Y)$$

where the duals are strong duals and $\mathcal{L}(X, Y^*)$ has the topology of uniform convergence on bounded sets.

Obviously, the category of nuclear spaces and, especially, the categories of NF and DNF spaces are extremely well behaved. Of course, we haven't yet shown that there are any nuclear spaces other than finite dimensional spaces. However, in the next chapter we will show that the spaces of importance in distribution theory and holomorphic function theory are all nuclear spaces and many of them are NF or DNF spaces.

Exercises

(1) Prove that the composition of a compact linear map with a continuous linear map (on either side) is compact.
(2) Prove that a nuclear space which is a Banach space is necessarily finite dimensional.
(3) Prove that the projective tensor produce of two nuclear spaces X and Y is nuclear (Hint: If $U \subset X$ and $V \subset Y$ are 0-neighborhoods such that $X \to X_U$ and $Y \to Y_V$ are nuclear maps, then so is $X \otimes_\pi Y \to X_U \otimes_\pi Y_V$. Furthermore, $X_U \otimes_\pi Y_V = (X \otimes_\pi Y)_W$ where W is the convex hull of U and V.).
(4) Prove that if $X \to Y$ is a continuous linear and open map and X is a DF space, then every bounded subset of Y is the image of some bounded subset of X.
(5) Prove that a complete nuclear DF space is reflexive and, hence, is the strong dual of a Frechet space.
(6) Prove that the strong dual of a complete nuclear DF space is nuclear.
(7) Conclude form (5) and (6) that a complete nuclear DF space is a DNF space.
(8) Verify part (4) of Theorem 5.25.

6. Distributions

In this section we introduce the function spaces that play the central role in distribution theory. All of these are nuclear spaces. We will begin by showing that one such space is a nuclear space and then we construct all others from this one using operations that we know preserve nuclear spaces.

Let T denote the unit circle and $\mathcal{D}(T)$ the space of infinitely differentiable functions on T. Of course, if we identify T with $\mathbb{R}/2\pi$ then $\mathcal{D}(T)$ may be identified with the space of infinitely differentiable functions on $\mathbb{R}$ which are periodic of period 2π. For each non-negative interger k we define a seminorm p_k on $\mathcal{D}(T)$ by

$$p_k(f) = \sup\{|f^{(j)}(t)| : t \in T,\ j \leq k\}$$

The topology of $\mathcal{D}(T)$ is the topology determined by the family of seminorms $\{p_k\}$. Since this family of seminorms is countable, $\mathcal{D}(T)$ is metrizable. But $\mathcal{D}(T)$ is obviously complete and, hence, it is a Frechet space.

Note that p_k is actually a norm on the space $\mathcal{C}^k(T)$ of k times continuously differentiable functions – a norm defining the topology of uniform convergence of functions along with their derivatives up to degree k. The space $\mathcal{C}^k(T)$ is clearly complete under the p_k norm and, hence, is a Banach space. It is easy to see that the space $\mathcal{D}(T)$ is dense in $\mathcal{C}^k(T)$ in this topology (Problem 6.1). Thus, $\mathcal{C}^k(T)$ is the Banach space completion of $\mathcal{D}(T)$ under the seminorm p_k. we denite the inclusion $\mathcal{D}(T) \to \mathcal{C}^k(T)$ by ϕ_k. We shall show that this map is nuclear for $k > 2$, which clearly implies that $\mathcal{D}(T)$ is a nuclear space. We first require a couple of lemmas.

Let $\gamma_n \in \mathcal{D}^*(T)$ denote the linear functional which assigns to each $f \in \mathcal{D}(T)$ its n^{th} Fourier series coefficient. That is,

$$\gamma_n(f) = (2\pi)^{-1} \int_0^{2\pi} f(t) \mathrm{e}^{-int}\, dt$$

6.1 Lemma. *For each integer k, the family $\{|n|^k \gamma_n\}_{n=-\infty}^{\infty}$ is an equicontinuous family in $\mathcal{D}^*(T)$.*

Proof. By repeated integration by parts we have for every n and k that:

$$|n|^k \gamma_n(f) = (2\pi)^{-1} \left| \int_0^{2\pi} f^k(t) \mathrm{e}^{-int}\ dt \right| \leq \sup_T |f^k(t)| \leq p_k(f)$$

Thus, the set $\{|n|^k \gamma_n\}_{n=-\infty}^{\infty}$ lies in the polar of the 0-neighborhood $\{f \in \mathcal{D}(T) : p_k(f) \leq 1\}$ and is, hence, equicontinuous.

For each integer n let e_n denote the element of $\mathcal{C}^k(T)$ given by

$$e_n(t) = \mathrm{e}^{-int}$$

Then:

6.2 Lemma. *For each non-negative integer k, the sequence $\{|n|^{-k}e_n\}_{n=-\infty}^{\infty}$ is bounded in $\mathcal{C}^k(T)$.*

Proof. For $j \le k$ we have
$$|e_n^{(j)}(t)| = |n|^j \le |n|^k$$
for all t and all n and so
$$p_k(|n|^{-k}e_n) \le 1 \quad \forall\, n$$
This completes the proof.

Proposition 6.3. *The space $\mathcal{D}(T)$ is nuclear.*

Proof. For each $k \ge 0$, the sequence $\{|n|^{k+2}\gamma_n\}_{n=-\infty}^{\infty}$ is an equicontinuous sequence in the dual of $\mathcal{D}(T)$, the sequence $\{|n|^{-k}e_n\}_{n=-\infty}^{\infty}$ is bounded in $\mathcal{C}^k(T)$ and the sequence $\{|n|^{-2}\}_{n=-\infty}^{\infty}$ is summable. It follows that

$$\psi(f) = \sum_{n=-\infty}^{\infty} |n|^{-2}\; |n|^{k+2}\gamma_n(f)\; |n|^{-k}e_n$$

defines a nuclear map from $\mathcal{D}(T)$ to $\mathcal{C}^k(T)$. One checks that this map is, in fact, the inclusion of $\mathcal{D}(T)$ into $\mathcal{C}^k(T)$ by observing that, for $f \in \mathcal{D}(T)$, $\psi(f)$ is the sum of a Fourier series which converges in the topology of $\mathcal{C}^k(T)$ by the above conditions, and that, in fact, it is the Fourier series of f. Thus, $\psi(f)$ and f have the same Fourier coeficients and, hence, must be the same function. Thus, we have proved that the natural map ϕ_k from $\mathcal{D}(T)$ to its Banach space completion under p_k is nuclear for each $k \ge 0$. It follows that $\mathcal{D}(T)$ is a nuclear space.

We let T^n denote the n-fold Cartesian product of T with itself and denote by $D(T^n)$ the space of infinitely differentiable functions on T^n with the topology determined by the family of seminorms $\{p_k\}_{k=1}^{\infty}$, where

$$p_k(f) = \sup\left\{\left|\frac{\partial^\alpha f(t)}{\partial t^\alpha}\right| : t \in T^n,\ \alpha = (\alpha_1, \cdots, \alpha_n),\ \sum \alpha_i \le k\right\}$$

The space $D(T^n)$ is obviously a Frechet space. One could generalize the above Fourier series argument for the nuclearity of $\mathcal{D}(T)$ to prove that $\mathcal{D}(T^n)$ is nuclear, but instead we will do this by using what we know about topological tensor products:

Proposition 6.4. *If $n = i + j$ then $\mathcal{D}(T^n)$ is canonically isomorphic to $\mathcal{D}(T^i)\hat{\otimes}\mathcal{D}(T^j)$. Furthermore, $\mathcal{D}(T^n)$ is a nuclear space for each positive integer n.*

Proof. We proceed by induction on n. Suppose we know that $\mathcal{D}(T^k)$ is a nuclear space whenever $k < n$. Let i and j be any two positive integers with $n = i + j$. There is a bilinear map $\mathcal{D}(T^i) \times \mathcal{D}(T^j) \to \mathcal{D}(T^n)$ defined by

$$(f, g) \to \{(s, t) \to f(s)g(t)\} \text{ for } s \in T^i,\ t \in T^j$$

This is clearly a continuous bilinear map and so it induces a continuous linear map

$$\phi : \mathcal{D}(T^i)\hat{\otimes}\mathcal{D}(T^j) \to \mathcal{D}(T^n).$$

On the other hand, there is a continuous linear map

$$\theta : \mathcal{D}(T^n) \to B_e(\mathcal{D}(T^i)^*_\sigma, \mathcal{D}(T^j)^*_\sigma)$$

defined as follows: If $f \in \mathcal{D}(t^n)$, and $t \in T^j$ then we set $f_t(s) = f(s,t)$ for all $s \in T^i$. Then $t \to f_t$ is an infinitely differentiable funtion on T^j with values in $\mathcal{D}(T^i)$. It follows that for each $u \in \mathcal{D}^*(T^i)$, the function $t \to u(f_t)$ belongs to $\mathcal{D}(T^j)$ and we may apply any element $v \in \mathcal{D}^*(T^j)$ to it. The result will be denoted $\theta(f)(u,v)$. Clearly $\theta(f)$ is bilinear and separately continuous in $(u,v) \in \mathcal{D}(T^i)^*_\sigma \times \mathcal{D}(T^j)^*_\sigma$. Note also that θ is injective since $\theta(f) = 0$ implies that $u(f_t)$ vanishes for every u and every t which clearly implies that $f = 0$.

Finally, the composition $\theta \circ \phi : \mathcal{D}(T^i)\hat{\otimes}\mathcal{D}(T^j) \to B_e(\mathcal{D}(T^i)^*_\sigma, \mathcal{D}(T^j)^*_\sigma)$ is just the map $f \otimes g \to \{(u,v) \to u(f)v(g)\}$, in other words, it is the canonical map of $\mathcal{D}(T^i)\hat{\otimes}\mathcal{D}(T^j)$ to $B_e(\mathcal{D}(T^i)^*_\sigma, \mathcal{D}(T^j)^*_\sigma)$ which is a topological isomorphism in this case since the spaces involved are complete nuclear spaces (Corollary 5.19). This implies that θ is surjective and, hence, also a topological isomorphism by the open mapping theorem. It follows that ϕ is a topological isomorphism. Thus, $\mathcal{D}(T^n)$ is a nuclear space since it is the completed tensor product of two nuclear spaces (Problem 5.2 and Corollary 5.10). This completes the proof.

Now for any compact subset K of $\mathbb{R}^n$, let $\mathcal{D}(K)$ denote the space of infinitely differentiable functions on $\mathbb{R}^n$ with support in K. Again the topology is defined by the family of seminorms $\{p_k\}$ where as before,

$$p_k(f) = \sup\left\{\left|\frac{\partial^\alpha f(x)}{\partial t^\alpha}\right| : x \in K,\ \alpha = (\alpha_1, \cdots, \alpha_n),\ \sum \alpha_i \le k\right\}$$

This space is, once again, a Frechet space. Furthermore, if $q > 0$ is chosen large enough so that K is contained in the square $\{x = (x_1, \cdots, x_n) \in \mathbb{R}^n : |x_i| < q,\ i = 1, \cdots, n\}$, then $D(K)$ may be regarded as a subspace of the space of infinitely differentiable functions which are periodic of period $2q$. Since the latter space is isomorphic to the nuclear space $\mathcal{D}(T^n)$ and a subspace of a nuclear space is nuclear (Prop. 5.16), we have proved:

6.5 Proposition. *For each compact subset K of $\mathbb{R}^n$, the space $\mathcal{D}(K)$ is a nuclear Frechet space.*

6.6 Definition. *If U is an open subset of $\mathbb{R}^n$ then $\mathcal{D}(U)$ will denote the space of infinitely differentiable functions with compact support in U.*

Clearly, as a vector space, $\mathcal{D}(U)$ is the inductive limit of the directed system $\{D(K)\}_{K\subset U}$ where K ranges over the family of compact subsets of U which is directed by inclusion. If $K \subset L$ then the corresponding map $D(K) \to D(L)$ is inclusion. This makes sense because a function with support in K is also a function with support in L. We give $\mathcal{D}(U)$ the

inductive limit topology defined by this directed system. Note that we can always choose a sequence of compact sets $\{K_n\}$ with $K_n \subset K_{n+1}^\circ$ and $\bigcup K_n = U$. Then each compact subset of U will be contained in some K_n and it follows that $\mathcal{D}(U)$ can be realized as the inductive limit of the sequence $\{\mathcal{D}(K_n)\}$. This is a strict inductive limit since each of the natural maps $\mathcal{D}(K_n) \to \mathcal{D}(K_{n+1})$ is just the inclusion. Furthermore, each $D(K_n)$ is clearly closed in $D(U)$. Thus, by Cor. 5.17 we have:

6.7. *For each open subset U of $\mathbb{R}^n$, the space $\mathcal{D}(U)$ is a nuclear space and, in fact, a strict inductive limit of a sequence of closed subspaces $\mathcal{D}(K_n)$, each of which is a nuclear Frechet space.*

We know that the dual of a nuclear Frechet space is nuclear (Prop. 5.20); but $\mathcal{D}(U)$ is not a Frechet space. However, we still prove that its dual is a nuclear space by exploiting the fact that $\mathcal{D}(U)$ is a strict inductive limit of Nuclear Frechet spaces. To do this, we need more information about duality between inductive and projective limits.

If $\{X_\alpha, \phi_\alpha\}$ is a directed system of l.c.s, then, clearly, $\{X_\alpha^*, \phi_\alpha^*\}$ is an inverse system of vector spaces. If $X = \varinjlim X_\alpha$ then, algebraically, there is a natural isomorphism

$$X^* \simeq \varprojlim X_\alpha^*$$

In fact, if $\phi_\alpha : X_\alpha \to X$ is the natural inclusion and for $f \in X^*$ we set $f_\alpha = f \circ \phi_\alpha$, then it follows from Prop. 2.8 that the linear map $f \to \{f_\alpha\} : X^* \simeq \varprojlim X_\alpha^*$ is an isomorphism.

What about topologies? It is easy to see that this isomophism is a topological isomorphism from X_σ^* to $\varprojlim (X_\alpha)_\sigma^*$. This is due to the fact that the topology of X_σ^* is the topology of uniform convergence on finite subsets of X and each finite subset of X is in the image of some X_α under ϕ_α. We would really like to have the analogous isomorphism for the strong topologies. This will hold if we are in a situation where we know that every bounded subset of X is the the image of a bounded subset of X_α for some α. This is not true in general, but it is true, by Prop. 2.12, for the strict inductive limit of a sequence of spaces $\{X_n\}$ with each X_n closed in X_{n+1}. Thus, we have:

6.8 Proposition. *If $X = \varinjlim X_n$ is the strict inductive limit of a sequence of its closed subspaces, then there is a natural topological isomorphism*

$$X_\beta^* \simeq \varprojlim (X_n)_\beta^*$$

6.9 Proposition. *The strong dual $\mathcal{D}^*(U)$ of $\mathcal{D}(U)$ is a nuclear space and is the projective limit of a sequence $\{\mathcal{D}^*(K_n)\}$ of DNF spaces.*

Proof. This follows immediately from the previous two propositions and the fact that a projective limit of nuclear spaces is nuclear.

The space $\mathcal{D}^*(U)$ is the space of distributions on U. The space $\mathcal{D}(U)$ is sometimes called the space of test functions for distributions, since a distribution is defined a continuous linear functional on $\mathcal{D}(U)$.

It turns out that $\mathcal{D}(U)$ is reflexive. Again, since $\mathcal{D}(U)$ is neither a Frechet space or the dual of a Frechet space, we can't immediately draw this conclusion from what we know about NF or DNF spaces. Instead, we prove it by analyzing the dual of a projective limit.

An inverse limit system $\{X_\alpha, \phi_{\alpha\beta}\}$ is called **reduced** if $\phi_\alpha X$ is dense in $X\alpha$ for all $\alpha < \beta$, where $X = \lim\limits_{\leftarrow} X_\alpha$ and $\phi_\alpha : X \to X_\alpha$ is the natural map. Any inverse limit system may be replaced by one which is reduced and has the same projective limit by simply replacing X_α by the closure of $\phi_\alpha(X)$ in X_α.

Note that, given an inverse system $\{X_\alpha, \phi_{\alpha\beta}\}$, the strong dual, $\{X_\alpha^*, \phi_{\alpha\beta}^*\}$, will be a direct limit system.

6.10 Proposition. *If $X = \lim\limits_{\leftarrow} X_\alpha$ is the projective limit of a reduced inverse system of l.c.s. and if the strong and Mackey topologies agree on each of the dual spaces X_α^*, then the natural map*

$$\psi : \lim_{\rightarrow}(X_\alpha)^* \to X^*$$

is a topological isomorphism if the duals are all given the strong topology.

Proof. We have that $\{X_\alpha^*, \phi_{\alpha\beta}^*\}$ is a direct limit system and the adjoint maps $\phi_\alpha^* : X_\alpha^* \to X^*$ are continuous and injective (since each ϕ_α has dense range). This system of maps clearly satisfies the compatibility condition necessary for it to define a continuous linear map ψ as in the Proposition. This map is clearly an algebraic isomorphism. To show that it is a topological isomorphism we must show that it is an open map.

Now a basic 0-neighborhood U in $\lim\limits_{\rightarrow}(X_\alpha)^*$ is a σ-closed, convex, balanced set with the property that $(\phi_\alpha^*)^{-1}(U)$ is a 0-neighborhood in X_α^* for each α. But this means that $U_\alpha = (\phi_\alpha^*)^{-1}(U)$ is a σ-closed convex balanced 0-neighborhood in X_α^* for each α. If K_α is the polar in X_α of U_α then each K_α is a σ-compact convex balanced subset of X_α, since we have assumed that the strong topology and Mackey topology agree for the duality (X_α^*, X_α). Furthermore, it follows from the Hahn-Banach theorem that $\phi_{\alpha\beta}(K_\beta) = K_\alpha$ for each $\beta > \alpha$ since $\phi_{\alpha\beta}^*$ is injective and $(\phi_{\alpha\beta}^*)^{-1}(U_\alpha) = U_\beta$. We set

$$K = \lim_{\leftarrow} K_\alpha = \{x \in X = \lim_{\leftarrow} X_\alpha : \phi_\alpha(x) \in K_\alpha \ \forall\ \alpha\}.$$

In general, without compactness, there is no guarantee that a set defined this way will be non-empty, even though we have $\phi_{\alpha\beta}(K_\beta) = K_\alpha$ for each $\beta > \alpha$. However, we claim that because each K_α is σ-compact, the set K is not only non-empty, it has the property that

$$\phi_\alpha(K) = K_\alpha \quad \forall\ \alpha$$

To prove this, let $y \in K_\alpha$ be fixed and for each $\beta > \alpha$ set

$$L_\beta = \{\{x_\gamma\} \in \prod K_\gamma : x_\gamma = \phi_{\gamma\beta}(x_\beta)\ \forall \gamma < \beta \text{ and } y = x_\alpha\}$$

The set L_β is a non-empty closed subset of the compact set $\prod X_\gamma : (X_\gamma)_\sigma$ and, hence, is compact. Furthermore, $L_{\beta_2} \subset L_{\beta_2}$ if $\beta_1 < \beta_2$ and so the family $\{L_\beta\}$ is directed downward by inclusion and, hence, has the finite intersection property. It follows that $\bigcap L_\beta \neq \emptyset$. An

element x of this intersection is clearly an element of $K = \varprojlim K_\beta$ with the property that $\phi_\alpha(x) = y$.

Next we claim that $K^\circ \subset \psi(U)$. This will finish the proof since it implies that $\psi(U)$ is a 0-neighborhood in the strong topology of X^* since K is weakly compact, hence, weakly bounded, hence, bounded in X. (Actually $K^\circ = \psi(U)$ but we only need the one containment to prove that $\psi(U)$ is a 0-neighborhood.)

If $g \in K^\circ$ then $g = \phi_\alpha^*(g_\alpha)$ for some element $g_\alpha \in X_\alpha^*$ with $|g_\alpha(\phi_\alpha(x))| \leq 1$ for all $x \in K$. Since $\phi_\alpha(K) = K_\alpha$, this implies that $g_\alpha \in U_\alpha = K_\alpha^\circ$. It follows that $g \in \psi(U)$. This completes the proof.

6.11 Proposition. *The spaces $\mathcal{D}(U)$ and $\mathcal{D}^*(U)$ are reflexive and complete.*

Proof. We have that $\mathcal{D}^*(U)$ is the projective limit $\varprojlim \mathcal{D}^*(K_n)$ of a reduced inverse limit system, by Proposition 6.9. Furthermore, each space $\mathcal{D}(K_n)$ is reflexive and so the strong and Mackey topologies agree on $\mathcal{D}^*(K_n)$. It follows from Proposition 6.10 that the natural maps give topological isomorphisms $\mathcal{D}^{**}(U) \simeq \varinjlim \mathcal{D}(K_n) \simeq \mathcal{D}(U)$. Hence, $\mathcal{D}(U)$ is reflexive, as is $\mathcal{D}^*(U)$.

The space $\mathcal{D}(U)$ is complete because it is the strict inductive limit of a sequence of complete spaces (Problem 2.6). Its dual, $\mathcal{D}^*(U)$ is complete because it is the projective limit of a family of complete spaces (Proposition 2.4).

Another space that is important in distribution theory is the space $\mathcal{C}^\infty(U)$. For simplicity of notation and consistency with common usage in distribution theory we will give this space another name:

6.12 Definition. *We will denote by $\mathcal{E}(U)$ the space of $\mathcal{C}^\infty$ functions on U with the topology of uniform convergence of functions and all their derivatives on compact subsets of U.*

The topology on $\mathcal{E}(U)$ is defined by a family of seminorms as follows: Given a compact set $K \subset U$ and a non-negative integer k, we define a seminorm $p_{K,k}$ by

$$p_{K,k}(f) = \sup \left\{ \left| \frac{\partial^\alpha f(x)}{\partial t^\alpha} \right| : x \in K,\ \alpha = (\alpha_1, \cdots, \alpha_n),\ \sum \alpha_i \leq k \right\}$$

The topology on $\mathcal{E}(U)$ is then the topology determined by this family of seminorms. If we choose a sequence $\{K_n\}$ of compact subsets of U with the property that every compact subset of U is contained in some K_n, then the countable family $\{p_{K_n,k}\}$ generates the same topology. Since it is obviously complete, this implies that $\mathcal{E}(u)$ is a Frechet space for every open set $U \subset \mathbb{R}^n$. In fact, we have the following:

6.13 Proposition. *For each open set $U \subset \mathbb{R}^n$, the space $\mathcal{E}(U)$ is a nuclear Frechet space.*

Proof. Choose a sequence $\{K_n\}$ of compact subsets of U so that $\bigcup K_n = U$ and $K_n \subset K_{n+1}^\circ$. Then every compact subset of U is contained in some K_n. For each n, choose a $\mathcal{C}^\infty$ function h_n on U which is identically 1 on K_n and has support in K_{n+1}. Then we have continous linear maps $\phi_n : \mathcal{E}(U) \to \mathcal{D}(K_{n+1})$ defined by

$$\phi_n(f) = h_n f$$

We also have that multiplication by h_{n-1} defines a map $\psi_n : \mathcal{D}(K_{n+1}) \to \mathcal{D}(K_n)$ for each n and $\phi_{n-1} = \psi_n \circ \phi_n$. Thus, $\{\mathcal{D}(K_n), \psi_n\}$ forms an inverse system and the sequence of maps $\{\phi_n : \mathcal{E}(U) \to \mathcal{D}(K_n)\}$ is consistent with this system. It follows that it induces a continuous linear map $\theta : \mathcal{E}(U) \to \varprojlim \mathcal{D}(K_n)$. This is clearly injective and surjective.

Note that a sequence in $\mathcal{E}(U)$ converges if and only if its image under ϕ_n converges in $\mathcal{D}(K_n)$ for each n. Thus, the map θ is a topological isomorphism and we have expressed $\mathcal{E}(U)$ as a projective limit of a sequence of nuclear spaces. It follows that $\mathcal{E}(U)$ is nuclear. We have already observed that it is a Frechet space.

6.14 Corollary. *The strong dual $\mathcal{E}^*(U)$ of $\mathcal{E}(U)$ is a DNF space.*

6.15 Corollary. *If U is an open set in $\mathbb{C}^n$, then the space $\mathcal{H}(U)$ of holomorphic functions on U is a nuclear Frechet space.*

Proof. The space $\mathcal{H}(U)$ is a closed subspace of $\mathcal{E}(U)$ since the Cauchy estimates show that the sup norm of any derivative of $f \in \mathcal{H}(U)$ on a compact subset of U is dominated by the sup norm of f itself on some slightly larger compact set.

Exercises

(1) Prove that $\mathcal{D}(T)$ is dense in $\mathcal{C}^k(T)$.
(2) Prove that the strict inductive limit of a sequence of reflexive spaces is reflexive (use Proposition 6.10).
(3) Let M be a $\mathcal{C}^\infty(M)$ manifold. Define $\mathcal{D}(M)$ and $\mathcal{E}(M)$ in the obvious way and prove that they are nuclear spaces.