Chapter 9

The Gamma and Zeta Functions

This chapter is devoted to developing some of the properties of two special functions of a complex variable – the gamma function and the zeta function. These functions are of great importance in modern mathematics. The developement of their properties provides a very instructive practical application of many of the techniques developed in the preceding chapters.

The zeta function is the subject of one of the most famous unsolved problems in mathematics - the Riemann Hypothesis. This conjecture arose from Riemann's attempt to settle an old conjecture concerning the rate of growth of the number $\pi(x)$ of primes less than or equal to x as the positive number x increases. In the process, Riemann developed (but did not completely prove) a formula for $\pi(x)$. This formula involves the zeroes of the zeta function in the strip $0 < \mathrm{Re}\,(z) < 1$, and its study led Riemann to conjecture that all these zeroes lie on the line $\mathrm{Re}\,(z) = 1/2$. If true, this would have been helpful in both the proof of Riemann's Formula and its use in analyzing the growth of $\pi(x)$.

The methods introduced by Riemann eventually led to proofs by others of the result on the growth of $\pi(x)$ that he was seeking. This result is now known as the Prime Number Theorem. These proofs use information about the location of the zeroes of the zeta function, but not, of course, the information proposed in the Riemann Hypothesis, since it has never been proved.

One of the reasons this chapter is included in the text is so that we may describe the Riemann Hypothesis and its connection to the Prime Number Theorem. For completeness, we conclude the chapter with a proof of the Prime Number Theorem.

9.1 Euler's Gamma Function

We define Euler's gamma function for $\mathrm{Re}(z) > 0$ by the integral formula

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt. \tag{9.1.1}$$

Of course, γ is defined by an improper integral and so we must show that this integral actually converges if $\mathrm{Re}(z) > 0$. In fact, it not only converges, but the resulting function of z is analytic.

Theorem 9.1.1. *The integral (9.1.1) converges and defines a holomorphic function $\Gamma(z)$ for $Re(z) > 0$.*

Proof. For $0 < r < s$ we define a function $\Gamma_{r,s}$ on the right half plane by

$$\Gamma_{r,s}(z) = \int_r^s e^{-t} t^{z-1}\, dt.$$

The function $\mathrm{e}^{-t} t^{z-1} = \mathrm{e}^{-t+(z-1)\ln t}$ is continuous as a function of (t, z) in $[r, s] \times \mathbb{C}$ and is analytic in z for each fixed value of t. By Exercise 3.2.16, $\Gamma_{r,s}$ is analytic on the entire plane. We will show that as $s \to \infty$ and $r \to 0$ the functions $\Gamma_{r,s}$ converge uniformly on each strip of the form

$$S = \{z : a \le \mathrm{Re}\,(z) \le b\} \quad \text{with} \quad 0 < a < b.$$

The limit function is then necessarily analytic on the right half plane and is, by definition, Euler's function Γ.

If $x = \mathrm{Re}(z)$, then

$$|e^{-t} t^{z-1}| = e^{-t} t^{x-1}.$$

Thus, if z is in the strip S, then

$$|e^{-t} t^{z-1}| \le t^{a-1} \quad \text{for} \quad t \le 1. \tag{9.1.2}$$

Now the function $e^{-t} t^{b+1}$ is continuous and has limit 0 at infinity. It is, therefore, bounded on $[1, \infty)$, by a positive number K. If $t \ge 1$, then

$$|e^{-t} t^{z-1}| \le e^{-t} t^{b-1} \quad \text{on} \quad S.$$

Thus,

$$|e^{-t} t^{z-1}| \le K t^{-2} \quad \text{for} \quad t \ge 1. \tag{9.1.3}$$

Since t^{a-1} is integrable on $(0, 1]$ and $K t^{-2}$ is integrable on $[1, \infty)$, Inequalities (9.1.2) and (9.1.3) imply that the improper integrals of $e^{-t} t^{z-1}$ on $(0, 1]$ and on $[1, \infty)$ both exist (see Theorem 5.2.2). Hence, the improper integral defining Γ exists for each $z \in S$.

To show that $\Gamma(z)$ is analytic, we will show that $\Gamma_{r,s}$ converges uniformly to Γ on each strip of the form S as $r \to 0$ and $s \to \infty$. In fact, from (9.1.2) and (9.1.3) we conclude

$$|\Gamma(z) - \Gamma_{r,s}(z)| \le \int_0^r t^{a-1}\, dt + \int_s^\infty K t^{-2}\, dt \le r^a/a + K/s.$$

Given $\epsilon > 0$ the right side of this inequality is less than ϵ whenever $r < (a\epsilon/2)^{1/a}$ and $s > 2K/\epsilon$. It follows from this that $\Gamma_{r,s}(z)$ converges uniformly to $\Gamma(z)$ for $z \in S$ as $r \to 0$ and $s \to \infty$. This completes the proof. □

Analytic Continuation of Gamma

We will continue Γ to a meromorphic function defined on the entire plane. The key to doing this is the fact that Γ satisfies a functional equation, as specified in the following theorem.

Theorem 9.1.2. *The gamma function satisfies the functional equation*

$$\Gamma(z+1) = z\Gamma(z)$$

for all z in the right half plane.

Proof. We have

$$\Gamma(z+1) = \int_0^\infty e^{-t}t^z\,dt.$$

Integrating by parts with $u = t^z$ and $dv = e^{-t}dt$ yields

$$\Gamma(z+1) = -t^z e^{-t}\Big|_0^\infty + \int_0^\infty e^{-t}zt^{z-1}\,dt = z\Gamma(z).$$

□

Corollary 9.1.3. *If n is a positive integer, then $\Gamma(n) = (n-1)!$.*

We leave the proof of this corollary as an exercise (Exercise 9.1.2).

Theorem 9.1.4. *The gamma function has a meromorphic continuation to the complex plane which has simple poles at the points $\{0, -1, -2, \cdots\}$.*

Proof. We prove by induction that Γ has a meromorphic continuation with the indicated poles and satisfying the functional equation $\Gamma(z+1) = z\Gamma(z)$ on the set $\{z : \mathrm{Re}(z) > -n\}$ for $n = 0, 1, 2, \cdots$. This is trivially true for $n = 0$. If it is true for n, then

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}$$

defines Γ on $\{z : \mathrm{Re}(z) > -n-1\}$ in a fashion which is consistent with its definition on the smaller set $\{z : \mathrm{Re}(z) > -n\}$, because of the functional equation. Clearly the poles of this continuation are as required and the functional equation continues to hold. □

Zeroes of Gamma

It turns out that Γ has no zeroes. To prove this requires deriving another functional equation. The derivation involves a pair of computational lemmas.

We define Euler's beta function, $B(z,w)$, by

$$B(z,w) = \int_0^1 (1-s)^{z-1} s^{w-1}\, ds,$$

for z and w with positive real part.

Lemma 9.1.5. *If z and w have positive real parts, then*

$$\frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)} = B(z,w).$$

Proof. For z with $\mathrm{Re}(z) > 0$, the substitution $t = u^2$ leads to

$$\Gamma(z) = \int_0^\infty t^{z-1} \mathrm{e}^{-t} dt = 2\int_0^\infty u^{2z-1} \mathrm{e}^{-u^2} du.$$

Then for two points z, w with positive real parts we have

$$\Gamma(z)\Gamma(w) = 4\int_0^\infty \int_0^\infty \mathrm{e}^{-(u^2+v^2)} u^{2z-1} v^{2w-1}\, du\, dv$$

We pass to polar coordinates, setting $u = r\cos(\theta)$ and $v = r\sin(\theta)$. Then

$$\begin{aligned}
\Gamma(z)\Gamma(w) &= 4\int_0^\infty \int_0^{\pi/2} \mathrm{e}^{-r^2} r^{2(z+w)-2} \cos^{2z-1}(\theta)\sin^{2w-1}(\theta)\ r\, dr\, d\theta \\
&= 2\int_0^\infty \mathrm{e}^{-r^2} r^{2(z+w)-1}\, dr \cdot 2\int_0^{\pi/2} \cos^{2z-1}(\theta)\sin^{2w-1}(\theta)\, d\theta \\
&= \Gamma(z+w)\cdot 2\int_0^{\pi/2} \cos^{2z-1}(\theta)\sin^{2w-1}(\theta)\, d\theta.
\end{aligned}$$

The substitution $s = \sin^2(\theta)$ leads to

$$2\int_0^{\pi/2} \cos^{2z-1}(\theta)\sin^{2w-1}(\theta)\, d\theta = \int_0^1 (1-s)^{z-1} s^{w-1}\, ds.$$

The latter expression is Euler's beta function $B(z,w)$. Thus,

$$\frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)} = B(z,w).$$

□

We will use the above identity in the case $w = 1-z$ to derive a functional equation for Γ. To do this, we will need to evaluate $B(z, 1-z)$. This involves the following integral evaluation.

Lemma 9.1.6. *If* $x \in (0,1)$, *then* $\displaystyle\int_0^\infty \frac{t^{-x}}{1+t}\,dt = \frac{\pi}{\sin \pi x}$.

Proof. We essentially proved this back in Chapter 5 where we discussed the Mellin Transform. In fact, the integral in the theorem is just the Mellin Transform of $\dfrac{1}{1+t}$ evaluated at $1-x$. In Example 5.3.8, as a special case of Theorem 5.3.7, we proved that the Mellin Transform of $\dfrac{1}{1+x}$ is

$$\int_0^\infty \frac{x^{t-1}}{1+x}\,dx = \frac{\pi}{\sin \pi t}.$$

If the roles of t and x are reversed, this becomes

$$\int_0^\infty \frac{t^{x-1}}{1+t}\,dt = \frac{\pi}{\sin \pi x}.$$

The identity of the theorem is then obtained by replacing x by $1-x$ and using the identity $\sin(\pi - \pi x) = \sin \pi x$. ☐

Theorem 9.1.7. *The Gamma function satisfies the functional equation*

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

Proof. If we set $z = x$ and $w = 1-x$ for $x \in (0,1)$, then since $\Gamma(1) = 1$, Lemma 9.1.5 implies

$$\Gamma(x)\Gamma(1-x) = \frac{\Gamma(x)\Gamma(1-x)}{\Gamma(1)} = B(x, 1-x) = \int_0^1 (1-s)^{x-1} s^{-x}\,ds.$$

Then the substitution $s = \dfrac{t}{t+1}$ leads to

$$\Gamma(x)\Gamma(1-x) = \int_0^\infty \left(1 - \frac{t}{t+1}\right)^{x-1} \frac{t^{-x}}{(t+1)^{-x}} \frac{dt}{(t+1)^2} = \int_0^\infty \frac{t^{-x}}{1+t}\,dt.$$

We use the previous lemma to evaluate the last integral and conclude that

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}.$$

Since this identity holds for $x \in (0,1)$, the Identity Theorem implies that it continues to hold when x is replaced by any complex number z for which the functions involved are defined. ☐

This theorem has the following corollary, the proof of which is left as an exercise (Exercise 9.1.4).

Corollary 9.1.8. *The gamma function has no zeroes.*

Product Formula for Γ

The fact that $e^{-t} = \lim_{n\to\infty}(1 - t/n)^n$ can be exploited to express Γ as an infinite product of the sort studied in the previous chapter. The first step in deriving this formula is to show the following:

Theorem 9.1.9. *The identity*

$$\Gamma(x) = \lim_{n\to\infty} \int_0^n (1 - t/n)^n t^{x-1}\, dt$$

holds for all $x > 0$.

Proof. The function $e^{-s} - 1 + s$ is 0 at $s = 0$ and has a positive derivative $(1 - e^{-s})$ for $s > 0$. It is, therefore positive for $s > 0$. Thus, $1 - s \le e^{-s}$ for $s > 0$. With $s = t/n$ this implies $1 - t/n < e^{-t/n}$ for $t > 0$ and, on taking nth powers,

$$(1 - t/n)^n \le e^{-t}.$$

Furthermore, an elementary calculus argument (Exercise 9.1.8) shows that

$$e^{-t} - (1 - t/n)^n \le \frac{1}{ne}, \tag{9.1.4}$$

for $t \ge 0$.

If we fix $a > 0$, then

$$\Gamma(x) - \int_0^n (1 - t/n)^n t^{x-1}\, dt \le \int_0^a (e^{-t} - (1 - t/n)^n) t^{x-1}\, dt + \int_a^\infty e^{-t} t^{x-1}\, dt.$$

for $n > a$. The first term on the right converges to 0 as $n \to \infty$ by (9.1.4) and the second term can be made less than any give ϵ by choosing a large enough, because the improper integral defining Γ converges. □

Theorem 9.1.10. *The entire function $1/\Gamma$ can be represented as the infinite product*

$$\frac{1}{\Gamma(z)} = z \prod_{k=1}^\infty \frac{1 + z/k}{(1 + 1/k)^z},$$

where this product converges uniformly on each disc of finite radius.

Proof. The integral in the previous theorem may be evaluated using a repeated application of integration by parts (Exercise 9.1.9). The result is

$$\Gamma(x) = \lim_{n\to\infty} \frac{n^x n!}{x(x+1)\cdots(x+n)}.$$

for $x > 0$.

If we invert this, divide both numerater and denomenator by $n!$ and note that $n^x = \prod_{k=1}^{n-1}(1 + 1/k)^x$ we obtain

$$\frac{1}{\Gamma(x)} = x \prod_{k=1}^\infty \frac{1 + x/k}{1 + 1/k}^x$$

for $x > 0$.

If we can show that this infinite produce converges, not just for $x > 0$, but uniformly on each disc of finite radius in the complex plane, then the result will be an entire function which agrees with the entire function $1/\Gamma(z)$ on the positive real axis. This implies the two entire functions agree on all of $\mathbb{C}$. Thus, the proof will be complete if we can show that

$$z \prod_{k=1}^{\infty} \frac{1 + z/k^{z}}{1 + 1/k} \tag{9.1.5}$$

converges uniformly on each compact disc. This product is very nearly a Weierstrass product, as studied in the previous chapter. This fact can be used to prove the uniform convergence on compact disc. The details are left to Exercise 9.1.10.

□

Exercise Set 9.1

1. Show that, for z real and positive, $\Gamma(z)$ is the Mellin Transform of a certain function. What function? (see Section 5.3).

2. Prove that $\Gamma(n) = (n-1)!$ if n is a positive integer.

3. Prove that $z(z+1)(z+2)\cdots(z+n) = \dfrac{\Gamma(z+n+1)}{\Gamma(z)}$.

4. Prove Corollary 9.1.8.

5. Prove that the residue of $\Gamma(z)$ at $-n$ is $\dfrac{(-1)^n}{n!}$ for $n = 0, 1, 2, \cdots$.

6. Prove that, for $r > 0$ and $\mathrm{Re}\,(z) > 0$, $\displaystyle\int_0^\infty \mathrm{e}^{-rt} t^{z-1}\, dt = r^{-z}\Gamma(z)$.

7. Prove that $\Gamma(z)\Gamma(-z) = \dfrac{-\pi}{z \sin \pi z}$.

8. Prove that $\mathrm{e}^{-t} - (1 - t/n)^n \le \dfrac{1}{n\mathrm{e}}$ for all $t \in [0, n]$. Hint: if $h(t) = \mathrm{e}^{-t} - (1 - t/n)^n$, show that the maximum of $h(t)$ on $[0, n]$ occurs at a point t_0 where $h(t_0) = \mathrm{e}^{-t_0} t_0/n$. Then show that this number is less than or equal to $\dfrac{1}{ne}$.

9. Using integration by parts, prove that if $x > 0$, then

$$\int_0^n \left(1 - \frac{t}{n}\right)^n t^{x-1}\, dt = \frac{n^x n!}{x(x+1)\cdots(x+n)}.$$

10. Prove that the infiinite product

$$\frac{1}{\Gamma(z)} = z \prod_{k=1}^{\infty} \frac{1 + z/k}{1 + 1/k}^{z}$$

converges uniformly on each compact disc. Hint: show that

$$\frac{1 + z/k}{(1 + 1/k)^z} = (1 + z/k)\mathrm{e}^{-z/k}\mathrm{e}^{a_k z},$$

where $a_k = 1/k - \log(1 + 1/k)$. Then show that $\prod_k (1 + z/k)\mathrm{e}^{-z/k}$ is a convergent Weierstrass product and $\sum_k |a_k|$ converges.

9.2 The Riemann Zeta Function

The Riemann zeta function is defined on the set $\{z : \mathrm{Re}(z) > 1\}$ by

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z}. \tag{9.2.1}$$

If $z = x+iy$, then $|n^{-z}| = n^{-x}$, and so this series converges uniformly absolutely on each set of the form $\{z : \mathrm{Re}(z) \geq r\}$ for $r > 1$. It follows that $\zeta(z)$ is defined and holomorphic on the set $\{z : \mathrm{Re}(z) > 1\}$.

A Product Formula for the Zeta Function

Let $\{p_1, p_2, p_3, \cdots\}$ be the set of prime numbers written in increasing order. Then we have

Theorem 9.2.1. *For Re*$(z) > 1$, $\zeta(z) = \prod_{n=1}^{\infty} (1 - p_n^{-z})^{-1}$.

Proof. The fact that the infinite product converges for $\mathrm{Re}(z) > 1$ follows from Theorem 8.1.4 and the fact that $\sum_{n=1}^{\infty} p_n^{\mathrm{Re}(z)}$ converges if $\mathrm{Re}\,(z) > 1$. Then

$$\zeta(z)(1 - 2^{-z}) = \sum_{1}^{\infty} n^{-z} - \sum_{1}^{\infty} (2n)^{-z} = \sum_{n \in S_1} n^{-z}$$

where S_1 is the set of odd natural numbers. An induction argument using the same technique then shows that for each natural number k

$$\zeta(z) \prod_{n=1}^{k} (1 - p_n^{-z}) = \sum_{n \in S_k} n^{-z}$$

where S_k is the set of natural numbers not divisible by any of the first k primes. Since the right side of this equation has limit 1 as $k \to \infty$, the theorem follows. □

This theorem has the following two corollaries. We leave the proofs to the exercises (Exercises 9.2.2 and 9.2.3).

Corollary 9.2.2. *There are infinitely many primes.*

Corollary 9.2.3. *The zeta function has no zeroes in the region* $Re(z) > 1$.

The ξ Function

Our next goal is to extend the zeta function to be a meromorphic function on the entire plane. We will do this by expressing the zeta function in terms of the Gamma function and a certain entire function ξ.

We begin the development of ξ by making the substitution $t = n^2 s^2 \pi$ in the formula (9.1.1) defining Γ. The result is

$$\Gamma(z) = 2n^{2z}\pi^z \int_0^\infty \mathrm{e}^{-n^2 s^2 \pi} s^{2z} \frac{ds}{s}.$$

If we divide by $n^{2z}\pi^z$ and sum over $n = 1, 2, 3, \cdots$ we obtain

$$\zeta(2z)\Gamma(z)\pi^{-z} = 2\sum_{n=1}^\infty \int_0^\infty \mathrm{e}^{-n^2 s^2 \pi} s^{2z} \frac{ds}{s} \quad \text{if} \quad \mathrm{Re}\,(z) > 1. \tag{9.2.2}$$

We will use the result of Exercise 9.2.7 to prove that it is legitimate to move the summation inside the integral in the expression on the right. We estimate the size of each integrand in this series as follows

$$\left|\mathrm{e}^{-n^2 s^2 \pi} s^{2z-1}\right| \le \mathrm{e}^{-ns^2} s^{2\mathrm{Re}(z)-1}.$$

The functions on the right are positive and their sum is

$$\sum_{n=1}^\infty \mathrm{e}^{-ns^2} s^{2\mathrm{Re}(z)-1} = \frac{s^{2\mathrm{Re}(z)-1}}{\mathrm{e}^{s^2} - 1}. \tag{9.2.3}$$

For each z, this series converges uniformly on each closed subinterval of $(0, \infty)$. Furthermore, since $e^{s^2} - 1 \ge s^2$, if $\mathrm{Re}\,(z) > 2$, the function on the right in (9.2.3) is less than or equal to $s^{2\mathrm{Re}(z)-3}$ and, hence, has finite integral over $[0, 1]$ if $\mathrm{Re}\,(z) > 2$. Since $e^{s^2} - 1 \ge \mathrm{e}^{s^2}/2$ if $s \ge 1$, the function on the right in (9.2.3) is less than or equal to $2\mathrm{e}^{-s^2} s^{\mathrm{Re}(z)-1}$ on $[1, \infty)$ and, hence, has finite integral on $[1, \infty)$. It follows that this function has finite integral on $[0, \infty)$. Thus, by the result of Exercise 9.2.7, the sum can be taken inside the integral in (9.2.2).Thus,

$$\zeta(2z)\Gamma(z)\pi^{-z} = 2\int_0^\infty \sum_{n=1}^\infty \mathrm{e}^{-n^2 s^2 \pi} s^{2z} \frac{ds}{s} \quad \text{for} \quad \mathrm{Re}\,(z) > 2. \tag{9.2.4}$$

If we replace z by $z/2$ and set

$$H(s) = \sum_{n=1}^\infty e^{-n^2 s^2 \pi}, \tag{9.2.5}$$

then (9.2.4) may be rewritten as

$$\zeta(z)\Gamma(z/2)\pi^{-z/2} = 2\int_0^\infty H(s)s^z\,\frac{ds}{s} \quad \text{for} \quad \mathrm{Re}\,(z) > 2. \tag{9.2.6}$$

The function ξ is obtained by multiplying this expression by $z(z-1)/2$. Thus, for $\mathrm{Re}\,(z) > 2$,

$$\xi(z) = \frac{z(z-1)}{2}\zeta(z)\Gamma(z/2)\pi^{-z/2} = z(z-1)\int_0^\infty H(s)s^z\,\frac{ds}{s}. \tag{9.2.7}$$

The Poisson Summation Formula

We pause to prove a technical result about Fourier Transforms (see section 5.3). It will be used in the up coming proof that ξ is extends to an entire function.

Theorem 9.2.4. *If f is a continuous function on $\mathbb{R}$ with the property that the series $\sum_{n=-\infty}^{\infty} f(x+2\pi n)$ converges absolutely and uniformly for $x \in [-\pi, \pi]$ and the series $\sum_{n=-\infty}^{\infty} \hat{f}(n)$ converges absolutely, then*

$$\sum_{n=1}^{\infty} f(2\pi n) = \frac{1}{\sqrt{2\pi}}\sum_{n=1}^{\infty} \hat{f}(n)$$

where $\hat{f}$ is the Fourier Transform of f.

Proof. We set $g(\theta) = \sum_{n=1}^{\infty} f(\theta+2\pi n)$ whenever $\theta \in [-\pi, \pi]$, and then integrate this function against the Poisson kernel (see section 6.5)

$$P_r(\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2} = \sum_{-\infty}^{\infty} r^{|n|}\mathrm{e}^{in\theta}$$

over the interval $[-\pi, \pi]$. The result is

$$\begin{aligned}
\int_{-\pi}^{\pi} g(\theta)P_r(\theta)\,d\theta &= \int_{-\pi}^{\pi}\sum_{-\infty}^{\infty} f(\theta+2\pi n)P_r(\theta)\,d\theta \\
&= \sum_{-\infty}^{\infty}\int_{-\pi}^{\pi} f(\theta+2\pi n)P_r(\theta)\,d\theta \\
&= \sum_{-\infty}^{\infty}\int_{(2n-1)\pi}^{(2n+1)\pi} f(\theta)P_r(\theta)\,d\theta \\
&= \int_{-\infty}^{\infty} f(\theta)P_r(\theta)\,d\theta \\
&= \sqrt{2\pi}\sum_{-\infty}^{\infty} r^{|n|}\hat{f}(n).
\end{aligned} \tag{9.2.8}$$

Note that the third step in this calculation uses the hypothesis that the series defining g converges uniformly absolutely .

As $r \to 1$ the integral on the left in (9.2.8) converges to

$$2\pi g(0) = 2\pi \sum_{n=1}^{\infty} f(2\pi n),$$

by Lemma 6.5.5, while the sum on the right converges to

$$\sqrt{2\pi} \sum_{-\infty}^{\infty} \hat{f}(n)$$

since, by hypothesis, the series $\sum_{-\infty}^{\infty} \hat{f}(n)$ converges absolutely. We conclude that

$$\sum_{n=-\infty}^{\infty} f(2\pi n) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \hat{f}(n),$$

as required. □

The Symmetry of ξ

We can now prove that ξ extends to be an entire function and that it satisfies the symmetry relation $\xi(z) = \xi(1-z)$. We first derive a symmetry relation for H.

Lemma 9.2.5. *The function H satisfies the relation*

$$H(s^{-1}) = sH(s) + (s-1)/2 \tag{9.2.9}$$

Proof. Note that $H(s) = (G(s) - 1)/2$, where

$$G(s) = \sum_{-\infty}^{\infty} e^{-n^2 s^2 \pi} = 1 + 2 \sum_{n=1}^{\infty} e^{-n^2 s^2 \pi}.$$

If $g(x) = \dfrac{1}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2}$ is the normal distribution function from Example 5.3.6, then

$$G(s) = \sqrt{2\pi} \sum_{-\infty}^{\infty} g(ns\sqrt{2\pi}).$$

If we apply the previous theorem to the function f defined by $f(x) = g(xs/\sqrt{2\pi})$ we conclude that

$$G(s) = \sum_{-\infty}^{\infty} f(2\pi n) = \frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} \hat{f}(n).$$

A change of variables in the integral defining the Fourier Transform shows that $\hat{f}(n) = s^{-1}\sqrt{2\pi}\hat{g}(ns^{-1}\sqrt{2\pi})$. But the function g is its own Fourier Transform, by Example 5.3.6. Thus,

$$G(s) = \sum_{-\infty}^{\infty} s^{-1}\sqrt{2\pi}g(ns^{-1}\sqrt{2\pi}) = s^{-1}G(s^{-1}).$$

The identity (9.2.9) follows from this. □

The consequence for the function ξ is the following:

Theorem 9.2.6. *The function ξ extends to an entire function which is symmetric about the point $z = 1/2$; that is $\xi(1-z) = \xi(z)$.*

Proof. If we break the integral on the right side of (9.2.6) into an integral over $[1, \infty)$ and an integral over $[0, 1]$ and make the substitution $s \to s^{-1}$ in the latter integral, the result is

$$\int_1^\infty H(s)s^z \frac{ds}{s} + \int_1^\infty H(s^{-1})s^{-z}\frac{ds}{s}.$$

Using (9.2.9), this becomes

$$\begin{aligned}&\int_1^\infty H(s)s^z \frac{ds}{s} + \int_1^\infty (sH(s) + (s-1)/2)s^{-z}\frac{ds}{s}\\ =&\int_1^\infty H(s)s^z \frac{ds}{s} + \int_1^\infty H(s)s^{(1-z)}\frac{ds}{s} + \frac{1}{2}\int_1^\infty (s^{(1-z)} - s^{-z})\frac{ds}{s}.\end{aligned}$$

Since

$$\frac{1}{2}\int_1^\infty (s^{(1-z)} - s^{-z})\frac{ds}{s} = \frac{1/2}{z(z-1)} \quad \text{if} \quad \mathrm{Re}\,(z) > 1,$$

we conclude that

$$\xi(z) = 1/2 - z(1-z)\int_1^\infty H(s)(s^z + s^{1-z})\,\frac{ds}{s} \quad \text{for} \quad \mathrm{Re}\,(z) > 2. \qquad (9.2.10)$$

However, the right side of this equation is defined and analytic on the entire plane, since $H(s)$ times any power of s is absolutely integrable on $[1, \infty)$. It is also obviously symmetric about $z = 1/2$. Thus, ξ has an extension to the whole plane with the required properties. □

Meromorphic Extension of ζ

The next theorem is an immediate consequence of the preceding theorem and (9.2.7).

Theorem 9.2.7. *The function ζ has a meromorphic extension to the plane given by the formula*

$$\zeta(z) = \frac{2\pi^{z/2}\xi(z)}{z(z-1)\Gamma(z/2)}. \qquad (9.2.11)$$

It is useful to note that the above formula can be put in a slightly different form by using Theorem 9.1.2. This theorem, with z replaced by $z/2$ implies that

$$z/2\Gamma(z/2) = \Gamma(z/2+1).$$

Then (9.2.11) becomes

$$\zeta(z) = \frac{\pi^{z/2}\xi(z)}{(z-1)\Gamma(z/2+1)}. \tag{9.2.12}$$

Exercise Set 9.2

1. Show that $\lim_{x\to\infty} \zeta(x+iy) = 1$ and the convergence is uniform in y.

2. Use Theorem 9.2.1 to prove Corollary 9.2.2.

3. Use Theorem 9.2.1 to prove Corollary 9.2.3.

4. If $z = s + it$ with $s > 1$, prove that

$$\left|\frac{1}{\zeta(z)}\right| < \zeta(s).$$

Hint: Use Theorem 9.2.1.

5. Use the result of the previous exercise to prove that

$$\left|\frac{1}{\zeta(z)}\right| < \zeta(2)$$

if $\mathrm{Re}\,(z) \geq 2$.

6. Let $u(t) = \sum_{n=1}^{\infty} u_n(t)$ be the sum of a series of positive continuous functions on $(0,\infty)$ and suppose this series converges uniformly on closed bounded intervals of $(0,\infty)$. Prove that

$$\sum_{n=1}^{\infty} \int_0^1 u_n(t)\,dt = \int_0^1 u(t)\,dt$$

if the integral on the right is finite.

7. Let $h(t) = \sum_{n=1}^{\infty} h_n(t)$ be the sum of an infinite series of continuous functions on $(0,\infty)$ and suppose

$$|h_n(t)| \leq u_n(t) \quad \text{for all} \quad n, t$$

where $\sum_{n=1}^{\infty} u_n$ is a positive termed series which satisfies the conditions of the previous exercise. Prove that the improper integral of h on $\mathbb{R}$ converges and

$$\int_0^{\infty} h(t)\,dt = \sum_{n=1}^{\infty} \int_0^{\infty} h_n(t)\,dt.$$

8. Show that the Poisson Summation Formula can, under appropriate hypotheses on f and $\hat{f}$, be reformulated as
$$\sum_{n=-\infty}^{\infty} f(n) = \sqrt{2\pi} \sum_{n=-\infty}^{\infty} \hat{f}(2\pi n).$$

9. Use the form of the Poisson Summation Formula derived in the previous exercise to show that
$$\sum_{-\infty}^{\infty} \frac{1}{1+n^2} = \pi \frac{e^{2\pi}+1}{e^{2\pi}-1}.$$
Hint: The Fourier Transform of $\dfrac{1}{1+x^2}$ is calculated in Example 5.3.3.

The next five exercises outline an alternative to the approach used in Theorem 9.2.7 to prove that ζ extends to be meromorphic in the plane.

10. Formula (9.2.4) was developed using the substitution $t = n^2 s^\pi$ in the integral defining Γ. Use the substitution $t = ns$ in a similar way to derive the formula
$$\zeta(z)\Gamma(z) = \int_0^\infty \frac{1}{e^s - 1} s^{z-1}\, ds \quad \text{for} \quad \mathrm{Re}\,(z) > 1.$$

11. For complex numbers w and z, define $(-w)^{z-1}$ to be $e^{(z-1)\log(-w)}$, where log is the principal branch of the log function. Show that this function is analytic except for a cut on the positive real line, that its limit as w approaches the positive real number s from above is $e^{(z-1)(\log s - \pi i)}$, and that its limit as w appproaches s from below is $e^{(z-1)(\log s + \pi i)}$.

12. Using this definition for $(-w)^{z-1}$, consider the contour integral
$$\eta(z) = \int_{\gamma_r} \frac{1}{e^w - 1} (-w)^{z-1}\, dw. \tag{9.2.13}$$
where γ_r is the contour indicated in Figure 9.1, $r < 2\pi$ is the radius of the indicated circle, and the two horizontal lines are a distance $\epsilon \le r$ above and below the positive real axis. Prove that this integral exists for all z, is independent of r and ϵ, and defines an entire function $\eta(z)$.

13. By passing to the limit as $\epsilon \to 0$ and $r \to 0$ in the integral 9.2.13, prove that $\eta(z) = -2\pi i \sin(\pi z)\Gamma(z)\zeta(z)$ if $\mathrm{Re}\,(z) > 1$.

14. Use the previous exercise and an identify involving Γ to prove that
$$\zeta(z) = -\frac{1}{2\pi i}\Gamma(1-z)\eta(z) \quad \text{for} \quad \mathrm{Re}\,(z) > 1.$$
Conclude that ζ has a meromorphic extension to the plane with a single simple pole at $z = 1$.

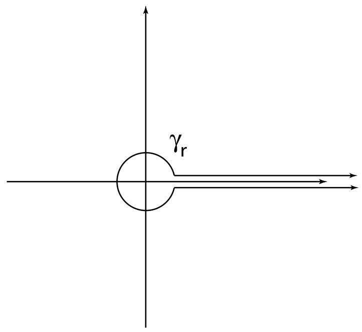

Figure 9.1: Contour for Exercise 12

9.3 Properties of ζ

The expressions (9.2.11) and (9.2.12) for ζ and the properties of Γ and ξ lead to a wealth of information about ζ. Ultimately, this information will lead to a proof of the Prime Number Theorem in the last section of this chapter.

Zeroes and Poles

Theorem 9.3.1. *The zeta function has a simple pole at $z = 1$, with residue 1, and no other poles.*

Proof. The function $\Gamma(z/2+1)$ has no zeroes, and the function ξ is entire. Thus, (9.2.12) implies that the only pole of ζ is at $z = 1$ and it is a simple pole. The fact that the residue is 1 follows from the fact, proved in the exercises, that $\Gamma(1/2) = \sqrt{\pi}$, and from (9.2.10), which implies $\xi(1) = 1/2$. □

Theorem 9.3.2. *The zeta function has a zero of order one at each negative even integer.*

This is Exercise 9.3.1.

Corollary 9.3.3. *The zeta function has no zeroes outside the strip*

$$0 \le Re(z) \le 1$$

except the ones that occur at negative even integers.

Proof. Except for the zeroes of ζ that occur at negative even integers (due to the poles of Γ), the functions ζ and ξ have the same zeroes. Since ζ has no zeroes in the region $\mathrm{Re}\,(z) > 1$ by Corollary 9.2.3, and since ξ is symmetric about 1/2, it follows that ζ has no zeroes outside the strip $0 \le \mathrm{Re}(z) \le 1$ except the negative even integers.T □

This result can be strengthened to exclude the existence of zeroes of ζ on the lines $\mathrm{Re}\,(z) = 0, 1$. The proof uses the following lemma:

Lemma 9.3.4. *For Re$(z) > 1$ there is an analytic logarithm for $\zeta(z)$, defined by*

$$\log \zeta(z) = \sum_p \sum_{m=1}^{\infty} \frac{p^{-mz}}{m}. \tag{9.3.1}$$

The derivative of this function is

$$\frac{\zeta'(z)}{\zeta(z)} = -\sum_p \sum_{m=1}^{\infty} p^{-mz} \log p. \tag{9.3.2}$$

Here, in both equations, the summation on the left is over all primes p.

Proof. If Re$(z) > 1$, then $|p^{-z}| < 1/2$ for all primes p and so $\log(1 - p^{-z})$ is defined and analytic if log is the principal branch of the log function. Furthermore, by Theorem 9.2.1 we have for Re$(z) > 1$

$$\exp\left(-\sum_p \log(1 - p^{-z})\right) = \prod (1 - p^{-z})^{-1} = \zeta(z). \tag{9.3.3}$$

Hence, $-\sum_p \log(1 - p^{-z})$ is an analytic logarithm for $\zeta(z)$ on Re$(z) > 1$. If we expand this function in a power series in p^{-z}, the result is (9.3.1). On differentiating (9.3.1), we obtain (9.3.2). □

Theorem 9.3.5. *The zeta function has no zeroes outside the strip*

$$0 < Re\,(z) < 1$$

except those which occur at negative even integers.

Proof. We first note that the inequality

$$0 \le 3 + 4\cos\theta + \cos 2\theta \tag{9.3.4}$$

holds for all real numbers θ due to the identity

$$3 + 4\cos\theta + \cos 2\theta = 2(1 + \cos\theta)^2,$$

which follows from $\cos 2\theta = 2\cos^2\theta - 1$.

If $z = x + iy$, then Re$(p^{-mz}) = p^{-mx}\cos(-my\log p)$. It follows from (9.3.4) that, for $x > 1$,

$$0 \le 3p^{-mx} + 4\mathrm{Re}\left(p^{-mx-myi}\right) + \mathrm{Re}\left(p^{-mx-2myi}\right),$$

which, when combined with (9.3.1), implies that, for $x > 1$,

$$3\mathrm{Re}\,(\log\zeta(x)) + 4\mathrm{Re}\,(\log\zeta(x+iy)) + \mathrm{Re}\,(\log\zeta(x+i2y)) \ge 0,$$

or, on exponentiating,

$$|\zeta(x)|^3 |\zeta(x+iy)|^4 |\zeta(x+i2y)| \ge 1. \tag{9.3.5}$$

We divide both sides of this inequality by $x-1$ and write the result in the form

$$|(x-1)\zeta(x)|^3 \left|\frac{\zeta(x+iy)}{x-1}\right|^4 |\zeta(x+i2y)| \geq \frac{1}{x-1}. \tag{9.3.6}$$

Since $\zeta(z)$ has a simple pole at $z=1$, $(1-z)\zeta(z)$ has a removable singularity at $z=1$. This implies that the first factor on the left in (9.3.6) is bounded as $x \to 1$. Since there are no other poles of ζ, the factor on the right is also bounded as $x \to 1$, provided $y \neq 0$. If ζ has a zero at $1+iy$, then the middle factor is also bounded as $x \to 1$. Since the right side of (9.3.6) is not bounded as $x \to 1$, we conclude that there can be no zero of ζ at $z = 1+iy$.

Now that we know that there are no zeroes of ζ on the line $\mathrm{Re}\,(z) = 1$, we use the fact that the set of zeroes of ζ in the strip $0 \leq \mathrm{Re}\,(z) \leq 1$ is symmetric about the line $\mathrm{Re}\,(z) = 1/2$ (since these are also the zeroes of ξ) to conclude that there are no zeroes on the line $\mathrm{Re}\,(z) = 0$. In view of Corollary 9.3.3, this completes the proof. □

Estimate on the growth of ξ

Integration by parts in the integral appearing in (9.2.10) leads to

$$\begin{aligned}\xi(z) &= 1/2 + H(1) + \int_1^\infty H'(s)((1-z)s^z + zs^{1-z})\,ds \\ &= 1/2 + H(1) + \int_1^\infty s^2 H'(s)((1-z)s^{(z-1)} + zs^{-z})\,\frac{ds}{s}.\end{aligned}$$

Another integration by parts leads to

$$\xi(z) = 1/2 + H(1) + 2H'(1) + \int_1^\infty (s^2 H'(s))'(s^{z-1} + s^{-z})\,ds.$$

If we differentiate (9.2.9) and set $s=1$, the result is

$$\frac{1}{2} + H(1) + 2H'(1) = 0.$$

Thus,

$$\xi(z) = \int_1^\infty (s^2 H'(s))'(s^{z-1} + s^{-z})\,ds. \tag{9.3.7}$$

Since

$$\begin{aligned}(s^{z-1} + s^{-z}) &= s^{-1/2}(s^{z-1/2} + (s^{-z+1/2}) \\ &= 2s^{-1/2}\cosh\left((z-1/2)\ln(s)\right),\end{aligned}$$

(9.3.7) can be rewrittten as

$$\xi(z) = \int_1^\infty (s^2 H'(s))' s^{-1/2} \cosh\left((z-1/2)\ln(s)\right) ds. \tag{9.3.8}$$

Theorem 9.3.6. *If ξ is expanded as a power series in $z - 1/2$, the coefficients of the power series are all real and non-negative.*

Proof. A direct calculation using (9.2.5) shows that

$$(s^2 H'(s))' s^{-1/2} = \sum_{n=1}^{\infty} (4n^4\pi^2 s^4 - 6n^2\pi s^2) s^{-1/2} \mathrm{e}^{-n^2 s^2 \pi}.$$

The terms of this series are clearly positive for $s \geq 1$ and so the function itself is positive. Also, the power series coefficients in the expansion of $\cosh w$ about 0 are real and non-negative. Since $\ln(s) \geq 0$ for $s \geq 1$, the theorem follows (see Exercise 9.36). □

Theorem 9.3.7. *There is a constant R such that $|\xi(1/2+z)| \leq r^r$ for all $z \in \mathbb{C}$ with $r > R$, where $r = |z|$.*

Proof. Since $\xi(z+1/2)$ has a power series expansion in z with real non-negative coefficients, its maximum absolute value on any disc $D_r(0)$ is achieved at $z = r$. However, if n is an integer such that $1/2 + r \leq 2n \leq 5/2 + r$, then by (9.2.7) and the fact that ξ is increasing on the positive real line (Exercise 9.3.2)

$$\xi(1/2 + r) \leq \xi(2n) = n(2n-1)\zeta(2n)\Gamma(n)\pi^{-n}.$$

Now ζ is decreasing on $(1, \infty)$. Thus,

$$\zeta(2n) \leq \zeta(2) \quad \text{if} \quad n \geq 1,$$

and

$$\Gamma(n) = (n-1)!$$

if n is a positive integer. Thus,

$$\xi(1/2 + r) \leq 2n\, n!\, \zeta(2) \leq 2\zeta(2)\, n^{n+1} \leq r^r$$

if r is sufficiently large (since $n \leq 5/4 + r/2$) (Exercise 9.3.8). □

A Product Expansion for ξ

In what follows, it will be convenient to make the change of variables $w = z - 1/2$. Then

$$\xi(z) = \xi(1/2 + w).$$

Since ξ is symmetric about 1/2, the function $\xi(1/2 + w)$ is symmetric about 0.

It follows from the previous theorem that $\xi(1/2+w)$ is a an entire function of finite order at most 1 (see Section 8.3 and Exercise 8.3.8). Hence, the Hadamard Factorization Theorem applies with $\lambda = 1$. It tells us that ξ has a factorization of the form

$$\xi(1/2 + w) = \mathrm{e}^{q(w)} \prod_{\sigma} (1 - w/\sigma) \mathrm{e}^{w/\sigma}$$

where q is a polynomial of degree at most 1, and the product is over all zeroes σ of $\xi(1/2+w)$. However, the zeroes of this function are symmetric about 0 and so, if a zero σ appears in this product, then so does its negative. The exponential factors $e^{w/\sigma}$ and $e^{-w/\sigma}$ cancel and so

$$\xi(1/2+w) = e^{q(w)} \prod_\sigma (1 - w/\sigma)$$

as long as it is understood that factors involving a given σ and its negative $-\sigma$ are to be grouped together. If this is done, then the product expansion becomes

$$\xi(1/2+w) = e^{q(w)} \prod_{\mathrm{Im}(()\sigma)>0} (1 - w^2/\sigma^2).$$

It is this product that actually converges (see the discussion of the product expansion of $\sin(\pi z)$ in Example 8.2.6).

Now $\xi(1/2+w)$ is an even function of w (symmetric about 0) and so is the above product. It follows that the polynomial q must also be an even function. Since the only even polynomials of degree at most 1 are constants, we conclude that

$$\xi(1/2+w) = c\prod_\sigma (1 - w/\sigma) \tag{9.3.9}$$

for some constant c.

If we recall that $w = z - 1/2$ and $\sigma = \rho - 1/2$ and we use the identity

$$1 - \frac{(z-1/2)}{(\rho-1/2)} = \left(1 - \frac{z}{\rho}\right)\left(1 + \frac{1/2}{\rho - 1/2}\right) \tag{9.3.10}$$

(Exercise 9.3.9), then (9.3.9) becomes

$$\xi(z) = \prod_\rho \left(1 - \frac{z}{\rho}\right)\left(1 + \frac{1/2}{\rho - 1/2}\right).$$

Since the product

$$c_1 = c\prod_\rho \left(1 + \frac{1/2}{\rho - 1/2}\right) \tag{9.3.11}$$

converges as long as factors involving $\rho - 1/2$ and its negative are grouped together (Exercise 9.3.10), we obtain an infinite product expansion

$$\xi(z) = c_1 \prod_\rho \left(1 - \frac{z}{\rho}\right).$$

By evaluating at $z = 0$, we see that $c_1 = \xi(0) = 1/2$. This proves the following theorem

Theorem 9.3.8. *The function ξ has the following infinite product expansion which converges uniformly on each disc of finite radius:*

$$\xi(z) = \frac{1}{2}\prod_{\rho}\left(1 - \frac{z}{\rho}\right)$$

where the numbers ρ are the zeroes of $\xi(z)$ and the factors are arranged so that ρ and $1-\rho$ are grouped together. The product converges uniformly on each disc of finite radius.

This, in turn, implies that the zeta function has the following infinite product expansion:

Theorem 9.3.9. *The zeta function satisfies*

$$\zeta(z) = \frac{1}{z(z-1)}\frac{\pi^{z/2}}{\Gamma(z/2)}\prod_{\rho}\left(1 - \frac{z}{\rho}\right).$$

where the product is over the zeroes ρ of ζ in the strip $0 < Re(z) < 1$, and factors involving ρ and $1-\rho$ are grouped together in the infinite product.

Exercise Set 9.3

1. Prove that ζ has a zero of order 1 at each negative even integer.
2. Prove that ξ is increasing on the positive real line.
3. Show that $\xi(0) = 1/2$ even though (9.2.7) suggests it should be 0. Why is there no contradiction here?
4. Calculate $\zeta(0)$ using (9.2.12).
5. Calculate $\zeta(2)$ directly from the definition of ζ, using results from Section 5.4.
6. Suppose f is an entire function and the coefficients of the power series expansion of f about 0 are all non-negative. Show that if $g(t)$ and $h(t)$ are positive continuous functions on $[1,\infty)$, then

$$\int_1^\infty g(t)f(zh(t))\,dt$$

is also an entire function of z with non-negative coefficients in its power series expansion about 0, provided this integral exists for all positive real values of z.

7. Show that

$$\int_1^\infty e^{-t}t^z\frac{dt}{t}$$

is an entire function of z with all non-negative coefficients in its power series expansion about 0.

8. Verify the claim made in the last sentence of the proof of Theorem 9.3.7 - that is, show that $2\zeta(2)n^{n+1} \leq r^r$ if r is sufficiently large and $n \leq 5/4 + r/2$.

9. Prove the identity (9.3.10).

10. Show why the product in (9.3.11) converges if the terms are grouped as indicated.

9.4 The Riemann Hypothesis and Prime Numbers

The Riemann Hypothesis is the conjecture that all zeroes of the zeta function in the strip $0 < \mathrm{Re}(z) < 1$ lie on the line $\mathrm{Re}(z) = 1/2$. The significance of this conjecture lies in its connection with the problem of estimating the density of the prime numbers in the set of natural numbers. In this section we will discuss some of the history of the two problems and attempt to illustrate the connection between them without going into too much computational detail. For a more comprehensive and detailed account of the subject see the book by H. M. Edwards (reference).

We let $\pi(x)$ denote the number of primes less than or equal to the positive real number x. It was recognized by Riemann that there is a connection between the rate of growth of $\pi(x)$ as x increases and the zeroes of the zeta function. That there is some connection between the growth of $\pi(x)$ and the zeta function is seen in the proof that there are infinitely many primes (Exercise 9.2.2).

Based on experimental evidence, Gauss and Legendre conjectured that

$$\lim_{x\to\infty} \frac{\pi(x)}{x} \log x = 1. \tag{9.4.1}$$

This means that the fraction of the natural numbers up to x that are prime, $\dfrac{\pi(x)}{x}$, is asymptotic to $\dfrac{1}{\log x}$ in the sense that their ratio has limit 1.

In his famous 1859 paper Riemann introduced a formula for $\pi(x)$. For a certain constant c, the function Li is defined to be

$$Li(x) = \begin{cases} 0 & x \leq 2 \\ \displaystyle\int_2^x \frac{dt}{\log t} + c & x > 2. \end{cases}$$

The formula of Riemann for $\pi(x)$ is then

$$\pi(x) = \sum_{n=1}^{\infty} Li(x^{1/n}) + \sum_{\rho} \sum_{n=1}^{\infty} Li(x^{\rho/n}) + other, \tag{9.4.2}$$

where the "other" terms are of lower order in x, and ρ ranges over all zeroes of ζ in the strip $0 < \mathrm{Re}\,(z) < 1$. Note that the sums over n are actually finite sums for each fixed x and ρ, since $x^{1/n}$ and $x^{\rho/n}$ are less than 2 if n is large enough.

An integration by parts argument shows that the first (and presumably dominant) term in the expansion (9.4.2) may be rewritten as

$$Li(x) = \frac{x}{\log x} - \int_2^x \frac{dt}{(\log t)^2} + c.$$

The second term in this expression, when divided by $\frac{x}{\log x}$ has limit 0 at infinity (Exercise 9.4.1). Thus, if the remaining terms in Riemann's Formula for $\pi(x)$, when divided by $\frac{x}{\log x}$, also have limit 0 at infinity, and if it is legitimate to take the limit inside the sum in the first term, then (9.4.1) follows.

In dealing with the terms involving the zeroes ρ of ζ in Riemann's Formula, it would be useful if the zeroes of ζ in the strip $0 < \operatorname{Re}(z) < 1$ all satisfied $\operatorname{Re}(z) < r$ for some $r < 1$. Riemann suspected that this was true and, in fact, he conjectured that all such zeroes actually lie on the line $\operatorname{Re}(z) = 1/2$. This is the Riemann Hypothesis.

Actually, Riemann did not give a complete proof of his formula (9.4.2) for $\pi(x)$ and, in fact, he did not even prove that the infinite series in this formula converges. Both facts were eventually proved, but the difficulties involved in these proofs and in determining the contribution of the terms involving the zeroes of ζ to the asymptotic behavior of $\pi(x)$ led to the introduction of another function $\psi(x)$ which also measures the density of primes and which satisfies a simpler and more natural formula analogous to (9.4.2).

Eventually, Hadamard and de la Vallee Poussin in 1896 proved (9.4.1) It is now known as the Prime Number Theorem. The proofs of Hadamard and de la Vallee Poussin as well as other classical proofs of this result are based on the fact that there are no zeroes of the zeta function on the line $\operatorname{Re}(z) = 1$. We will present one such proof in the next section.

The Function ψ

As mentioned above, the function $\pi(x)$ is closely related to another function which also measures the density of primes and which has a more straightforward connection to the zeta function. It is:

$$\psi(x) = \sum_{p^m \le x} \log p \tag{9.4.3}$$

where the summation on the right is over all integers $p^m \le x$ which are positive powers of primes p. Thus, for each prime p, the summand $\log p$ appears as many times in this sum as there are powers of p less than or equal to x. We will show that if the function ψ satisfies $\lim_{x\to\infty} \psi(x)/x = 1$, then the Prime Number Theorem follows.

Lemma 9.4.1. *Let λ be any function from $(1, \infty)$ to itself. Then*

$$\frac{\psi(x)}{\log x} \le \pi(x) \le \lambda(x) + \frac{\psi(x)}{\log \lambda(x)}$$

for all $x > 1$.

Proof. We have

$$\begin{aligned}\pi(x) &\leq \pi(\lambda(x)) + \sum_{\lambda(x)<p\leq x} 1 \\ &\leq \lambda(x) + \sum_{\lambda(x)<p\leq x} \frac{\log p}{\log \lambda(x)},\end{aligned} \tag{9.4.4}$$

where the sums are over primes p in the indicated range.

If p is a prime less than or equal to x and m_p is the positive integer such that $p^{m_p} \leq x < p^{m_p+1}$, then $\log p$ appears exactly m_p times in the sum (9.4.3) defining ψ. It follows that

$$\psi(x) = \sum_{p\leq x} m_p \log p = \sum_{p\leq x} \log p^{m_p}. \tag{9.4.5}$$

The sum on the right satisfies the inequalities

$$\sum_{\lambda(x)<p\leq x} \log p \leq \sum_{p\leq x} \log p^{m_p} \leq \pi(x) \log x,$$

and so, by (9.4.5),

$$\sum_{\lambda(x)<p\leq x} \log p \leq \psi(x) \leq \pi(x) \log x$$

If we combine this with (9.4.4), the result is

$$\frac{\psi(x)}{\log x} \leq \pi(x) \leq \lambda(x) + \frac{\psi(x)}{\log \lambda(x)},$$

as claimed in the lemma. □

Theorem 9.4.2. *If* $\lim_{x\to\infty} \frac{\psi(x)}{x} = 1$, *then* $\lim_{x\to\infty} \frac{\pi(x)}{x} \log x = 1$.

Proof. We use the previous lemma with

$$\lambda(x) = \frac{x}{\log^2 x}.$$

According to that lemma

$$\frac{\psi(x)}{\log x} \leq \pi(x) \leq \frac{x}{\log^2 x} + \frac{\psi(x)}{\log x - 2\log\log x},$$

and so

$$1 \leq \pi(x)\frac{\log x}{\psi(x)} \leq \frac{1}{\log x}\frac{x}{\psi(x)} + \frac{\log x}{\log x - 2\log\log x}.$$

The first term on the right has limit 0 as $x \to \infty$, while the second term has limit 1. This is because

$$\lim_{x\to\infty} \frac{\psi(x)}{x} = 1,$$

by hypothesis, while

$$\lim_{x\to\infty} \frac{\log x}{\log x - 2\log\log x} = 1.$$

The latter statement is left as an exercise (Exercise 9.4.4). It follows that

$$\lim_{x\to\infty} \pi(x)\frac{\log x}{\psi(x)} = 1,$$

and, from this, that

$$\lim_{x\to\infty} \frac{\pi(x)}{x} \log x = 1.$$

This completes the proof. □

There are direct proofs that $\lim_{x\to\infty} \psi(x)/x = 1$. However, they involve serious difficulties with improper integrals and conditionally convergent series. It turns out the these difficulties can be made to disappear if we follow a similar approach, but use the integral of ψ rather than ψ itself as the main focus of attention. Thus, we set

$$\phi(x) = \int_1^x \psi(u)\,du. \tag{9.4.6}$$

This is another function which measures the density of primes. Furthermore, the prime number follows from an appropriate estimate on its asymptotic behavior.

Properties of ϕ

It turns out that,

$$\lim_{x\to\infty} \frac{\psi(x)}{x} = 1,$$

from which the Prime Number Theorem follows, provided

$$\lim_{x\to\infty} \frac{\phi(x)}{x^2} = \frac{1}{2}, \tag{9.4.7}$$

This follows from a kind of reverse L'Hôpital's Rule. Specifically:

Lemma 9.4.3. *Let f be a positive, increasing function on $[1,\infty)$ and suppose $r > 0$. Then*

$$\lim_{x\to\infty} \frac{f(x)}{x^r} = \lim_{x\to\infty} \frac{r+1}{x^{r+1}} \int_1^x f(u)\,du,$$

provided the limit on the right exists and is finite.

Proof. If we were to assume that the limit on the left exists, then the equality would follow from applying L'Hôpital's rule to the limit on the right. However, we are assuming only that the limit on the right exists, and so we must proceed differently.

Using the fact that f is increasing and positive, we conclude that, for $\alpha < 1$ and $\beta > 1$,

$$\frac{1}{(1-\alpha)x}\int_{\alpha x}^{x} f(u)\,du \le f(x) \le \frac{1}{(\beta-1)x}\int_{x}^{\beta x} f(u)\,du.$$

On dividing by x^r, this becomes

$$\frac{1}{(1-\alpha)x^{r+1}}\int_{\alpha x}^{x} f(u)\,du \le \frac{f(x)}{x^r} \le \frac{1}{(\beta-1)x^{r+1}}\int_{x}^{\beta x} f(u)\,du.$$

If we set $F(x) = \displaystyle\int_1^x f(u)\,du$, then this can be re-written as

$$\frac{F(x)-F(\alpha x)}{(1-\alpha)x^{r+1}} \le \frac{f(x)}{x^r} \le \frac{F(\beta x)-F(x)}{(\beta-1)x^{r+1}},$$

or as

$$\frac{1}{1-\alpha}\left(\frac{F(x)}{x^{r+1}} - \alpha^{r+1}\frac{F(\alpha x)}{(\alpha x)^{r+1}}\right) \le \frac{f(x)}{x^r} \le \frac{1}{\beta-1}\left(\beta^{r+1}\frac{F(\beta x)}{(\beta x)^{r+1}} - \frac{F(x)}{x^{r+1}}\right)$$

If we set

$$L = \lim_{x\to\infty}\frac{F(x)}{x^{r+1}} = \lim_{x\to\infty}\frac{F(\alpha x)}{(\alpha x)^{r+1}} = \lim_{x\to\infty}\frac{F(\beta x)}{(\beta x)^{r+1}},$$

then the above inequality implies that

$$\frac{1-\alpha^{r+1}}{1-\alpha}L \le \liminf_{x\to\infty}\frac{f(x)}{x^r} \le \limsup_{x\to\infty}\frac{f(x)}{x^r} \le \frac{\beta^{r+1}-1}{\beta-1}L.$$

The lemma then follows from this on taking the limit as α and β approach 1, since both $\dfrac{1-\alpha^{r+1}}{1-\alpha}$ and $\dfrac{\beta^{r+1}-1}{\beta-1}$ have limit $r+1$. □

This leads directly to the following theorem. The details are left to the exercises.

Theorem 9.4.4. *If* $\displaystyle\lim_{x\to\infty}\frac{\phi(x)}{x^2} = 1/2$, *then* $\displaystyle\lim_{x\to\infty}\frac{\pi(x)}{x}\log x = 1$.

Exercise Set 9.4

1. Prove that $\displaystyle\lim_{x\to\infty}\frac{\log x}{x}\int_2^x \frac{dt}{(\log t)^2} = 0$.

2. Prove Part (a) of Lemma 9.5.1.

3. Prove Part (b) of Lemma 9.5.1.
4. Prove that $\lim_{x\to\infty} \dfrac{\log x}{\log x - 2\log\log x} = 1$.
5. Prove Theorem 9.4.4.

9.5 A Proof of the Prime Number Theorem

In view of Theorems 9.4.2 and 9.4.4, to prove the Prime Number Theorem, it suffices to show that $\lim_{x\to\infty} \phi(x)/x^2 = 1/2$. The strategy for doing this involves expressing $\phi(x)$ as an integral involving ζ'/ζ. This is where the zeroes of the zeta function come in.

The integral formula relating ζ'/ζ and ϕ is derived from the series expansion 9.3.2 and the following integral formula.

Lemma 9.5.1. *Suppose $p(z)$ is a non-constant polynomial and b a real number such that no zero of p lies on the line $Re(z) = b$. If $y > 1$ let $A = \{z_1, z_2, \cdot, z_n\}$ be the set of zeroes of $p(z)$ that lie to the left of the line $Re(z) = b$. Then*

$$\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \frac{y^z}{p(z)}\, dz = \sum_{k=1}^{n} Res(y^z/p(z), z_k), \tag{9.5.1}$$

If the set A is empty, then the integral is zero.

If $y < 1$ a similar formula holds, the only differences being: A is replaced by the set of zeroes to the right of $Re(z) = b$ and the expression on the right is multiplied by -1.

Proof. In Chapter 5 we showed how to use residue theory to calculate the Fourier Transforms of certain functions (Theorem 5.3.2). The integral that appears in (9.5.1) is actually the Fourier Transform of a function to which Theorem 5.3.2 applies. To see this, we write

$$\frac{y^z}{p(z)} = \frac{e^{z\log y}}{p(z)} = \frac{y^b}{p(b+it)} e^{it\log(y)}.$$

for $z = b + it$. Then

$$\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \frac{y^z}{p(z)}\, dz = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t) e^{it\log y}\, dt = \frac{1}{\sqrt{2\pi}} \hat{f}(-\log(y)),$$

where f is the restriction to the real line of the meromorphic function

$$f(z) = \frac{y^b}{p(b+iz)}.$$

The function f has limit 0 at infinity since p is a non-constant polynomial. Thus, by Theorem 5.3.2, if $y > 1$, then

$$\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \frac{y^z}{p(z)}\, dz = i \sum_{w\in B} Res(f(z) e^{iz\log y}, w)$$

where B is the set of zeroes of f in the upper half plane. Since

$$f(z)e^{z \log y} = \frac{y^{b+iz}}{p(b+iz)},$$

a calculation of the effect on a residue of the change of variables $z \to b + iz$ (exercise 9.5.6) shows that

$$\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \frac{y^z}{p(z)}\,dz = \sum_{\lambda \in A} Res(f(z)e^{iz \log y}, \lambda),$$

where $A = \{\lambda = b + iw) : w \in B\}$ − that is, $A = \{\lambda \in \mathbb{C} : \mathrm{Re}\,(\lambda) < b\}$. This completes the proof in the case $y > 1$. The proof in the case $y < 1$ proceeds in the same way. □

Example 9.5.2. Prove that if $b > 0$ and $y > 0$, then

$$\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \frac{y^z}{z(z+1)}\,dz = \begin{bmatrix} 1 - 1/y & \text{if} & y > 1 \\ 0 & \text{if} & y < 1 \end{bmatrix}.$$

Solution: Since $b > 0$, by the previous lemma, if $y > 1$ the integral is the sum of the residues of $\dfrac{y^z}{z(z+1)}$ at 0 and −1, which is $1 - 1/y$. If $y < 1$ the lemma implies that the integral is 0.

Theorem 9.5.3. *If $x > 0$ is not a power of a prime and $b > 1$, then*

$$\phi(x) = -\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \frac{x^{z+1}}{z(z+1)}\frac{\zeta'(z)}{\zeta(z)}\,dz.$$

Proof. We multiply equation (9.3.2) by $\dfrac{x^{z+1}}{z(z+1)}$ and integrate. The result is

$$\begin{aligned} &-\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \frac{x^{z+1}}{z(z+1)}\frac{\zeta'(z)}{\zeta(z)}\,dz \\ &\qquad = \sum_{p^m}\left(\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty}\left(\frac{x}{p^m}\right)^z \frac{x\,dz}{z(z+1)}\right)\log p, \end{aligned} \tag{9.5.2}$$

provided the integrals exist and the integral can be moved inside the summation on the right. Assuming these things for the moment, we have, by the previous example,

$$-\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \frac{x^{z+1}}{z(z+1)}\frac{\zeta'(z)}{\zeta(z)}\,dz = \sum_{p^m \le x}(x - p^m)\log p.$$

The expression on the right is $\phi(x) = \displaystyle\int_1^x \psi(u)\,du$ (Exercise 9.5.1), and so the proof will be complete if we can verify that the integrals in (9.5.2) exist and the integral can be brought inside the summation on the right.

The integrand corresponding to $n = p^m$ on the right in (9.5.2) is less than or equal in modulus to

$$\frac{1}{n^b}\frac{x^{b+1}}{b^2+t^2}$$

on the vertical line $z = b + it$. This has the form $c_n f(t)$, where f is a positive integrable function of t on $(-\infty, \infty)$ and $\sum_1^n c_n$ is a convergent series of positive numbers (a p-series with $p = b > 1$). By Exercise 9.5.4 this implies that the series of integrals on the right in (9.5.2) converges and it converges to the integral on the left, □

A Series Expansion of ϕ

The Prime Number Theorem will follow directly from the following infinite series expansion of ϕ.

Theorem 9.5.4. *There are constants A and B such that*

$$\phi(x) = \frac{x^2}{2} - \sum_{k=1}^{\infty} \frac{x^{1-2k}}{2k(2k-1)} - \sum_{\rho} \frac{x^{\rho+1}}{\rho(\rho+1)} - Ax + B,$$

where ρ ranges over the zeroes of ζ in the strip $0 < Re(z) < 1$.

Proof. The integral that appears in Theorem 9.5.3 can also be evaluated by using the infinite product expansion of Theorem 9.3.9. This theorem implies that the logarithmic derivative of ζ can be written as

$$\frac{\zeta'(z)}{\zeta(z)} = \frac{1}{1-z} - \frac{1}{z} + \frac{\log\pi}{2} - \frac{\Gamma'(z/2)}{\Gamma(z/2)} + \sum_{\rho} \frac{1}{z-\rho},$$

where, in the last sum, terms involving ρ and $1-\rho$ must be grouped together for the series to converge. The product formula for $1/\Gamma$ given in Theorem 9.1.10 leads to

$$-\frac{\Gamma'(z/2)}{\Gamma(z/2)} = \frac{1}{z} + \sum_{k=1}^{\infty}\left(\frac{1}{2k+z} - \frac{1}{2}\log(1+1/k)\right).$$

Thus,

$$\frac{\zeta'(z)}{\zeta(z)} = -\frac{1}{z-1} + \frac{\log\pi}{2} + \sum_{k=1}^{\infty}\left(\frac{1}{z+2k} + \frac{1}{2}\log(1+1/k)\right) + \sum_{\rho}\frac{1}{z-\rho}.$$

This simplifies significantly if we subtract $\zeta'(0)/\zeta(0)$:

$$\frac{\zeta'(z)}{\zeta(z)} - \frac{\zeta'(0)}{\zeta(0)} = -1 - \frac{1}{z-1} + \sum_{k=1}^{\infty}\left(\frac{1}{z+2k} - \frac{1}{2k}\right) + \sum_{\rho}\left(\frac{1}{z-\rho} + \frac{1}{\rho}\right),$$

or

$$\frac{\zeta'(z)}{\zeta(z)} = -\frac{z}{z-1} - \sum_{k=1}^{\infty} \frac{z}{2k(z+2k)} + \sum_{\rho} \frac{z}{\rho(z-\rho)} + \frac{\zeta'(0)}{\zeta(0)}. \tag{9.5.3}$$

We next multiply equation (9.5.3) by $\dfrac{x^{z+1}}{z(z+1)}$ and integrate. Assuming for the moment that the integral can be taken inside each of the infinite sums, the result is

$$\begin{aligned}\phi(x) &= -\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \frac{x^{z+1}}{z(z+1)}\frac{\zeta'(z)}{\zeta(z)}\,dz \\ &= \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \frac{x^{z+1}}{(z-1)(z+1)}\,dz + \sum_{k=1}^{\infty}\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \frac{x^{z+1}}{2k(z+2k)(z+1)}\,dz \\ &\quad - \sum_{\rho}\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \frac{x^{z+1}}{\rho(z-\rho)(z+1)}\,dz - \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \frac{\zeta'(0)}{\zeta(0)}\frac{x^{z+1}}{z(z+1)}\,dz.\end{aligned}$$

Each of these integrals can be evaluated using Lemma 9.5.1. This leads to

$$\phi(x) = \frac{x^2}{2} - \sum_{k=1}^{\infty}\frac{x^{1-2k}-1}{2k(2k-1)} - \sum_{\rho}\frac{x^{\rho+1}-1}{\rho(\rho+1)} - \frac{\zeta'(0)}{\zeta(0)}x,$$

or

$$\phi(x) = \frac{x^2}{2} - \sum_{k=1}^{\infty}\frac{x^{1-2k}}{2k(2k-1)} - \sum_{\rho}\frac{x^{\rho+1}}{\rho(\rho+1)} - Ax + B,$$

where $A = \dfrac{\zeta'(0)}{\zeta(0)}$ and $B = \displaystyle\sum_{k=1}^{\infty}\frac{1}{2k(2k-1)} + \sum_{\rho}\frac{1}{\rho(\rho+1)}$.

It remains to prove that the integral can be taken inside the infinite sums. The kth term of the first sum is

$$\frac{x^{z+1}}{2k(z+2k)(z+1)}. \tag{9.5.4}$$

The numerator of this fraction is bounded on the vertical line $\mathrm{Re}\,(z) = b$. With $z = b + it$, we estimate the middle factor of the denominator as follows:

$$\begin{aligned}|z+2k|^2 &= t^2 + (b+2k)^2 = t^2 + b^2 + 4bk + 4k^2 \\ &\geq t^2 + b^2 + 4k^2 = |z|^2 + (2k)^2 \geq 4|z|k.\end{aligned}$$

Thus,

$$|z+2k| \geq 4|z|^{1/2}k^{1/2}.$$

The right factor of the denominator satisfies $|z+1| \geq |z|$ since $z = b + it$ has positive real part. Hence, the fraction(9.5.4) has modulus less than or equal to a constant times $|z|^{-3/2}k^{-3/2}$. It follows that, in the first infinite sum the integral of each term over $\mathrm{Re}\,(z) = b$ exists and the integral of the sum is the sum of the integrals and the latter sum is absolutely convergent (see Exercise 9.5.4). The same result for the second infinite sum can be proved in a similar fashion, but the proof requires information about the exponent of convergence of the sequence of zeroes $\{\rho\}$. The details are left to the exercises. □

The Prime Number Theorem

Theorem 9.5.5. *If $\pi(x)$ the number of primes less than or equal to x, then*

$$\lim_{x\to\infty} \pi(x)\frac{\log x}{x} = 1.$$

Proof. By Theorems 9.4.4 and 9.4.2, it suffices to prove that $\lim_{x\to\infty} \phi(x)/x^2 = 1/2$.

By Theorem 9.5.4,

$$\frac{\phi(x)}{x^2} = \frac{1}{2} - \sum_{k=1}^{\infty} \frac{x^{-1-2k}}{2k(2k-1)} - \sum_{\rho} \frac{x^{\rho-1}}{\rho(\rho+1)} - \frac{A}{x} + \frac{B}{x^2}. \qquad (9.5.5)$$

Each of the infinite sums in this expression involves only negative powers of x and each of them is abosolutely convergent at $x = 1$. It follows that both infinite series converge uniformly in x on $[1, \infty)$. Thus, in taking the limit of $\phi(x)/x^2$ as $x \to \infty$, we may take the limit inside the infinite sums. Since each term on the right side of (9.5.5) has limit 0 except the term 1/2, the theorem is proved. □

Exercise Set 9.5

1. Verify that $\displaystyle\int_1^x \psi(u)\, du = \sum_{p^m \le x} (x - p^m) \log p$, where p is prime and m is a positive integer.

2. Prove that if p and r are arbitrary positive numbers, there is a constant C such that $\log^p(t) \le Ct^r$ for all $t > 1$.

3. Give a direct proof of Lemma 9.5.1, using residue theory, but without interpreting the integal as a Fourier Transform and applying Theorem 5.3.2.

4. Prove that if $g_n(t)$ is a continuous function on $(-\infty, \infty)$ for each n and, for each n, $|g_n(t)| \le c_n f(t)$, where f is a positive integrable function of t on $(-\infty, \infty)$ and $\displaystyle\sum_1^n c_n$ is a convergent series of positive numbers , then

$$\sum_{n=1}^{\infty} \int_{-\infty}^{\infty} g_n(t)\,, dt = \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} g_n(t)\,, dt,$$

and both sides of this equation exist. Hint: Use Exercise 9.2.7.

5. Verify the statement about taking the limit inside the integral in the proof of Theorem 9.5.5. That is, prove that if a series $\sum_{n=1}^{\infty} u_n(x)$ of functions on $[1, \infty)$ converges uniformly absolutely on $[1, \infty)$, then

$$\lim_{x \to \infty} \sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} \lim_{x \to \infty} u_n(x),$$

provided each limit on the right converges.

6. Prove that if f is a function analytic in an open set containing $b + iw$ and $g(z) = f(b + iz)$, then g is analytic in an open set containing w and

$$\mathrm{Res}(g, w)) = -i\mathrm{Res}(f, b + iw).$$

7. Let $\{\rho_k\}$ be a list of the zeroes of ζ in the strip $0 < \mathrm{Re}\,(z) < 1$, indexed by increasing order of modulus. Prove that the exponent of convergence of this sequence is at most 1 (see Section 8.4).

8. In the proof of Theorem 9.5.4 we assumed that

$$\frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \sum_{\rho} \frac{x^{z+1}}{\rho(z-\rho)(z+1)}\, dz = \sum_{\rho} \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \frac{x^{z+1}}{\rho(z-\rho)(z+1)}\, dz.$$

Prove this using the result of the previous exercise and a method similar to the one used to prove the analogous result for the other infinite sum of integrals that appears in the proof of Theorem 9.5.4.