

# Foundations of Analysis

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# Preface

This text evolved from notes we developed for use in a two semester undergraduate course on foundations of analysis at the University of Utah. The course is designed for students who have completed three semesters of calculus and one semester of linear algebra. For most of them, this is the first mathematics course in which everything is proved rigorously and they are expected to not only understand proofs, but to also create proofs.

The course has two main goals. The first is to develop in students the mathematical maturity and sophistication they will need when they move on to senior or graduate level mathematics courses. The second is to present a rigorous development of the calculus, beginning with a study of the properties of the real number system.

We have tried to present this material in a fashion which is both rigorous and concise, with simple, straightforward explanations. We feel that the modern tendency to expand textbooks with ever more material, excessively verbose explanations, and more and more bells and whistles, simply gets in the way of the student's understanding of the material.

The exercises differ widely in level of abstraction and level of difficulty. They vary from the simple to the quite difficult and from the computational to the theoretical. There are exercises that ask students to prove something or to construct an example with certain properties. There are exercises that ask students to apply theoretical material to help do a computation or to solve a practical problem. Each section contains a number of examples designed to illustrate the material of the section and to teach students how to approach the exercises for that section.

This text, in its various incarnations, has been used by the author and his colleagues for several years at the University of Utah. Each use has led to improvements, additions, and corrections.

The topics covered in the text are quite standard. Chapters 1 through 6 focus on single variable calculus and are normally covered in the first semester of the course. Chapters 7 through 11 are concerned with calculus in several variables and are normally covered in the second semester.

Chapter 1 begins with a section on set theory. This is followed by the introduction of the set of natural numbers as a set which satisfies Peano's axioms. Subsequent sections outline the construction, beginning with the natural numbers, of the integers, the rational numbers, and finally the real numbers. This

is only an outline of the construction of the reals beginning with Peano's axioms and not a fully detailed development. Such a development would be much too time consuming for a course of this nature. What is important is that, by the end of the chapter: (1) students know that the real number system is a complete, Archimedean, ordered field; (2) they have some practice at using the axioms satisfied by such a system; and (3) they understand that this system may be constructed beginning with Peano's axioms for the counting numbers.

Chapter 2 is devoted to sequences and limits of sequences. We feel sequences provide the best context in which to first carry out a rigorous study of limits. The study of limits of functions is complicated by issues concerning the domain of the function. Furthermore, one has to struggle with the student's tendency to think that the limit of  $f(x)$  as  $x$  approaches  $a$  is just a pedantic way of describing  $f(a)$ . These complications don't arise in the study of limits of sequences.

Chapter 3 provides a rigorous study of continuity for real valued functions of one variable. This includes proving the existence of minimum and maximum values for a continuous function on a closed bounded interval as well as the Intermediate Value Theorem and the existence of a continuous inverse function for a strictly monotone continuous function. Uniform continuity is discussed as is uniform convergence for a sequence of functions.

The derivative is introduced in Chapter 4 and the main theorems concerning the derivative are proved. These include the Chain Rule, the Mean Value Theorem, existence of the derivative of an inverse function, the monotonicity theorem, and L'Hôpital's Rule.

In Chapter 5 the definite integral is defined using upper and lower Riemann sums. The main properties of the integral are proved here along with the two forms of the Fundamental Theorem of Calculus. The integral is used to define and develop the properties of the natural logarithm. This leads to the definition of the exponential function and the development of its properties.

Infinite sequences and series are discussed in Chapter 6 along with Taylor's Series and Taylor's Formula.

The second half of the text begins in Chapter 7 with an introduction to  $d$ -dimensional Euclidean space,  $\mathbb{R}^d$ , as the vector space of  $d$ -tuples of real numbers. We review the properties of this vector space while reminding the students of the definition and properties of general vector spaces. We study convergence of sequences of vectors and prove the Bolzano-Weierstrass Theorem in this context. We describe open and closed sets and discuss compactness and connectedness of sets in Euclidean spaces. Throughout this chapter and subsequent chapters we follow a certain philosophy concerning abstract versus concrete concepts. We briefly introduce abstract metric spaces, inner product spaces, and normed linear spaces, but only as an aside. We emphasize that Euclidean space is the object of study in this text, but we do point out now and then when a theorem concerning Euclidean space does or does not hold in a general metric space or inner product space or normed vector space. That is, the course is grounded in the concrete world of  $\mathbb{R}^d$ , but the student is made aware that there are more exotic worlds in which these concepts are important.

Chapter 8 is devoted to the study of continuous functions between Euclidean

spaces. We study the basic properties of continuous functions as they relate to open and closed sets and compact and connected sets. The third section is devoted to sequences and series of function and the concept of uniform convergence. The last two sections comprise a review of the topic of linear functions between Euclidean spaces and the corresponding matrices. This includes the study of rank, dimension of image and kernel and invertible matrices. We also introduce representations of linear or affine subspaces in parametric form as well as solution sets of systems of equations.

The most important topic in the second half of the course is probably the study, in Chapter 9, of the total differential of a function from  $\mathbb{R}^p$  to  $\mathbb{R}^q$ . This is introduced in the context of affine approximation of a function near a point in its domain. The chain rule for the total differential is proved in what we believe is a novel and intuitively satisfying way. This is followed by applications of the total differential and the chain rule, including the multivariable Taylor's formula and the inverse and implicit function theorems.

Chapter 10 is devoted to integration over Jordan regions in  $\mathbb{R}^d$ . The development, using upper and lower sums, is very similar to the development of the single variable integral in Chapter 5. Where the proofs are virtually identical to those in Chapter 5, they are omitted. The really new and different material here is that on Fubini's Theorem and the change of variables formula. We give rigorous and detailed proofs of both results along with a number of applications.

The chapter on vector calculus, Chapter 11 uses the modern formalism of differential forms. In this formalism, the major theorems of the subject – Green's Theorem, Stoke's Theorem, and Gauss' s Theorem – all have the same form. We do point out the classical forms of each of these theorems, however. Each of the main theorems is proved first on a rectangle or cube and then extended to more complicated domains through the use of transformation laws for differential forms and the change of variables formula for multiple integrals. Most of the chapter focuses on integration over sets in  $\mathbb{R}$ ,  $\mathbb{R}^2$  or  $\mathbb{R}^3$  which can be parameterized by smooth maps from an interval, a square or a cube, or sets which can be partitioned into sets of this form. However, in an optional section at the end, we introduce integrals over p-chains and p-cycles and state the general form of Stoke's Theorem

There are topics which could have been included in this text, but were not. Some of our colleagues suggested that we include an introductory chapter or section on formal logic. We considered this but decided against it. Our feeling is that logic at this simple level is just language used with precision. Students have been using language for most of their lives, perhaps not always with precision, but that doesn't mean that they are incapable of using it with precision if required to do so. Teaching students to be precise in their use of the language tools that they already possess is one of the main objectives of the course. We do not believe that beginning the course with a study of formal logic would be of much help in this regard and, in fact, might just get in the way.

We could also have included a chapter of Fourier Series. However, we fell that the material that has been included makes for a text that is already a challenge to cover in a two semester course. We feel it unrealistic to think that

an additional chapter at the end would often get covered. In any case, the study of Fourier series is most naturally introduced at the undergraduate level in a course in differential equations.