Chapter 8

Infinite Products

As is amply demonstrated by power series expansions, a highly useful technique in complex analysis is to express an analytic function as an infinite sum of much simpler functions. Likewise, it can be useful to express an analytic function as an infinte product of simpler functions. This is especially true in the study of the zeroes of analytic functions. For example, a polynomial p of degree n can be written as a product

$$p(z) = a \prod_{j=1}^{n} (z - z_j),$$

where $\{z_1, z_2, \cdot, z_n\}$ are the zeroes of p. It turns out that product expansions of a similar type (but with infinitely many factors) are possible for other analytic functions.

Since the exponential function converts sums to products, we can expect that the theory of infinite products will be closely related to the theory of infinite sums.

8.1 Convergence of Infinite Products

In the following discussion, log will be the principal branch of the log function.

Definition 8.1.1. If $\{u_k\}$ is a sequence of complex numbers and

$$p_n = \prod_{k=1}^{n} u_k, \tag{8.1.1}$$

then we will say that the infinite product

$$\prod_{k=1}^{\infty} u_k \tag{8.1.2}$$

converges to the complex number p if $\lim_{n\to\infty} p_n = p$.

Theorem 8.1.2. *If $\{u_k\}$ is a sequence of complex numbers, then the infinite product (8.1.2) converges to a non-zero number p if and only if the infinite sum*

$$\sum_{k=1}^{\infty} \log u_k \tag{8.1.3}$$

converges to a number λ. In this case, $p = e^{\lambda}$. Furthermore, if the infinite series converges absolutely, then the infinite product is unchanged by a rearrangement of the factors.

Proof. We have to be careful here, because the log function converts products to sums only up to $\pm 2\pi i$. However, it is true that $\log uv = \log u + \log v$ if u and v have positive real part since, in this case, $\log u$ and $\log v$ have imaginary parts in $(-\pi/2, \pi/2)$ and $\log uv$ has imaginary part in $(-\pi, \pi)$. Thus, $\log uv$ and $\log u + \log v$ cannot differ by a non-zero multiple of $2\pi i$.

We define the partial products p_n as in (8.1.1). If $p = \lim\limits_{n\to\infty} p_n$ exists and is non-zero, then

$$\lim_{n\to\infty} \log(p_n/p) = 0,$$

since log is continuous at 1. In particular, there is an N such that

$$-\pi/4 < \operatorname{Im}(\log(p_n/p)) < \pi/4 \quad \text{whenever} \quad n \geq N.$$

It follows that $p_n/p_m = (p_n/p)(p_m/p)^{-1}$ is in the right half plane for $n, m \geq N$. In particular, $u_{n+1} = p_{n+1}/p_n$ is in the right half plane for $n \geq N$. Thus,

$$\log(p_{n+1}/p_N) = \log((p_n/p_N)u_{n+1}) = \log(p_n/p_N) + \log u_{n+1}$$

whenever $n \geq N$. This equation and an induction argument beginning with $n = N$ show that

$$\log(p_n/p_N) = \sum_{k=N+1}^{n} \log u_k$$

for all $n > N$. Since the left side of this equality converges as $n \to \infty$ so does the right side. This implies the convergence of the series (8.1.3).

Conversely, if this series converges and we let

$$\lambda_n = \sum_{k=1}^{n} \log u_n,$$

be its nth partial sum, then the sequence $\{\lambda_n\}$ converges to a number λ. Since

$$p_n = \mathrm{e}^{\lambda_n},$$

and the exponential function is continuous, the sequence $\{p_n\}$ converges to e^{λ}.

If the series (8.1.3) converges absolutely, then each of its rearrangements converges to the same number. It follows that each rearrangement of the infinite product (8.1.2) also converges to the same number. □

Uniform Convergence of Products

We will be primarily interested in infinite products of analytic functions. In this situation, whether or not the product converges uniformly is of critical importance. We say that an the infinite product of a sequence of functions $\{u_k\}$ converges uniformly on a set S if the sequence p_n of partial products converges uniformly on S.

Theorem 8.1.3. *Let u_k be a sequence of complex valued functions defined and bounded on a set S. If the series*

$$\sum_{k=1}^{\infty} \log u_k(z)$$

converges uniformly to $\lambda(z)$ on S, then the infinite product

$$\prod_{k=1}^{\infty} u_k(z)$$

converges uniformly to $e^{\lambda(z)}$ on S.

Proof. Let $\lambda_n(z)$ be the nth partial sum of the infinite sum and $p_n(z)$ the nth partial product of the infinite product. Then the uniform convergence of the series on S implies that $\lambda_n(z) - \lambda(z)$ converges uniformly to 0 on S. Since the exponential function is continuous at 0, this implies that

$$\frac{p_n(z)}{p(z)} = e^{\lambda_n(z) - \lambda(z)}$$

converges uniformly to 1.

The fact that each λ_n is bounded on S and the convergence is uniform implies that λ is bounded on S and, hence, that $p(z) = e^{-\lambda(z)}$ is also bounded on S. Hence, $p_n = (p_n/p)p$ converges uniformly to p on S. □

Theorem 8.1.4. *Let $\{a_k(z)\}$ is a sequence of complex valued functions defined on a set S. If the series*

$$\sum_{k=1}^{\infty} |a_k(z)| \tag{8.1.4}$$

converges uniformly on S, then the infinite product

$$\prod_{k=1}^{\infty} (1 + a_k(z)) \tag{8.1.5}$$

converges uniformly on S. Each rearrangement of the infinite product converges to the same function. If the infinite product converges to $p(z)$, then each zero of $p(z)$ is a zero, with the same order, of some finite product of the factors $1 + a_k(z)$.

Proof. If $|w| < 1/2$, then (Exercise 8.1.1)

$$\frac{2}{3}|w| \leq |\log(1+w)| \leq 2|w|. \tag{8.1.6}$$

If the series (8.1.4) converges uniformly on S, then there is a K such that $|a_k(z)| \leq 1/2$ for $k \geq K$ and for all $z \in S$. If we use (8.1.6) with $w = a_k$, it follows that one of the two series

$$\sum_{k=K}^{\infty} |\log(1 + a_k(z))| \quad \text{and} \quad \sum_{k=K}^{\infty} |a_k(z)|$$

converges uniformly on S if and only if the other one does also. Hence, if (8.1.4) converges uniformly then

$$\sum_{k=K}^{\infty} \log(1 + a_k(z))$$

converges uniformly and absolutely. By the previous two theorems, this is implies the uniform convergence of

$$\prod_{k=K}^{\infty} (1 + a_k(z))$$

to a function on S with no zeroes. It follows that (8.1.5) converges uniformly on S, the limit is unaffected by by rearrangements of the factors, and each of its zeroes is a zero, with the same order, of the product of the factors $1 + a_k(z)$ for $k < K$. □

Example 8.1.5. Prove that the infinite product

$$\prod_{k=1}^{\infty}(1 - z^2/k^2) \tag{8.1.7}$$

converges uniformly on each bounded subset of $\mathbb{C}$.

Solution: We have $|-z^2/k^2| \leq R^2/k^2$ for all z in the disc $D_R(0)$. Since the positive termed series

$$\sum_{k=1}^{\infty} \frac{R^2}{k^2}$$

converges, it follows that the series

$$\sum_{k=1}^{\infty} \frac{|z^2|}{k^2}$$

converges uniformly on $D_R(0)$. Hence, by the previous theorem, the infinite product (8.1.7) also converges uniformly on $D_R(0)$ for each R and, hence, on each bounded subset of $\mathbb{C}$.

Logarithmic Derivative of a Product

If an analytic function f on an open set U has an analytic logarithm g on U – that is, if $f = e^g$ on U with g analytic – then $g' = f'/f$. The expression f'/f is independent of which logarithm is chosen for f. Furthermore, as long as f is not identically zero on any component of U, f'/f exists (as a meromorphic function on U) even if f does not have an analytic logarithm on U. Note that f cannot have an analytic or even a meromorphic logarithm in any neighborhood of a point where it has the value zero (Exercise 8.1.4).

Definition 8.1.6. Let f be an analytic function on an open set U and suppose that f is not identically 0 on any component of U. Then the meromorphic function f'/f is called the *logarithmic derivative* of f on U.

Logarithmic derivative is quite a well behaved notion. The logarithmic derivative of the product of two functions is the sum of there logrithmic derivatives (Exercise 8.1.5). Furthermore, the following theorem states that logarithmic derivative is preserved by uniform limits. The proof is left to the exercises (Exercise 8.1.6).

Theorem 8.1.7. *Let $\{f_n\}$ be a sequence of analytic functions on a connected open set U. If this sequence converges uniformly to f on U then the sequence $\{f_n'/f_n\}$ converges uniformly to f'/f on compact subsets of $U \setminus S$, where S is the set of zeroes of f.*

When applied to infinite products, this immediately implies the following corollary.

Corollary 8.1.8. *Let $\{u_k\}$ be a sequence of analytic functions on a connected open set U. If the product*

$$f(z) = \prod_{k=1}^{\infty} u_k(z)$$

converges uniformly on compact subsets of U to a function f which is not identically 0, then the infinite sum

$$\sum_{k=1}^{\infty} \frac{u_k'(z)}{u_k(z)}$$

converges uniformly to f'/f on compact subsets of $U \setminus S$, where S is the set of zeroes of f.

Example 8.1.9. Show that the function

$$f(z) = \pi z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right)$$

has a logarithmic derivative which can be written as

$$\frac{f'(z)}{f(z)} = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{2z}{z^2 - k^2} \tag{8.1.8}$$

or as

$$\frac{f'(z)}{f(z)} = \lim_{n\to\infty} \sum_{k=-n}^{n} \frac{1}{z-k} \tag{8.1.9}$$

Solution Note that the infinite product in the expression for f converges uniformly on each compact disc in the plane by Example 8.1.5.

By the previous theorem,

$$\frac{f'(z)}{f(z)} = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{-2z/k^2}{1-z^2/k^2} = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{2z}{z^2-k^2}.$$

This proves (8.1.8). Since

$$\frac{2z}{z^2-k^2} = \frac{1}{z-k} + \frac{1}{z+k},$$

the nth partial sum of the series (8.1.8) can be re-written as the sum that appears in (8.1.9).

The logarithmic derivative of f, as computed in the above example, will be used in the problem set to prove that $f(z) = \sin \pi z$. That is,

$$\sin(\pi z) = \pi z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right). \tag{8.1.10}$$

Exercise Set 8.1

1. Prove that if w is a complex number with $|w| \le 1/2$, then

$$\frac{2}{3}|w| \le |\log(1+w)| \le 2|w|.$$

2. Does the infinite product

$$\prod_{k=1}^{\infty} \left(1 + \frac{1}{k}\right)$$

converge? How about the product

$$\prod_{k=1}^{\infty} \left(1 + \frac{1}{k^{3/2}}\right)?$$

3. Show that the infinite product

$$\prod_{k=1}^{\infty} \left(1 - \frac{z}{k}\right) e^{z/k}$$

converges uniformly on compact subsets of the plane.

4. Prove that if f is analytic on U and has a zero at $z_0 \in U$, then there is no meromorphic function g defined in a neighborhood V of z_0 such that $f = \mathrm{e}^g$ on V.

5. Prove that the logarithmic derivative of the product fg of two analytic functions is the sum of the logarithmic derivative of f and the logarithmic derivative of g. Also prove the analogous statement for the quotient f/g.

6. Prove Theorem 8.1.7. Hint: first prove that it is true on any disc in U on which f has no zeroes.

7. Prove that the logarithmic derivative of a meromorphic function f on $\mathbb{C}$ is also a meromorphic function on $\mathbb{C}$ and is odd (even) if f is odd (even).

8. If f is the function defined in Example 8.1.9, prove that the logarithmic derivative of f is an odd meromorphic function which is periodic of period 1. Observe that the logarithmic derivative of $\sin \pi z$ has the same properties.

9. Prove that if f is the function of the previous exercise, and we set
$$g(z) = \frac{\sin(\pi z)}{f(z)},$$
then g is an entire function with no zeroes and, hence, has a logarithm h which is entire. Then, $\sin(\pi z) = f(z)\mathrm{e}^{h(z)}$.

10. Prove that if f, g and h are the functions of the previous exercise, then the logarithmic derivative of g is $h'(z) = \pi \cot \pi z - f'(z)/f(z)$.

11. With h as above, prove that h' is bounded on the strip $0 \le \mathrm{Re}\,(z) \le 1$ (use (8.1.8)). Show that this implies it is bounded on the entire plane and, hence, is constant.

12. With h as above, prove that $h'(0) = 0$ and, hence, that h' is identically 0 and h is a constant. Then use the fact that $\lim_{z\to 0} z^{-1} \sin z = 1$ to show that this constant is 0. Conclude that
$$\sin(\pi z) = \pi z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right).$$

8.2 Weierstrass Products

In this section we will show that, given any sequence of points of an open set $U \subset \mathbb{C}$, with no limit point in U, there is an analytic function on U with exactly the points of this sequence as its zeroes, with each zero having order equal to the number of times it appears in the sequence. The analytic function will be constructed as an infinite product of certain simple functions, each of which has exactly one zero. These simple functions are constructed as follows.

For $p = 0, 1, 2, \cdots$ we define entire functions $E_p(z)$ by $E_0(z) = 1 - z$ and

$$E_p(z) = (1-z)e^{z+z^2/2+\cdots+z^p/p} \quad \text{for} \quad p > 0.$$

Note that $z + z^2/2 + \cdots + z^p/p$ is the pth partial sum for the power seriies expansion of $-\log(1-z)$ about $z = 0$ and so, although $E_p(1) = 0$, the sequence $E_p(z)$ will converge uniformly to $(1-z)(1-z)^{-1} = 1$ on each disc of radius less than 1 centered at 0. More precisely:

Theorem 8.2.1. *Each $E_p(z)$ is an entire function with the following properties:*

(a) the only zero of $E_p(z)$ occurs at $z = 1$;

(b) if $|z| \leq 1$, then $|E_p(z) - 1| < |z|^{p+1}$.

Proof. Part (a) is obvious. To prove Part (b), we note that the derivative of $1 - E_p(z)$ is (Exercise 8.2.1)

$$(1 - E_p(z))' = -E_p'(z) = z^p e^{z+z^2/2+\cdots+z^p/p}. \tag{8.2.1}$$

Since this has a zero of order p at $z = 0$, the function $1 - E_p(z)$ has a zero of order $p + 1$ at $z = 0$.

The function (8.2.1) has a power series expansion about 0 with all of its coefficients non-negative real numbers, since this is true of the exponential function and the function $z + z^2/2 + \cdots + z^p/p$. It follows that the function

$$h(z) = \frac{1 - E_p(z)}{z^{p+1}},$$

also has non-negative real numbers as coefficients for its power series expansion about 0. This implies that the maximum value achieved by $|h(z)|$ for $|z| \leq 1$ is $h(1) = 1$. That is,

$$\left| \frac{1 - E_p(z)}{z^{p+1}} \right| \leq 1 \quad \text{for} \quad |z| \leq 1.$$

Part (b) follows from this. □

If f is an analytic function on U, then we will say that a sequence $\{z_k\} \subset U$ is a list of the zeroes of f *counting multiplicity* if each z_k is a zero of f, and if each zero w of f occurs $m(w)$ times in this sequence, where $m(w)$ is the order of the zero w.

Let $\{z_k\}$ be a sequence of non-zero complex numbers converging to ∞. The next theorem show how to use scaled versions of the functions E_p to construct an entire function with this sequence as a list of its zeroes counting multiplicity. The resulting product is called a *Weierstrass product*.

Theorem 8.2.2. *Let A be a subset of $\mathbb{C}$. If $\{z_k\}$ is a sequence of non-zero complex numbers and $\{p_k\}$ is a sequence of integers such that*

$$\sum_{k=1}^{\infty} \left| \frac{r}{z_k} \right|^{p_k+1} < \infty \quad \textit{for all} \quad r > 0, \tag{8.2.2}$$

then the Weierstrass product

$$f(z) = \prod_{k=1}^{\infty} E_{p_k}(z/z_k), \tag{8.2.3}$$

converges uniformly on compact subsets of $\mathbb{C}$ *to an entire function which has* $\{z_k\}$ *as a list of its zeroes counting multiplicity.*

Proof. By part (b) of the previous theorem, we have

$$|E_{p_k}(z/z_k) - 1| \le \left|\frac{z}{z_k}\right|^{p_k+1} \quad \text{if} \quad |z| \le |z_k|.$$

The condition $|z| \le |z_k|$ must be satisfied for all sufficiently large k if the series (8.2.2) converges. The theorem follows by applying Theorem 8.1.4 with $a_k(z) = E_{p_k}(z/z_k) - 1$. □

The Weierstrass Theorem

Theorem 8.2.3. *If* $\{z_k\}$ *is any sequence of complex numbers converging to infinity, then there is an entire function with* $\{z_k\}$ *as a list of its zeroes counting multiplicity.*

Proof. Suppose m of the z_k are equal to 0 (m might be 0). We may as well assume these are the first m terms of the sequence. Then $\{z_k\}_{k=m+1}^{\infty}$ is a sequence of non-zero complex numbers.

If $R > 0$, then, since $z_k \to \infty$, there is a $K > m$ such that $|z_k| > 2R$ for all $k \ge K$. Then the series

$$\sum_{k=m+1} \left|\frac{z}{z_k}\right|^k$$

converges uniformly on $|z| \le R$ by comparison with the geometric series with ratio 1/2. Thus, the hypotheses of the previous theorem are satisfied if we choose $p_k = k - 1$ for each k. The resulting Weierstrass product

$$\prod_{k=m+1}^{\infty} E_p(z/z_k)$$

converges uniformly on compact subsets of $\mathbb{C}$ to an entire function which has $\{z_k\}_{k=m+1}^{\infty}$ as a list of its zeroes counting multiplicity. Then

$$f(z) = z^m \prod_{k=m+1}^{\infty} E_p(z/z_k)$$

is an entire function with $\{z_k\}_{k=1}^{\infty}$ as a list of its zeroes counting multiplicity. □

Example 8.2.4. Find an entire function which has a zero of order k at each positive integer k.

Solution: We construct a sequence

$$1, 2, 2, , 3, 3, 3, 4, 4, 4, 4, \cdots$$

in which each n appears k times and the terms are arranged in increasing order. Then, for this sequence $\{z_k\}$ and a given positive integer p, we have

$$\sum_{k=1}^{\infty} \frac{1}{|z_k|^{p+1}} = \sum_{k=1}^{\infty} k \frac{1}{k^{p+1}} = \sum_{k=1}^{\infty} \frac{1}{k^p}$$

If we choose $p = 2$, then the right side is the convergent series

$$\sum_{k=1}^{\infty} \frac{1}{k^2}.$$

The Weierstrass product for the sequences $\{z_k\}$ and $\{p_k = 2\}$ is

$$\prod_{k=1}^{\infty} \left((1 - z/k) \mathrm{e}^{z/k + z^2/(2k^2)} \right)^k.$$

By Theorem 8.2.2 this infinite product converges to an entire function with the required zeroes.

Weierstrass Factorization

The Weierstrass Theorem for the plane leads immediately to the Weierstrass Factorization Theorem for entire functions:

Theorem 8.2.5. *Let f be an entire function which is not identically zero. Let m be the order of the zero of f at 0, and let $\{z_k\}$ be a list of the non-zero zeroes of f counting multiplicity. Then there exists non-negative integers $p_1, p_2, \cdots$ and an entire function h such that*

$$f(z) = e^{h(z)} z^m \prod_{k=1}^{\infty} E_{p_k}(z/z_k).$$

The sequence $\{p_k\}$ may be chosen in any way which satisfies (8.2.2).

Proof. The product

$$g(z) = z^m \prod_{k=1}^{\infty} E_{p_k}(z/z_k)$$

converges uniformly on compact sets if $\{p_k\}$ is chosen such that (8.2.2) holds ($p_k = k - 1$ is one choice which always works, but there may be better choices for a given f). Furthermore, the resulting function g has the same zeroes as f

with the same multiplicities. Thus, fg^{-1} is an entire function with no zeroes (after removable singularities are removed). It follows that

$$fg^{-1} = \mathrm{e}^h$$

for some entire function h. The theorem follows from this. □

In many cases, the sequence $\{p_k\}$ can be chosen to be constant.

Example 8.2.6. Find a Weierstrass factorization for $\sin(\pi z)$.

Solution: This function has a zero of order 1 at each integer and has no other zeroes. Since

$$\sum_{k=1}^{\infty} \frac{1}{k^2} < \infty,$$

the condition (8.2.2) holds if we choose $p_k = 1$ for every k. Then the above theorem tells us that

$$\sin(\pi z) = \mathrm{e}^{h(z)} z \prod_{k \neq 0} E_1(z/k) = \mathrm{e}^{h(z)} z \prod_{k \neq 0} (1 - z/k)\mathrm{e}^{z/k},$$

where the product is over all non-zero integers k. Note that if the factors for k and $-k$ in this product are paired, the result is

$$(1 - z/k)\mathrm{e}^{z/k}(1 + z/k)\mathrm{e}^{-z/k} = 1 - z^2/k^2.$$

We conclude from Exercise 8.1.12 that $\mathrm{e}^{h(z)} = \pi$, that

$$\sin(\pi z) = \pi z \prod_{k \neq 0} (1 - z/k)\mathrm{e}^{z/k}$$

is a Weierstrass factorization of $\sin(\pi z)$, and that this factorization is equivalent to the factorization

$$\sin(\pi z) = \pi z \prod_{k=1}^{\infty} (1 - z^2/k^2).$$

The General Weierstrass Theorem

If $\mathbb{C}$ is replaced by an arbitrary non-empty, proper open subset of S^2, the analogue of Theorem 8.2.3 holds with only a slightly more complicated proof.

Theorem 8.2.7. *Let U be a non-empty, proper open subset of S^2. If $\{z_k\}$ is any sequence of points of U with no limit points in U, then there is an analytic function f on U with $\{z_k\}$ as a list of its zeroes counting mulltiplicity.*

Proof. Either U or its image under some linear fractional transformation will contain ∞. Thus, we may as well assume $\infty \in U$. Then the complement of U in S^2 is a compact subset K of the plane.

Since $\{z_k\}$ has no limit point in U, the distance between z_k and K must approach 0 as $k \to \infty$. It follows that we may choose a sequence $\{w_k\}$ of points of K such that $\lim |z_k - w_k| = 0$.

We set

$$f(z) = \prod_{n=1}^{\infty} E_k\left(\frac{z_k - w_k}{z - w_k}\right).$$

The product converges uniformly on compact subsets of U, since $\lim |z_k - w_k| = 0$ implies the uniform convergence on compact subsets of U of

$$\sum_{k=1}^{\infty} \left|\frac{z_k - w_k}{z - w_k}\right|^{k+1}.$$

The function f is analytic in U as has $\{z_k\}$ as a list of its zeroes counting multiplicity. □

Meromorphic Functions

On a connected open set U, the set of analytic functions forms an integral domain – that is, it is a commutative ring with the property that the product of two elements is zero if and only if one of them is zero. The set of meromorphic functions of U forms a field – that is, a commutative ring in which every non-zero element has an inverse. The next theorem shows that the field of meromorphic function is actually the quotient field of the ring of analytic functions. That is, every meromorphic function is the quotient f/g of two analytic functions.

Theorem 8.2.8. *If U is a connected open subset of $\mathbb{C}$, then each meromorphic function on U has the form f/g, where f and g are analytic on U and g is not identically zero.*

Proof. Let h be a meromorphic function on U and let $\{z_k\}$ be a sequence consisting of the poles of h, with each z_k listed as many times as the order of the pole at z_k. By Theorem 8.2.7 there is an analytic function g on U with $\{z_k\}$ as a list of its zeroes counting multiplicity. Then, after removing removable singularities, $f = gh$ is an analytic function on U. Thus, $h = f/g$ with f and g analytic on U. □

The Mittag-Leffler Theorem

The Weierstrass Theorem (Theorem 8.2.7) gives the existence of an analytic function with a specified list of zeroes counting multiplicity. The Mittag-Leffler Theorem is a companion theorem. It gives the existence of a meromorphic function with a specified list of poles and principal parts. We prove it only for discs, although it is true for general open sets.

Theorem 8.2.9. *Let R be a positive number or ∞. Let S be a discrete set of points of $D_R(0)$ and $\{h_w : w \in S\}$ a set of polynomials with no constant terms. Then there exists a meromorphic function f with a pole at w with principal part $h_k((z-w)^{-1})$ for each $w \in S$ and with no other poles.*

Proof. We choose an increasing sequence of radii $\{r_n\}$ with $r_n \to R$ and we let S_1 be the subset of S which lies in $\overline{D}_{r_1}(0)$ and, for $n > 1$, and let

$$S_n = \{w \in S : r_{n-1} < |w| \le r_n.$$

Then, for each n,

$$g_n(z) = \sum_{w \in S_n} h_k((z-w)^{-1})$$

is a meromorphic function on the plane with a pole at w with the required principal part for each $w \in S_n$ and with no other poles.

We might hope to construct the function we are after by simply taking the infinite sum of the functions g_n. Unfortunately, there is no reason to think this sequence should converge on $D_R(0)$. However, we can modify each g_n, without changing its poles and pricipal parts, in such a way as to end up with an infinite series which does converge.

For each $n > 1$,the function g_n is analytic on an open set containing the closed disc $\overline{D}_{r_{n-1}}(0)$. Hence, it is the uniform limit on this closed disc of its power series at 0. It follows that there is a polynomial p_n such that

$$|g_n(z) - p_n(z)| < 2^{-n} \quad \text{for} \quad |z| \le r_{n-1}.$$

If we set $f_1 = g_1$ and $f_n = g_n - p_n$ for $n > 1$, then, for each $m > 1$, the series

$$\sum_{n=m+1}^{\infty} f_n(z)$$

converges uniformly to an analytic function on $D_{r_m}(0)$. This means that

$$f(z) = \sum_{n=1}^{\infty} f_n(z)$$

is defined as a meromorphic function on $D_{r_m}(0)$ and has the required poles and principal parts at those points of S which lie in this disc. Since this is true for each m, and $\lim r_m = R$, f is meromorphic on all of $D_R(0)$ and has the required poles and principal parts. □

Exercise Set 8.2

1. Show that the derivative of $E_p(z)$ is $z^p e^{z+z^2/2+\cdots+z^p/p}$.

2. Compute the logarithmic derivative of $E_p(z)$.

3. Find an entire function (given by a Weierstrass product) that has a zero of order 1 at $\sqrt{n}$ for $n = 1, 2, 3, \cdots$ and no other zeroes.

4. Find an entire function (given by a Weierstrass product) that has a zero of order 2 at $\sqrt{n}$ for $n = 1, 2, 3, \cdots$ and has no other zeroes.

5. Find an entire function (given by a Weierstrass product) that has a zero of order n at n^2 for $n = 1, 2, 3, \cdots$ and has no other zeroes.

6. If f is an entire function, show that $f = g^n$ for some entire function g if and only if the order of each zero of f is divisible by n.

7. Suppose f is an entire function such that $\{z_k\}$ is a list of its non-zero zeroes counting multiplicity and suppose that
$$\sum_{k=1}^{\infty} \frac{1}{|z_k|} < \infty.$$
Describe the simplest Weierstrass factorization of f.

8. Suppose f is an odd entire function, the order of the zero at 0 is m, and $\{z_k\}$ is a list of the other zeroes of f counting multiplicity. Show that m is positive and odd. If
$$\sum_{k=1}^{\infty} \frac{1}{|z_k|^2} < \infty,$$
prove that f has a factorization of the form
$$f(z) = z^m \mathrm{e}^{h(z)} \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{z_k^2}\right)$$
where h is an entire function.

9. Show that, if U is any non-empty open subset of the plane, then there is an analytic function on U which cannot be extended to be analytic on any larger open set. Hint: Use the general Weierstrass Theorem to construct an analytic function on U with a lot of zeroes.

10. Prove that, given a sequence $\{z_k\}$ of complex numbers converging to infinity and a sequence $\{n_k\}$ of integers, there is an entire function f with given values for f and its derivatives up to order n_k at z_k for each k. Hint: Use the Mittag-Leffler and Weierstrass Theorems together.

11. Prove that if f_1 and f_2 are two entire functions with no common zeroes, then there exist entire functions g_1 and g_2 such that
$$g_1 f_1 + g_2 f_2 = 1.$$
Hint: Use the Mittag-Leffler Theorem to show that an entire function g_2 can be chosen so that, at each zero of f_1, the function $1 - g_2 f_2$ has a zero of order at least as large.

12. Let $f_1, f_2, \cdots f_n$ be entire functions. Show that there are entire functions $h_1, h_2, \cdot, h_n$, and u such that $f_j = u h_j$ for $j = 1, \cdots, n$ and the functions $h_1, h_2, \cdots, h_n$ have no common zeroes. Hint: Use the Weierstrass Theorem.

13. Let $f_1, f_2, \cdots f_n$ be entire functions with no common zero. Use induction and the preceding two exercises to show that there are entire functions $g_1, g_2, \cdots, g_n$ such that

$$g_1 f_1 + g_2 f_2 + \cdots + g_n f_n = 1.$$

14. Those who are familiar with commutative ring theory may want to do this exercise. Let $\mathcal{E}$ be the ring of entire functions. Show that the following ring theoretic properties of $\mathcal{E}$ are consequences of the preceding two exercises:

    (a) every finitely generated ideal of $\mathcal{E}$ is a principle ideal;

    (b) every finitely generated maximal ideal of $\mathcal{E}$ is of the form

$$M_w = \{f \in \mathcal{E} : f(w) = 0\} \quad \text{for some} \quad w \in \mathbb{C}.$$

15. The conclusions of the last four exercises actually hold for the ring of analytic functions on any open subset of the plane. However, to prove them all in this generality would require a stronger form of the Mittag-Leffler Theorem than the one proved here. Prove these results for the largest class of open sets that you can using the machinery developed in this text.

8.3 Entire Functions of Finite Order

Definition 8.3.1. An entire function f is said to be of *finite order* if there is a number t such that

$$|f(z)| \le e^{|z|^t}$$

for all z with $|z|$ sufficiently large. The infimum of all such numbers t is called the *order* of f.

For each non-negative integer p, the function e^{z^p} is an entire function of finite order p. More generally:

Example 8.3.2. Show that $e^{h(z)}$ is an entire function of finite order p if h is a polynomial of degree p.

Solution: If $t > p$, then $\lim_{z\to\infty} |z|^{-t}|h(z)| = 0$. This implies that there is an $R > 0$ such that

$$|h(z)| < |z|^t \quad \text{for} \quad |z| > R.$$

Then

$$|e^{h(z)}| \le e^{|z|^t} \quad \text{for} \quad |z| > R. \tag{8.3.1}$$

Since such a statement is true for all $t > p$, by definition $e^{h(z)}$ has finite order at most p.

On the other hand, if $t < p$, then $\lim_{z\to\infty} |z|^{-t}|h(z)| = +\infty$. Hence, there is no R for which (8.3.1) holds. We conclude that the order of f is at least p and, hence, is equal to p.

Non-vanishing Entire Functions of Finite Order

It turns out that the functions $e^{h(z)}$ of the preceding example are the only entire functions of finite order which are non-vanishing.

To prove this, we will need the following theorem of Borel-Carathéodory relating the growth of the real part of an analytic function to the growth of the the absolute value of the function.

Theorem 8.3.3. *Suppose $0 < r < R$ and let g be a function analytic on an open set containing $D_R(0)$. Then*

$$|g(z)| \le \frac{2r}{R-r}\sup\{Re\,(g(w)) : |w| = R\} + \frac{R+r}{R-r}|g(0)| \quad \textit{if} \quad |z| \le r.$$

Proof. We suppose first that $g(0) = 0$. We set

$$m = \sup\{\mathrm{Re}\,(g(w)) : |w| = R\}.$$

Note that the Mean Value Theorem for harmonic functions implies that $m \ge 0$.

If $|w| = R$ and $u = \mathrm{Re}\,(g(w))$, then $u \le m$ and

$$u - 2m \le u \le 2m - u.$$

Thus, $|u| \le |2m - u|$, from which it follows that

$$|g(w)| \le |2m - g(w)|,$$

since the numbers $g(w)$ and $2m - g(w)$ have the same imaginary parts and have real parts u and $2m - u$, respectively.

We conclude from the above, that the function

$$h(z) = \frac{g(z)}{z(2m - g(z))}$$

satisfies the inequality

$$|h(w)| \le \frac{1}{R} \quad \text{for} \quad |w| = R.$$

Since the analytic function h has a removable singularity at 0, this inequality holds throughout the disc $D_R(0)$ by the Maximum Modulus Theorem. Thus,

$$\frac{|g(z)|}{r|2m - g(z)|} \le \frac{1}{R} \quad \text{whenever} \quad |z| = r,$$

which implies

$$|g(z)| \le \frac{r}{R}(2m + |g(z)|).$$

If we collect terms involving $|g(z)|$ on the left and divide by $1 - r/R$, the result is

$$|g(z)| \le \frac{2r}{R-r}m \quad \text{for} \quad |z| \le r.$$

This concludes the proof in the case where $g(0) = 0$. This general case follows from applying this result to the function $g_0(z) = g(z) - g(0)$. The details are left to the exercises. □

Theorem 8.3.4. *An entire function f with no zeroes has finite order p if and only if p is a non-negative integer and f has the form*

$$f(z) = e^{h(z)},$$

where h is a polynomial of degree p.

Proof. In view of Example 8.3.2, we need only show that every non-vanishing entire function f of finite order p has the above form.

Since f has no zeroes and the plane is simply connected, there is an entire function h such that

$$f(z) = \mathrm{e}^{h(z)} \quad \text{for all} \quad z \in \mathbb{C}.$$

Since f has finite order p, for each $t > p$ there is an $M > 0$ such that

$$\mathrm{e}^{\mathrm{Re}(h(z))} = |f(z)| \leq \mathrm{e}^{|z|^t} \quad \text{for} \quad |z| \geq M.$$

This implies

$$\mathrm{Re}\,(h(z)) \leq |z|^t \quad \text{for} \quad |z| \geq M.$$

We apply the previous theorem with $r > M$ and $R = 2r$ to conclude

$$|h(z)| \leq 2|z|^t + 3|h(0)| \quad \text{if} \quad |z| = r.$$

Since this is true for all $r > M$, Exercise 3.3.9 implies that h must be a polynomial of degree at most t. Since t was an arbitrary number greater than p, we conclude that h is a polynomial of degree at most p. If it were a polynomial of degree less than p, then f would have order less than p. Hence, the degree of the polynomial h is exactly p. This, of course, implies that p is a non-negative integer. □

Canonical Products

Given a sequence $\{z_k\}$, we let μ be the inf of the numbers t such that

$$\sum_{k=1}^{\infty} \frac{1}{|z_k|^t} < \infty. \tag{8.3.2}$$

If there is no such t, then we set $\mu = \infty$. The number μ is called the *exponent of convergence* for the sequence $\{z_k\}$.

If $\{z_k\}$ has finite exponent of convergence μ, then we can write down a convergent Weierstrass product (8.2.3), using $\{z_k\}$, in which the sequence $\{p_k\}$ is a constant p. We choose p to be the smallest integer such that $\mu < p + 1$. Then the condition

$$\sum_{k=1}^{\infty} \frac{1}{|z_k|^{p+1}} < \infty, \tag{8.3.3}$$

is satisfied. Hence, by Theorem 8.2.2, the Weierstrass product

$$f(z) = \prod_{k=1}^{\infty} E_p(z/z_k) \tag{8.3.4}$$

converges. This is called the *canonical product* for the sequence $\{z_k\}$.

The significance of the choice of p made for the canonical product is that, with this choice, the resulting product is an entire function with order λ equal to the exponent of convergence μ of the sequence $\{z_k\}$. The next theorem yields part of what is needed to prove this. The remainder of the proof will come in the next section.

Theorem 8.3.5. *The canonical product for a sequence $\{z_k\}$, with finite exponent of convergence μ, is an entire function of finite order $\lambda \le \mu$.*

Proof. We choose p to be the smallest integer such that $\mu < p + 1$, and let t be any number in the range $\mu < t < p + 1$.

We claim that there is a positive constant A such that

$$|E_p(z)| \le \mathrm{e}^{A|z|^t} \tag{8.3.5}$$

for all z.

If $|z| \le 1/2$, this follows from (8.1.6) with $w = E_p(z) - 1$ and Theorem 8.2.1. These combine to show that

$$|\log E_p(z)| \le 2|z|^{p+1} \le 2|z|^t,$$

and this implies (8.3.5) holds with $A = 2$.

If $|z| > 1/2$, then $|z|^k \le 2^{t-k}|z|^t$, and so

$$\begin{aligned}\log|E_p(z)| &= \log|1 - z| + \sum_{k=1}^{p} \frac{\mathrm{Re}\left(z^k\right)}{k} \\ &\le |z| + \sum_{k=1}^{p} |z|^k \le (p+1)2^t|z|^t.\end{aligned}$$

Thus, (8.3.5) holds with $A = (p + 1)2^t$ in this case.

To prove the theorem, we note that, if f is given by the canonical product (8.3.4), then by (8.3.5),

$$|f(z)| \le \prod_{k=1}^{\infty} \mathrm{e}^{A|z/z_k|^t} = \mathrm{e}^{B|z|^t},$$

where

$$B = A\sum_{k=1}^{\infty} 1/|z_k|^t.$$

The series in this expression converges because t is larger than the exponent of convergence μ.

Since for any $s > t$, we have $B|z|^t \le |z|^s$ for $|z|$ sufficiently large, it follows that f has finite order at most t. Since t was an arbitrary number strictly between μ and $p + 1$, we conclude that f has order at most μ. □

One might guess, based on Theorem 8.3.4 that the order of an entire function of finite order must be a non-negative integer. This is not the case, as is shown by the following example.

Example 8.3.6. Find an entire function with finite order 1/2.

Solution: The function

$$\frac{\sin \pi z}{\pi z} = \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right)$$

has order 1 (Exercise 8.3.3). It seems reasonable that if we replace z^2 by z in this product that the result would be an entire function of order 1/2. In fact, the resulting function has a zero of order 1 at k^2 for each positive integer k and

$$\sum_{k=1}^{\infty} \frac{1}{(k^2)^t} < \infty$$

for every $t > 1/2$ and for no smaller values of t. Hence, the sequence $\{1/k^2\}$ has exponent of convergence 1/2. Since 0 is the smallest integer p such that $1/2 < p + 1$, the preceding theorem implies that the canonical product

$$f(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z}{k^2}\right)$$

is an entire function of finite order at most 1/2.

In fact, it is easy to directly compute the order of f if we note that

$$f(z) = \frac{\sin \pi \sqrt{z}}{\pi \sqrt{z}}.$$

This expression on the right is entire and is independent of the choice of the square root function because the function $(\pi z)^{-1} \sin \pi z$ is an even function. It is easy to see from this that, since $(\pi z)^{-1} \sin \pi z$ has order 1, f has order 1/2 (Exercise 8.3.5).

Exercise Set 8.3

1. Finish the proof of Theorem 8.3.3 by showing that, if it is true in the case where $g(0) = 0$, then it is true in general.

2. Show that a polynomial has finite order 0.

3. Show that $\sin z$, $z^{-1} \sin z$, and $\cos z$ all have finite order 1.

4. If f is an entire function of order $\lambda(f)$, k is a non-negative integer, and $g(z) = f(z^k)$, then prove that $\lambda(g) = k\lambda(f)$, where $\lambda(g)$ is the order of g.

5. Prove that if $g(z)$ is an even entire function of finite order λ and $f(z) = g(\sqrt{z})$, then f is an entire function of finite order $\lambda/2$. In particular, show that $\cos \sqrt{z}$ has order 1/2.

6. Prove that the order of the sum or product of two entire functions is less than or equal to the maximum of the orders of the two functions.

7. What is the order of the entire function $e^{\sin z}$?

8. Suppose f is an entire function which satisfies the inequality $|f(z)| \le |z|^{|z|}$ for $|z|$ sufficiently large. Prove that f has finite order at most 1.

9. Find the exponent of convergence of the following sequences: $\{2^k\}$, $\{k^r\}$ $(r > 0)$, $\{\log k\}$.

10. Given an arbitrary non-negative real number μ, show that there is a sequence of complex numbers $\{z_k\}$ with exponent of convergence μ.

11. Does the order of an entire function necessarily have to be the same as the exponent of convergence of its sequence of zeroes? Justify your answer.

8.4 Hadamard's Factorization Theorem

Our goal in this section is to complete the characterization of entire functions of finite order λ. We will prove a theorem of Hadamard which asserts that every such function factors as a power of z times a canonical product of order at most λ times the exponential of a polynomial of degree at most λ. The key ingredient in the proof is Jensen's Formula relating the density of the zeroes of an entire function to the rate of growth at infinity of the function.

Jensen's Formula

Theorem 8.4.1. *If f is analytic in an open set containing the disc $\overline{D}_r(0)$, f has no zeroes on the boundary of this disc, $f(0) \neq 0$, and $z_1, z_2, \cdots z_n$ are the zeroes, counting multiplicity, of f in $D_r(0)$, then*

$$\log\left(\frac{|f(0)|r^n}{|z_1|\cdot|z_2|\cdot\cdots|z_n|}\right) = \frac{1}{2\pi}\int_0^{2\pi} \log(|f(re^i\theta)|)\, d\theta.$$

Proof. We first prove this in the case where $r = 1$. We divide f by a product of linear fractional transformations which preserve the unit circle and have zeroes at the points z_i. This yields a function

$$g(z) = f(z)\frac{1-\overline{z}_1 z}{z-z_1}\frac{1-\overline{z}_2 z}{z-z_2}\cdots\frac{1-\overline{z}_n z}{z-z_n}.$$

This function is analytic and non-vanishing in an open set containing the closed unit disc $\overline{D}$, and has the same modulus on the unit circle as does f. Thus, g has an analytic logarithm in an open set containing $\overline{D}$. Then $\log|g(x)|$ is the real part of an analytic function in this set and, hence, is harmonic. The Mean Value Theorem for harmonic functions implies that

$$\log\left(\frac{|f(0)|}{|z_1|\cdot|z_2|\cdot\cdots|z_n|}\right) = \log|g(0)| = \frac{1}{2\pi}\int_0^{2\pi}\log|f(e^{i\theta})|\, d\theta. \qquad (8.4.1)$$

To prove the theorem for general r, it suffices to apply (8.4.1) with f replaced by the function $f(rz)$. If f has zeroes at $z_1, z_2, \cdots, z_n$ in the disc $D_r(0)$, then $f(rz)$ has zeroes $z_1/r, z_2/r, \cdots, z_n/r$ in the unit disc D. Thus, the equation of the theorem follows directly from (8.4.1) applied to $f(rz)$. □

This leads to the following estimate on the number of zeroes of an entire function inside a disc $D_r(0)$.

Theorem 8.4.2. *If f is an entire function with $|f(0)| = 1$, $n(r)$ is the number of zeroes of f inside a disc $D_r(0)$, and $M(2r)$ is the supremum of $|f(z)|$ on the boundary of $D_{2r}(0)$, then*

$$n(r) \leq \frac{\log M(2r)}{\log 2}.$$

Proof. Let $n = n(r)$ and $m = n(2r)$, and let $z_1, z_2, \cdots z_m$ be the zeroes of f inside the disc $D_{2r}(0)$ ordered so that $|z_j| \leq |z_k|$ for $j \leq k$. Then Jensen's Theorem with r replaced by $2r$ implies that

$$\log \left| f(0) \frac{2r}{z_1} \frac{2r}{z_2} \cdots \frac{2r}{z_n} \cdots \frac{2r}{z_m} \right| \leq \log M(2r).$$

Since

$$2 < \frac{2r}{|z_j|} \quad \text{if} \quad j \leq n \quad \text{and} \quad 1 < \frac{2r}{|z_j|} \quad \text{if} \quad j > n,$$

this implies that

$$\log |f(0)2^n| \leq \log M(2r),$$

or

$$\log |f(0)| + n \log 2 \leq \log M(2r).$$

The theorem follows from this, since $\log |f(0)| = 0$. □

Zeroes of Functions of Finite Order

The preceding theorem has the following consequence for entire functions of finite order.

Theorem 8.4.3. *Let f be an entire function of finite order λ and with $f(0) \neq 0$. Let $\{z_k\}$ be a list of the zeroes of f, counted according to multiplicity and indexed in order of increasing modulus, and let μ be the exponent of convergence of $\{z_k\}$, then $\mu \leq \lambda$.*

Proof. We claim that, for each $t > \lambda$, there are constants N, $C > 0$ and $q > 1$ such that

$$|z_k|^t \geq Ck^q \quad \text{for all} \quad k \geq N. \tag{8.4.2}$$

Assuming this, we conclude that the series

$$\sum_{k=1}^{\infty} \frac{1}{|z_k|^t}$$

converges for all $t > \lambda$, by comparison with the series

$$\sum_{k=1}^{\infty} \frac{1}{k^q},$$

which converges for $q > 1$. This, in turn, implies the exponent of convergence μ is at most λ.

To complete the proof, we must verify the claim concerning (8.4.2). In doing this, we may as well assume that $|f(0)| = 1$, since, if this is not so, we may make it so by replacing f by f divided by a constant times a power of z. Such a replacement will have no effect on whether the above claim is true.

Let $r_k = |z_k|$. Since the zeroes are indexed in such a way that the modulus is a non-decreasing function of k, there are at least k zeroes of f with modulus less than or equal to r_k. By Theorem 8.4.2,

$$k \le \frac{\log M(2r_k)}{\log 2},$$

where $M(2r_k)$ is the sup of $|f(z)|$ on the circle $|z| = 2r_k$.

We choose s with $\lambda < s < t$. Since f has order λ, there is an R such that $r_k \ge R$ implies

$$M(2r_k) \le \mathrm{e}^{(2r_k)^s}.$$

Hence, for $r_k \ge R$,

$$k \le \frac{(2r_k)^s}{\log 2}.$$

This implies

$$r_k^t \ge \frac{(\log 2)^{t/s}}{2^t} k^{t/s} = Ck^q,$$

where

$$C = \frac{(\log 2^{t/s})}{2^t} \quad \text{and} \quad q = \frac{t}{s} > 1.$$

This is true provided $r_k = |z_k| > R$. However, since $\lim z_k = \infty$, there is an N such that $k > N$ implies $|z_k| > R$. This completes the proof □

The above theorem, when combined with Theorem 8.3.5, yields the following corollary.

Corollary 8.4.4. *The canonical product for a sequence $\{z_k\}$ with exponent of convergence μ has finite order $\lambda = \mu$.*

Hadamard's Theorem

In the proof of the next theorem, we will need the following estimates on the size of the inverse $E_p^{-1}(z)$ of the function $E_p(z)$.

Lemma 8.4.5. *If p is a non-negative integer, $p \le t \le p+1$, and $z \in \mathbb{C}$, then there is a constant A such that*

$$\frac{1}{|E_p(z)|} \le e^{A|z|^t} \tag{8.4.3}$$

if $|z| \ge 2$ or $|z| \le 1/2$.

Proof. If $|z| \ge 2$, then $|1-z| \ge 1$ and $|z^k/k| \le |z|^t$ for $k \le p$. Hence,

$$|E_p^{-1}(z)| = |1-z|^{-1}|\mathrm{e}^{-z-z^2/2-\cdots-z^p/p}| \le \mathrm{e}^{p|z|^t}$$

and so (8.4.3) holds with $A = p$ in this case.

On the other hand, if $|z| \le 1/2$, then

$$\log E_p(z) = \log(1-z) + \sum_{k=1}^{p} z^k/k = -\sum_{k=p+1}^{\infty} z^k/k,$$

and so

$$|\log E_p(z)| \le |z|^{p+1} \sum_{j=0}^{\infty} |z|^k \le 2|z|^{p+1} \le 2|z|^t$$

Thus, (8.4.3) holds with $A = 2$ in this case. If we choose $A = \max\{2, p\}$, then (8.4.3) holds in both cases. □

We are now in a position to prove Hadamard's Theorem characterizing entire functions of finite order. This will be used in the proof of the Prime Number Theorem in the next chapter.

Theorem 8.4.6. *If f is an entire function of order λ, and p is the smallest integer such that $p+1 > \lambda$, then f factors as*

$$f(z) = z^m e^{h(z)} \prod_{k=1}^{\infty} E_p(z/z_k), \tag{8.4.4}$$

where m is the order of the zero of f at 0, $\{z_k\}$ is a list of the other zeroes of f counting multiplicity, and $h(z)$ is a polynomial of degree at most p.

Proof. According to the Weierstrass Factorization Theorem (Theorem 8.2.5) f has a factorization of the form (8.4.4), where h is an entire function. Thus, the only thing to be proved is that h is a polynomial of degree at most p. This will follow from Theorem 8.3.4 if we can show that the function

$$g(z) = \mathrm{e}^{h(z)} = \frac{f(z)}{z^m \prod_{j=1}^{\infty} E_p(z/z_k)}$$

has finite order at most λ.

Let t be any number with $\lambda < t \leq p+1$ and let $r \geq 1$ be any radius which is not one of the numbers $|z_k|$. We factor $g(z)$ as $g(z) = g_1(z)g_2(z)$, where

$$g_1(z) = f(z)z^{-m} \prod_{|z_k| \leq 2r} E_p^{-1}(z/z_k), \tag{8.4.5}$$

and

$$g_2(z) = \prod_{|z_k| > 2r} E_p^{-1}(z/z_k). \tag{8.4.6}$$

Suppose $|z| = 4r = R$. Then $|z/z_k| \geq 2$ for all k with $|z_k| \leq 2r$. By the previous lemma, there is a positive constant A_1 such that

$$|g_1(z)| \leq |f(z)| \prod_{|z_k| \leq 2r} \mathrm{e}^{A_1 |z/z_k|^t}.$$

Since f has finite order λ, for sufficiently large r we have

$$|f(z)| \leq \mathrm{e}^{|z|^t}$$

and, hence,

$$|g_1(z)| \leq \mathrm{e}^{B_1 r^t} \tag{8.4.7}$$

where

$$B_1 = 4^t \left(1 + A_1 \sum_{k=1}^{\infty} \frac{1}{|z_k|^t}\right).$$

The infinite series in this expression converges by Theorem 8.4.3. Since $g_1(z)$ is an entire function (once the removable singularities at the z_k with $|z_k| < 2r$ are removed), if the inequality (8.4.7) holds for $|z| = 4r = R$ it must hold for all z in the disc $|z| \leq R$, by the maximum modulus principle. In particular, this inequality holds for all z with $|z| = r$.

Also if $|z| = r$, then $|z/z_k| < 1/2$ if $|z_k| > 2r$, and the previous lemma implies that there is a constant A_2 such that

$$|g_2(z)| \leq \prod_{|z_k| > 2r} \mathrm{e}^{A_2 |z/z_k|^t} \leq \mathrm{e}^{B_2 r^t}, \tag{8.4.8}$$

where

$$B_2 = A_2 \sum_{k=1}^{\infty} \frac{1}{|z_k|}.$$

If we set $B = B_1 + B_2$ and combine (8.4.7) and (8.4.8), we obtain

$$|g(z)| \leq \mathrm{e}^{B|z|^t}.$$

Since t is an arbitrary number larger than λ and less than or equal to $p+1$, g has order at most λ. This completes the proof. $\square$

Exercise Set 8.4

1. What does Hadamard's Factorization Theorem say about an entire function of order $\lambda < 1$?

2. If a non-constant entire function of finite order λ has zeroes at the points $i\sqrt{n}$ what are the possible values for λ?

3. Show that an even entire function of order 1 has the form
$$Cz^m \prod_k \left(1 - \frac{z^2}{z_k^2}\right),$$
where m is even, C is a non-zero constant, and the sequence
$$\{z_1, -z_1, z_2, -z_2, \cdots, z_k, -z_k, \cdots\}$$
is a list of the zeroes of f counting multiplicity.

4. What is the exponent of convergence for the sequence of zeroes in the preceding exercise.

5. State and prove the analogues of the previous two exercises for odd entire functions of order 1.

6. Prove that if f is an entire function of order λ and λ is not an integer, then f has infinitely many zeroes.

7. Under the hypotheses of the preceding exercise, prove that f takes on every complex value infinitely many times.

8. Prove that if f and g are entire functions of finite order λ and if $f(z_k) = g(z_k)$ on a sequence which satisfies
$$\sum_{k=1} \frac{1}{|z_k|^t} = \infty$$
for some $t > \lambda$, then $f(z) = g(z)$ identically.

9. Use the previous exercise to prove that if two functions of finite order agree at the points of the sequence $\{\log n\}_{n=1}^{\infty}\}$, then they agree identically. Thus, e^z is the only entire function of finite order which has the value n at the point $\log n$ for $n = 1, 2, \cdots$.

10. Find an entire function which has zeroes at the points of the sequence $\{\log n\}_{n=1}^{\infty}$. Does it have finite order?

11. Suppose f is an entire function of finite order λ and μ is the exponent of convergence of the list of zeroes of f. Prove that if $\mu < \lambda$, then λ is an integer.

12. Is there an entire function of order $3/2$ which has the integers as its list of zeroes, counting multiplicity?