

## Chapter 7

# Convergence in Euclidean Space

With this chapter we begin our study of calculus in several variables. The first task is to define  $\mathbb{R}^d$  – Euclidean space of dimension  $d$ . We will then study convergence of sequences of points in this space and introduce the concepts of open and closed sets. These are generalizations to  $\mathbb{R}^d$  of the concepts of open and closed intervals in  $\mathbb{R}$ . In the final two sections we introduce the concepts of compact sets and connected sets. These are also generalizations to  $\mathbb{R}^d$  of properties of intervals in  $\mathbb{R}$ . These ideas will be of fundamental importance when we study continuous functions on  $\mathbb{R}^d$  in the next chapter.

In order to define and study convergence and continuity, we don't need to use all of the properties of  $\mathbb{R}^d$  – only the ones derived from the concept of distance between points. A set together with a well behaved notion of distance between pairs of points is called a *metric space*. In the coming pages, we will give a more precise definition of metric space and point out how many of the definitions and theorems we develop in this chapter are valid, not only in  $\mathbb{R}^d$ , but in any metric space.

### 7.1 Euclidean Space

The space  $\mathbb{R}^d$  is defined to be the set of all  $d$ -tuples of real numbers, where, by a  $d$ -tuple of real numbers, we mean an ordered set  $(x_1, x_2, \dots, x_d)$  of  $d$  real numbers. It is ordered because the numbers are listed in a certain order and, if this order is changed, then the new  $d$ -tuple is different from the old one (unless the change of order just interchanges identical numbers). For example,  $(5, 0, 7)$  and  $(0, 5, 7)$  are different 3-tuples and, hence, different points of  $\mathbb{R}^3$ .

The spaces  $\mathbb{R}^2$  and  $\mathbb{R}^3$  are familiar from calculus. The space  $\mathbb{R}^2$  is the set of all ordered pairs  $(x_1, x_2)$  of real numbers, while  $\mathbb{R}^3$  is the set of ordered triples  $(x_1, x_2, x_3)$  of real numbers. Often points of  $\mathbb{R}^2$  are denoted  $(x, y)$  rather than  $(x_1, x_2)$  and points of  $\mathbb{R}^3$  are denoted  $(x, y, z)$  rather than  $(x_1, x_2, x_3)$ .

### The Vector Space $\mathbb{R}^d$

We will often refer to a point of  $\mathbb{R}^d$  as a *vector* in  $\mathbb{R}^d$ , while a point of  $\mathbb{R}$  will often be referred to as a *scalar*.

There are natural operations of addition of vectors in  $\mathbb{R}^d$  and multiplication of vectors by scalars. That is, if  $x = (x_1, x_2, \dots, x_d)$  and  $y = (y_1, y_2, \dots, y_d)$  are vectors in  $\mathbb{R}^d$ , and  $a$  is a scalar, then we set

$$x + y = (x_1 + y_1, x_2 + y_2, \dots, x_d + y_d)$$

and

$$ax = (ax_1, ax_2, \dots, ax_d).$$

The zero vector (also called the *origin* of  $\mathbb{R}^d$ ) is the vector

$$0 = (0, 0, \dots, 0).$$

Note that we use the same symbol,  $0$ , to stand for both the scalar  $0$  and the vector  $0 \in \mathbb{R}^d$ . This shouldn't cause any confusion, since it will always be obvious from the context which is meant.

Given a vector  $x = (x_1, x_2, \dots, x_d)$  in  $\mathbb{R}^d$ , the *components* of  $x$  are the numbers  $x_1, x_2, \dots, x_d$ . The  $j$ th component is the number  $x_j$ . Two vectors are identical if and only if their  $j$ th components are identical for  $j = 1, 2, \dots, d$ .

As noted in the next theorem, addition in  $\mathbb{R}^d$  satisfies the associative and commutative laws and  $0$  has the appropriate properties. Also, scalar multiplication satisfies an associative law and two distributive laws.

**Theorem 7.1.1.** *Let  $u, v, w$  be points of  $\mathbb{R}^d$  and  $a$  and  $b$  real numbers. Then*

$$(a) \quad u + (v + w) = (u + v) + w;$$

$$(b) \quad u + v = v + u;$$

$$(c) \quad 0 + u = u;$$

$$(d) \quad 0u = 0 \text{ and } 1u = u;$$

$$(e) \quad a(bu) = (ab)u;$$

$$(f) \quad (a + b)u = au + bu;$$

$$(g) \quad a(u + v) = au + av.$$

*Proof.* Each statement asserts that two vectors are identical. Thus, each can be proved by proving that the  $j$ th components of the two vectors are identical for each  $j$ . In each case, this follows immediately from the definitions and the fact that  $\mathbb{R}$  satisfies the field axioms **A1** - **A4**, **M1** - **M4**, and **D** (see Section 1.3).  $\square$

A set together with operations of addition and scalar multiplication (where the scalars belong to some field  $F$ ), satisfying the properties listed in the above theorem, is called a *vector space* over  $F$  (see Section 1.3 for the definition of a field). Hence,  $\mathbb{R}^d$  is a vector space over the field  $\mathbb{R}$ .

Using only the vector space axioms listed in Theorem 7.1.1, one can easily derive all of the properties of general vector spaces.

**Example 7.1.2.** Using only the properties listed in Theorem 7.1.1, prove that if  $x$  is an element of a vector space, then  $(-1)x$  is an additive inverse for  $x$ . That is, prove that  $x + (-1)x = 0$ .

**Solution:** By Theorem 7.1.1 (d) and (f) we have

$$x + (-1)x = (1 + (-1))x = 0x = 0.$$

In view of this example,  $(-1)x$  is an additive inverse for  $x$  and so it makes sense to denote it simply by  $-x$ .

Other properties of vector spaces will be derived in the exercises.

## Inner Product

**Definition 7.1.3.** The Euclidean inner product of two vectors  $u = (u_1, \dots, u_d)$  and  $v = (v_1, \dots, v_d)$  in  $\mathbb{R}^d$  is the real number

$$u \cdot v = u_1v_1 + u_2v_2 + \dots + u_dv_d. \quad (7.1.1)$$

This has the following simple properties. The proof is left to the exercises.

**Theorem 7.1.4.** If  $u, v, w \in \mathbb{R}^d$  and  $a \in \mathbb{R}$ , then

- (a)  $u \cdot v = v \cdot u$ ;
- (b)  $(u + v) \cdot w = u \cdot w + v \cdot w$ ;
- (c)  $(au) \cdot v = a(u \cdot v)$ ;
- (d)  $u \cdot u > 0$  unless  $u = 0$  in which case  $u \cdot u = 0$ .

More generally, a function from pairs of vectors to scalars which satisfies (a) through (d) above is called an *inner product* on the vector space. A vector space together with an inner product on that vector space is called an *inner product space*. Thus,  $\mathbb{R}^d$  is an inner product space with the inner product described in Definition 7.1.3.

There are other inner products on  $\mathbb{R}^d$ . For example, if each term  $u_jv_j$  in (7.1.1) is replaced by  $a_ju_jv_j$ , where  $a_1, \dots, a_d$  are positive scalars, then the resulting sum defines a new inner product which is different from the original unless all the  $a_j$ 's are 1. In this text, the only inner product on  $\mathbb{R}^d$  that we will use is the Euclidean inner product as defined in (7.1.1).

Using (a) and (c) of Theorem 7.1.4, we easily show that  $u \cdot (av) = a(u \cdot v)$ . Thus, for a scalar  $a$  and vectors  $u$  and  $v$ , there is no ambiguity if we simply write  $au \cdot v$  in place of any one of the three equal products

$$a(u \cdot v), \quad (au) \cdot v, \quad u \cdot (av).$$

**Example 7.1.5.** If  $X$  is an inner product space,  $x, y \in X$  and  $a, b \in \mathbb{R}$ , then calculate the inner product of  $ax + by$  with itself.

**Solution:** By (b) and (c) of the previous theorem, we have

$$(ax + by) \cdot (ax + by) = ax \cdot (ax + by) + by \cdot (ax + by).$$

By (a), (b), and (c) we have

$$ax \cdot (ax + by) = a^2x \cdot x + abx \cdot y,$$

$$by \cdot (ax + by) = abx \cdot y + b^2y \cdot y.$$

Combining these yields

$$(ax + by) \cdot (ax + by) = a^2x \cdot x + 2abx \cdot y + b^2y \cdot y.$$

## Components of a Vector

We will typically denote by  $e_j$  the vector consisting of the  $d$ -tuple with all entries 0 except for the  $j$ th entry which is 1. Thus,  $e_j = (0, 0, \dots, 0, 1, 0, \dots, 0)$  with the 1 occurring in the  $j$ th position. Note that

$$e_j \cdot e_k = \delta_{jk},$$

where  $\delta_{jk}$  is 1 if  $j = k$  and is 0 otherwise. This means that  $\{e_j\}_{j=1}^d$  is an *orthonormal set* in  $\mathbb{R}^d$ .

Note that if  $x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$ , then the  $j$ th component  $x_j$  of  $x$  is given by  $x_j = x \cdot e_j$  for  $j = 1, \dots, d$ .

**Example 7.1.6.** Show that each vector in  $\mathbb{R}^d$  is a unique linear combination of the vectors  $e_j$  for  $j = 1, \dots, d$ .

**Solution:** If  $x = (x_1, x_2, \dots, x_d)$ , then

$$x = \sum_{j=1}^d x_j e_j = \sum_{j=1}^d (x \cdot e_j) e_j.$$

This is one way of expressing  $x$  as a linear combination of the  $e_j$ 's. On the other hand, if

$$x = \sum_{j=1}^d a_j e_j$$

is any such linear combination, then for  $k = 1, \dots, d$ ,

$$x_k = x \cdot e_k = \sum_{j=1}^d a_j e_j \cdot e_k = a_k,$$

since  $e_j \cdot e_k = 1$  if  $j = k$  and is 0 otherwise. Thus the coefficients  $a_j$  must be the numbers  $x_j$ .

## Norm and Distance

**Definition 7.1.7.** For an inner product space, we define the Euclidean norm  $\|x\|$  of a vector  $x$  to be the number

$$\|x\| = \sqrt{x \cdot x}.$$

The distance between two vectors  $x$  and  $y$  is defined to be  $\|x - y\|$ .

Note that, by Theorem 7.1.4 (d), the norm of a vector is always non-negative and is zero only if the vector is the zero vector. Thus, the distance between two vectors is always non-negative and is zero if and only if the vectors are equal.

In calculus, it is often shown that for two vectors  $u$  and  $v$  in  $\mathbb{R}^2$  or  $\mathbb{R}^3$  the inner product satisfies

$$u \cdot v = \|u\| \|v\| \cos \theta,$$

where  $\theta$  is the angle between  $u$  and  $v$ . Since  $|\cos \theta| \leq 1$ , this implies that

$$|u \cdot v| \leq \|u\| \|v\|.$$

As we show below, this inequality is true in  $\mathbb{R}^d$  and, in fact, in any inner product space. In this generality it is known as the Cauchy-Schwarz inequality.

**Theorem 7.1.8. (Cauchy-Schwarz Inequality)** *If  $X$  is an inner product space, then*

$$|u \cdot v| \leq \|u\| \|v\|$$

for all  $u, v \in X$ .

*Proof.* If we take the inner product of a vector with itself, the result is non-negative by (d) of Theorem 7.1.4. Thus, if  $u$  and  $v$  are vectors in  $X$  and  $t \in \mathbb{R}$  is a scalar, then

$$0 \leq (tu + v) \cdot (tu + v) = t^2 u \cdot u + 2tu \cdot v + v \cdot v = at^2 + 2bt + c,$$

where  $a = u \cdot u = \|u\|^2$ ,  $b = u \cdot v$ , and  $c = v \cdot v = \|v\|^2$ . The expression on the right is a quadratic function of  $t$  which is never negative. This means that the quadratic equation

$$at^2 + 2bt + c = 0$$

has at most one real root (since the graph of  $at^2 + 2bt + c$  cannot cross the  $t$ -axis). By the quadratic formula, the roots of this equation are

$$-b \pm \sqrt{b^2 - ac}.$$

Since there cannot be two real roots, it must be the case that  $b^2 \leq ac$ . On taking the square root of both sides of this inequality, we obtain the inequality of the theorem.  $\square$

Let  $u$  and  $v$  be vectors in an inner product space. In view of the above theorem, the number  $\frac{u \cdot v}{\|u\|\|v\|}$  is always between  $-1$  and  $1$  and, hence, is the cosine of some angle  $\theta$  with  $0 \leq \theta \leq \pi$ . This leads to the following extension to arbitrary inner product spaces of the notion of the angle between two vectors.

**Definition 7.1.9.** With  $u, v$  and  $\theta$  as above, we will call  $\theta$  the angle between  $u$  and  $v$ . This angle is  $\pi/2$  if and only if  $u \cdot v = 0$ . In this case we will say that  $u$  and  $v$  are mutually orthogonal and write  $u \perp v$ .

### The Triangle Inequality

The triangle inequality is just the vector space version of the statement that the length of one side of a triangle is always less than or equal to the sum of the lengths of the other two sides. It is stated more precisely in part (a) of the following theorem.

**Theorem 7.1.10.** *If  $X$  is an inner product space,  $x, y \in X$ , and  $a \in \mathbb{R}$ , then*

- (a)  $\|x + y\| \leq \|x\| + \|y\|$ ;
- (b)  $\|ax\| = |a| \|x\|$ ;
- (c)  $\|x\| = 0$  implies  $x = 0$ .

*Proof.* Using Example 7.1.5 and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \|x + y\|^2 &= (x + y) \cdot (x + y) = \|x\|^2 + 2x \cdot y + \|y\|^2 \\ &\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2. \end{aligned}$$

Part (a) of the theorem follows from this on taking square roots. Parts (b) and (c) follow from (c) and (d) of Theorem 7.1.4.  $\square$

Suppose  $u, v$ , and  $w$  are points in a vector space  $X$ . Then  $\|u - v\|, \|v - w\|$ , and  $\|u - w\|$  are the lengths of the sides of the triangle with vertices at  $u, v$ , and  $w$ . If we apply part (a) of the previous theorem to the vectors  $x = u - v$  and  $y = v - w$ , the result is the inequality

$$\|u - w\| \leq \|u - v\| + \|v - w\|, \quad (7.1.2)$$

which says that a side of a triangle always has length less than or equal to the sum of the lengths of the other two sides.

### Norms in General

The norm induced by an inner product is just one type of norm on a vector space. In general, a *norm* on a vector space  $X$  is a non-negative function  $\|\cdot\|$  which satisfies (a), (b), and (c) of the previous theorem. A *normed vector space* is a vector space  $X$  together with a norm on  $X$ . There are norms on  $\mathbb{R}^d$  which are different from the Euclidean norm (the norm induced by the Euclidean inner product).

**Definition 7.1.11.** If  $x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$ , we set

1.  $\|x\|_1 = |x_1| + |x_2| + \dots + |x_d|$ ;
2.  $\|x\|_\infty = \max\{|x_1|, |x_2|, \dots, |x_d|\}$ .

**Example 7.1.12.** Show that  $\|\cdot\|_1$  is a norm on  $\mathbb{R}^d$ .

**Solution:** If  $x = (x_1, x_2, \dots, x_d)$  and  $y = (y_1, y_2, \dots, y_d)$ , then

$$\|x + y\|_1 = \sum_{j=1}^n |x_j + y_j| \leq \sum_{j=1}^n (|x_j| + |y_j|),$$

by the triangle inequality for  $\mathbb{R}$ . The sum on the right is equal to

$$\sum_{j=1}^d |x_j| + \sum_{j=1}^d |y_j| = \|x\|_1 + \|y\|_1.$$

Thus,  $\|\cdot\|_1$  satisfies the triangle inequality ((a) of Theorem 7.1.10).

If  $a \in \mathbb{R}$ , then

$$\|ax\|_1 = \sum_{j=1}^d |ax_j| = \sum_{j=1}^d |a| |x_j| = |a| \|x\|_1.$$

Thus,  $\|\cdot\|_1$  also satisfies (b). That (c) holds as well is obvious, since  $\|x\|_1 = 0$  implies that  $x_j = 0$  for each  $j$  and, hence, that  $x = 0$ .

We leave to the exercises, the problem of showing that  $\|\cdot\|_\infty$  is also a norm on  $\mathbb{R}^d$ .

**Theorem 7.1.13.** *The three norms we have defined on  $\mathbb{R}^d$  are related as follows:*

$$d^{-1} \|x\|_1 \leq \|x\|_\infty \leq \|x\| \leq \|x\|_1$$

for each  $x \in \mathbb{R}^d$ .

The proof of this is also left to the exercises.

## The Normed Vector Space $C(I)$

In mathematics we deal with a great many normed vector spaces. One that does not look at all like  $\mathbb{R}^d$  is the space  $C(I)$ , where  $I$  is a closed bounded interval on the real line, and  $C(I)$  is the vector space of all continuous real valued functions on  $I$ . Addition is pointwise addition of functions and scalar multiplication is multiplication of a function by a constant. It is easy to see that  $C(I)$  is a vector space under these two operations (Exercise 7.1.10). There are many norms that can be put on this vector space, but perhaps the most useful is the sup norm,  $\|\cdot\|_\infty$ , defined by

$$\|f\|_\infty = \sup_I |f(x)|, \quad (7.1.3)$$

for  $f \in C(I)$ . The problem of showing that this is a norm is left to the exercises.

**Exercise set 7.1**

1. For the vectors  $x = (1, 0, 2)$  and  $y = (-1, 3, 1)$  in  $\mathbb{R}^3$  find
  - (a)  $2x + y$ ;
  - (b)  $x \cdot y$ ;
  - (c)  $\|x\|$  and  $\|y\|$ ;
  - (d) the cosine of the angle between  $x$  and  $y$ ;
  - (e) the distance from  $x$  to  $y$ .
2. Using only the properties listed in Theorem 7.1.1, prove that if  $u, v, w$  are vectors in a vector space and  $u + w = v + w$ , then  $u = v$ .
3. Using only the properties listed in Theorem 7.1.1, prove that if  $u$  is a vector in a vector space,  $a$  is a scalar, and  $au = 0$ , then either  $a = 0$  or  $u = 0$ .
4. Prove Theorem 7.1.4.
5. Prove the second form of the triangle inequality. That is, prove that

$$\left| \|x\| - \|y\| \right| \leq \|x - y\|$$

holds for any pair of vectors  $x, y$  in a normed vector space. Hint: use the first form (Theorem 7.1.10(a)) to prove the second form.

6. Prove that equality holds in the Cauchy-Schwarz inequality (Theorem 7.1.8) if and only if one of the vectors  $u, v$  is a scalar multiple of the other.
7. For a norm on a vector space  $X$ , defined by an inner product as in Definition 7.1.7, prove that the parallelogram law:

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2,$$

holds for all  $x, y \in X$ .

8. Prove that  $\|\cdot\|_\infty$ , as defined in Definition 7.1.11, is a norm on  $\mathbb{R}^d$ .
9. Prove Theorem 7.1.13
10. Prove that the space  $C(I)$ , defined in the previous subsection, is a vector space.
11. Prove that the sup norm as defined in 7.1.3 is really a norm on  $C(I)$ .
12. Prove that if  $\{x_k\}$  and  $\{y_k\}$  are sequences of real numbers such that

$$\sum_{k=1}^{\infty} x_k^2 < \infty \quad \text{and} \quad \sum_{k=1}^{\infty} y_k^2 < \infty, \quad \text{then} \quad \sum_{k=1}^{\infty} |x_k y_k| < \infty.$$

Hint: what can you say about the corresponding finite sums?

13. Find a non-zero vector in  $\mathbb{R}^3$  which is orthogonal to both  $(1, 0, 2)$  and  $(3, -1, 1)$ .
14. Prove that if  $u$  and  $v$  are vectors in an inner product space and  $u \perp v$ , then  $\|u + v\|^2 = \|u\|^2 + \|v\|^2$ .

## 7.2 Convergent Sequences of Vectors

In this section we study convergence of sequences of vectors in  $\mathbb{R}^d$ . The definitions and theorems in this topic are very similar to those of Chapter 2 on sequences of numbers.

### Metric Spaces

As long as we are working in a space with a reasonable notion of distance between points, we can define and study convergent sequences and continuous functions. Such a space is called a *metric space*. The precise conditions for a space to be a metric space are defined below.

**Definition 7.2.1.** Let  $X$  be a set and  $\delta$  a function which assigns to each pair  $(x, y)$  of elements of  $X$  a non-negative real number  $\delta(x, y)$ . Then  $\delta$  is called a *metric* on  $X$  if, for all  $x, y, z \in X$ , the following conditions hold:

- (a)  $\delta(x, y) = \delta(y, x)$ ;
- (b)  $\delta(x, y) = 0$  if and only if  $x = y$ ; and
- (c)  $\delta(x, z) \leq \delta(x, y) + \delta(y, z)$ .

A set  $X$ , together with a metric  $\delta$  on  $X$  is called a *metric space*.

Conditions (a) and (b) above are called the symmetry and identity conditions, while condition (c) is the triangle inequality for metric spaces.

We will show that  $\mathbb{R}^d$  is a metric space, as is any normed vector space.

**Theorem 7.2.2.** *If  $X$  is a normed vector space, then  $X$  is a metric space if its metric  $\delta$  is defined by*

$$\delta(x, y) = \|x - y\|.$$

*In particular,  $\mathbb{R}^d$  is a metric space in the Euclidean norm, as is  $C(I)$  in the sup norm.*

*Proof.* Parts (a), (b), and (c) of Theorem 7.1.10 are satisfied by the norm in any normed vector space. Part (b) with  $a = -1$  implies that  $\|x - y\| = \|y - x\|$  and so  $\delta$  is symmetric. Part (c) implies that  $\|x - y\| = 0$ , if and only if  $x = y$ , and so  $\delta$  satisfies the identity condition. Part (a) implies (7.1.2), which shows that  $\delta$  satisfies the triangle inequality. Thus,  $\delta$  is a metric on  $X$ .  $\square$

**Remark 7.2.3.** If  $X$  is a metric space with metric  $\delta$  and  $Y$  is any subset of  $X$ , then  $Y$  is also a metric space with the same metric  $\delta$ . Thus, any subset of  $\mathbb{R}^d$  is also a metric space if it is given the usual Euclidean metric.

There are a great many metric spaces other than subsets of  $\mathbb{R}^d$  that are important in mathematics. We will explore some of these in the exercises.

**Remark 7.2.4.** The following statements summarize the relationship between the types of spaces we have introduced so far:

1.  $\mathbb{R}^d$  is an inner product space;
2. every inner product space is a normed vector space, with norm defined by  $\|x\| = \sqrt{x \cdot x}$ ;
3. every normed vector space is a metric space, with metric defined by  $\delta(x, y) = \|x - y\|$ .

## Sequences

The definition of convergence for a sequence  $\{x_n\}$  in  $\mathbb{R}^d$  should look familiar:

**Definition 7.2.5.** If  $\{x_n\}$  is a sequence of vectors in  $\mathbb{R}^d$  and  $x \in \mathbb{R}^d$ , then we say  $\{x_n\}$  *converges* to  $x$  if for every  $\epsilon > 0$  there is an  $N \in \mathbb{R}$  such that

$$\|x - x_n\| < \epsilon \quad \text{whenever} \quad n \geq N.$$

In this case, we write  $\lim_{n \rightarrow \infty} x_n = x$  or  $\lim x_n = x$  or simply  $x_n \rightarrow x$ .

Note that we do not require the  $N$  that appears in this definition to be an integer.

Note also that the only thing we use about  $\mathbb{R}^d$  in making this definition is the notion of distance between points in  $\mathbb{R}^d$ . Quite clearly, the same definition can be made for any metric space  $X$  if we just replace  $\|x - x_n\|$  by  $\delta(x, x_n)$ , where  $\delta$  is the metric on  $X$ . Thus, the definition of convergence for a sequence in a general metric space is the following:

**Definition 7.2.6.** Let  $X$  be a metric space with metric  $\delta$ . If  $\{x_n\}$  is a sequence in  $X$  and  $x \in X$ , then we say  $\{x_n\}$  *converges* to  $x$  if for every  $\epsilon > 0$  there is an  $N \in \mathbb{R}$  such that

$$\delta(x, x_n) < \epsilon \quad \text{whenever} \quad n \geq N.$$

In this case, we write  $\lim_{n \rightarrow \infty} x_k = x$  or  $\lim x_n = x$  or simply  $x_n \rightarrow x$ .

We will not try to prove everything in this section in the context of general metric spaces; after all, the object of study here is  $\mathbb{R}^d$ . However, we will point out some theorems we prove for  $\mathbb{R}^d$  that can be proved in general metric spaces or normed vector spaces or inner product spaces, and some of the exercises will be devoted to verifying these claims.

**Example 7.2.7.** Let  $x_n = (1/n^2, 1 + 1/n) \in \mathbb{R}^2$ . Use Definition 7.2.5 to prove that the sequence  $\{x_n\}$  converges to  $x = (0, 1)$ .

**Solution:** We have  $x - x_n = (-1/n^2, -1/n)$  and so

$$\|x - x_n\| = \sqrt{1/n^4 + 1/n^2} \leq \sqrt{2/n^2} = \sqrt{2}/n.$$

Thus, given  $\epsilon > 0$ , if we choose  $N = \sqrt{2}/\epsilon$ , then

$$\|x - x_n\| < \sqrt{2}/n \leq \sqrt{2}/N = \epsilon \quad \text{whenever } n \geq N.$$

This completes the proof that  $\lim x_n = x$ .

Many limit proofs for sequences in  $\mathbb{R}^d$  follow the same pattern as in the above example. We showed that  $\|x - x_n\| < \sqrt{2}/n$  and then used the fact that  $\sqrt{2}/n$  can be made less than  $\epsilon$  by making  $n$  large enough – that is, we used the fact that  $\lim \sqrt{2}/n = 0$ . We can save some effort in future proofs by formalizing in a theorem the method that was used here. The theorem is a vector version of Theorem 2.3.1. In fact, it follows immediately from Theorem 2.3.1 and the fact (obvious from the definition of limit) that  $\lim x_n = x$  if and only if  $\lim \|x_n - x\| = 0$ .

**Theorem 7.2.8.** *Let  $\{x_n\}$  be a sequence in  $\mathbb{R}^d$  and let  $x$  be a vector in  $\mathbb{R}^d$ . If there is a sequence  $\{a_n\}$  of non-negative real numbers such that*

$$\|x - x_n\| \leq a_n \quad \text{for all } n$$

*and if  $\lim a_n = 0$ , then  $\lim x_n = x$ .*

Note that, since the proof of this theorem uses nothing about  $\mathbb{R}^d$  but the existence of a metric and the definition of limit, it holds in any metric space (if  $\|x - x_n\|$  is replaced by  $\delta(x, x_n)$ ).

**Example 7.2.9.** If  $x_n = (e^{-n} \sin n, e^{-n} \cos n) \in \mathbb{R}^2$ , prove that  $\lim x_n = 0$ .

**Solution:** We have

$$\|x_n - 0\| = \|x_n\| = \sqrt{e^{-2n}(\sin^2 n + \cos^2 n)} = e^{-n} = 1/e^n.$$

Since,  $\lim 1/e^n = 0$ , the previous theorem tells us that  $\lim x_n = 0$ .

## Limit Theorems

The following theorem says that the limit of a sequence, if it exists, is unique. Its proof is identical to the proof of Theorem 2.1.6. We won't repeat it here. The analogous theorem for metric spaces is also true and also has the same proof.

**Theorem 7.2.10.** *If  $\{x_n\}$  is a sequence in  $\mathbb{R}^d$  and  $x, y \in \mathbb{R}^d$  with  $x_n \rightarrow x$  and  $x_n \rightarrow y$ , then  $x = y$ .*

**Theorem 7.2.11.** *If  $\lim x_n = x$  for a sequence  $\{x_n\}$  in  $\mathbb{R}^d$ , then  $\lim \|x_n\| = \|x\|$ .*

*Proof.* The second form of the triangle inequality tells us that

$$|\|x\| - \|x_n\|| \leq \|x - x_n\|.$$

If  $\lim x_n = x$ , then the sequence of numbers on the right converges to 0. It follows that the one on the left also converges to 0. Thus,  $\lim \|x_n\| = \|x\|$ .  $\square$

The next theorem is the vector version of the Main Limit Theorem (Theorem 2.3.6) for sequences of real numbers.

**Theorem 7.2.12.** *If  $\{x_n\}$  and  $\{y_n\}$  are sequences of vectors in  $\mathbb{R}^d$  and  $a_n$  is a sequence of scalars, and if  $x_n \rightarrow x \in \mathbb{R}^d$ ,  $y_n \rightarrow y \in \mathbb{R}^d$  and  $a_n \rightarrow a$ , then*

$$(a) \quad x_n + y_n \rightarrow x + y;$$

$$(b) \quad a_n x_n \rightarrow ax; \text{ and}$$

$$(c) \quad x_n \cdot y_n \rightarrow x \cdot y.$$

*Proof.* (a) By the triangle inequality, we have

$$\|x + y - (x_n + y_n)\| \leq \|x - x_n\| + \|y - y_n\|.$$

Since  $x_n \rightarrow x$  and  $y_n \rightarrow y$  we have that  $\|x - x_n\| \rightarrow 0$  and  $\|y - y_n\| \rightarrow 0$ . Thus,  $\|x - x_n\| + \|y - y_n\| \rightarrow 0$  and it follows from Theorem 7.2.8 that  $x_n + y_n \rightarrow x + y$ .

(b) We have

$$\|ax - a_n x_n\| = \|a(x - x_n) + (a - a_n)x_n\| \leq |a| \|x - x_n\| + |a - a_n| \|x_n\|.$$

Since  $\|x - x_n\| \rightarrow 0$ ,  $|a - a_n| \rightarrow 0$  and  $\|x_n\| \rightarrow \|x\|$  (by the previous theorem), the expression on the right converges to 0. Hence, by Theorem 7.2.8 again,  $\lim a_n x_n = ax$ .

(c) The proof of this is similar to the proof of (b). The details are left to the exercises.  $\square$

Note that the proofs of (a) and (b) above use only properties of  $\mathbb{R}^d$  that are also true in any normed vector space, and so they hold in this much more general context. The proof of (c) uses only properties of  $\mathbb{R}^d$  that hold in any inner product space and so (c) is true in any inner product space.

The next theorem tells us that a sequence of vectors converges if and only if it converges componentwise.

**Theorem 7.2.13.** *A sequence  $\{x_n\}$  in  $\mathbb{R}^d$  converges to  $x \in \mathbb{R}^d$  if and only each component of  $\{x_n\}$  converges to the corresponding component of  $x$  – that is, if and only if  $\lim x_n \cdot e_j = x \cdot e_j$  for  $j = 1, \dots, d$ .*

*Proof.* If  $\lim_{n \rightarrow \infty} x_n = x$ , then  $\lim_{n \rightarrow \infty} x_n \cdot e_j = x \cdot e_j$  for each  $j$  by Theorem 7.2.12, part (c).

To prove the converse, we suppose  $\lim_{n \rightarrow \infty} x_n \cdot e_j = x \cdot e_j$  for each  $j$ . We note that this implies that  $\lim_{n \rightarrow \infty} |(x_n - x) \cdot e_j| = 0$  for each  $j$ . By Theorem 7.1.13,

$$\|x_n - x\| = \sum_{j=1}^d |(x_n - x) \cdot e_j|^2.$$

The sum on the right converges to 0 and, hence,  $\lim x_n = x$ .  $\square$

## The Bolzano-Weierstrass Theorem

The conclusion of the Bolzano-Weierstrass Theorem from Chapter 2 (Theorem 2.5.5) also holds for bounded sequences in  $\mathbb{R}^d$ . A sequence in  $\mathbb{R}^d$  is bounded if there is a number  $M$  such that  $\|x_n\| \leq M$  for all  $n$ .

**Theorem 7.2.14. (Bolzano-Weierstrass Theorem)** *Each bounded sequence in  $\mathbb{R}^d$  has a convergent subsequence.*

*Proof.* We will prove this by induction on the dimension  $d$  of the Euclidean space. It is, of course, true for  $d = 1$  by the single variable version of the Bolzano-Weierstrass Theorem (Theorem 2.5.5).

Suppose  $d > 1$  and the theorem is true for Euclidean space of dimension  $d - 1$ . Let  $\{x_n\}$  be a bounded sequence in  $\mathbb{R}^d$ . Then there is an  $M \in \mathbb{R}$  such that  $\|x_n\| \leq M$  for all  $n$ .

We identify  $\mathbb{R}^d$  with the Cartesian product  $\mathbb{R}^{d-1} \times \mathbb{R}$ . This is the space of all pairs  $(y, z)$ , where  $y \in \mathbb{R}^{d-1}$  and  $z \in \mathbb{R}$ . That is, if  $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ , then we identify  $x$  with the pair  $(y, z)$ , where  $y = (x_1, x_2, \dots, x_{d-1})$  and  $z = x_d$ . If this is done, notice that

$$\|y\| \leq \|x\| \quad \text{and} \quad |z| \leq \|x\|.$$

Thus, if we write each element of the sequence  $\{x_n\}$  in the form  $x_n = (y_n, z_n) \in \mathbb{R}^{d-1} \times \mathbb{R}$ , then  $\|y_n\| \leq \|x_n\| \leq M$  and  $|z_n| \leq \|x_n\| \leq M$ . This implies that the sequences  $\{y_n\}$  and  $\{z_n\}$  are both bounded.

By the induction assumption, the sequence  $\{y_n\}$  has a convergent subsequence  $\{y_{n_i}\}$ . The corresponding subsequence  $\{z_{n_i}\}$  of the sequence  $\{z_n\}$  is still bounded, and so it has a convergent subsequence. By replacing  $\{y_{n_i}\}$  by a (still convergent) subsequence of itself, we may assume that  $\{z_{n_i}\}$  itself converges.

The component sequences of  $\{x_{n_j}\}$  are those of  $\{y_{n_j}\}$ , which all converge since  $\{y_{n_j}\}$  converges, and the sequence  $\{z_{n_i}\}$ , which converges. Thus,  $\{x_{n_i}\}$  converges since all of its component sequences converge.

We conclude that every bounded sequence in  $\mathbb{R}^d$  has a convergent subsequence. This completes the induction and finishes the proof of the theorem.  $\square$

## Cauchy Sequences

Cauchy sequences in  $\mathbb{R}^d$  are defined in the same way as Cauchy sequences of numbers were defined in Definition 2.5.6.

**Definition 7.2.15.** A sequence  $\{x_n\}$  in  $\mathbb{R}^d$  is said to be a *Cauchy Sequence* if, for every  $\epsilon > 0$ , there is an  $N$  such that

$$\|x_n - x_m\| < \epsilon \quad \text{whenever } n, m \geq N.$$

The following theorem is proved using the Bolzano-Weierstrass Theorem in exactly the same way its single variable counterpart (Theorem 2.5.7) was proved. We won't repeat the proof.

**Theorem 7.2.16.** *A sequence  $\{x_n\}$  in  $\mathbb{R}^d$  is a Cauchy sequence if and only if it converges.*

To prove directly from the definition that a certain sequence converges, it is necessary to have in hand the element to which it converges. On the other hand, the definition of a Cauchy sequence involves only the elements of the sequence. Hence, the above theorem provides a way to prove that a sequence converges without having already identified the limit.

Clearly, Cauchy sequences can be defined in any metric space – simply replace “ $\|x_n - x_m\|$ ” in the above definition by “ $\delta(x_n, x_m)$ ”, where  $\delta$  is the metric. However, the analogue of Theorem 7.2.16 is not true in general for metric spaces. A metric space in which it is true is said to be *complete*. Thus,  $\mathbb{R}^d$  is a complete metric space. An example of a metric space which is not complete follows.

**Example 7.2.17.** Let the interval  $(0, 1)$  be considered a metric space with the usual distance between points as metric. Show that this is not a complete metric space.

**Solution:** The sequence  $\{1/n\}$  is a Cauchy sequence since it converges in  $\mathbb{R}$  to the point 0. However, since  $0 \notin (0, 1)$ , this sequence does not converge in the metric space  $(0, 1)$ . Hence,  $(0, 1)$  is not a complete metric space.

## Exercise Set 7.2

- Using only the definition of the limit of a sequence in  $\mathbb{R}^d$  prove that
 
$$\lim \left( \frac{n}{1+n}, \frac{1-n}{n} \right) = (1, -1).$$

In each of the next four problems, determine whether or not the sequence  $\{x_n\}$  converges and find its limit if it does converge. Use limit theorems to justify your answers.

- $x_n = \left( \frac{n^2 + n - 1}{3n^2 + 2}, \frac{n - 1}{n + 1} \right).$

3.  $x_n = (1 + (-1)^n, 1/n, 1 + 1/n)$ .
4.  $x_n = (2^{-n} \sin(n\pi/4), 2^{-n} \cos(n\pi/4))$ ;
5.  $x_n = (\ln(n + 1) - \ln n, \sin(1/n))$ .
6. Let  $\{x_n\}$  and  $\{y_n\}$  be sequences in  $\mathbb{R}^d$ . Prove that if  $\lim x_n = 0$  and  $\{y_n\}$  is bounded, then  $\lim x_n \cdot y_n = 0$ .
7. Let  $\{x_n\}$  be a bounded sequence in  $\mathbb{R}^d$  and  $a_n$  a bounded sequence of scalars. Prove that if either sequence has limit 0, then so does the sequence  $\{a_n x_n\}$ .
8. Prove that every convergent sequence in  $\mathbb{R}^d$  is bounded.
9. If  $x_n = (\sin n, \cos n, 1 + (-1)^n)$ , does the sequence  $\{x_n\}$  in  $\mathbb{R}^2$  have a convergent subsequence? Justify your answer.
10. Prove part (c) of Theorem 7.2.12.
11. If  $x_n = (1/n, \sin(\pi n/2))$ , find three convergent subsequences of  $\{x_n\}$  which converge to three different limits.
12. If, for  $x, y \in \mathbb{R}$ , we set  $\delta(x, y) = 0$  if  $x = y$  and  $\delta(x, y) = 1$  if  $x \neq y$ , prove that the result is a metric on  $\mathbb{R}$ . Thus,  $\mathbb{R}$  with this metric is a metric space – one that is quite different from  $\mathbb{R}$  with the usual metric.
13. What are the convergent sequences in the metric space described in the previous exercise.
14. Let  $a$  and  $b$  be points of  $\mathbb{R}^2$  and let  $X$  be the set of all smooth parameterized curves joining  $a$  to  $b$  in  $\mathbb{R}^2$ , with parameter interval  $[0, 1]$ . That is,  $X$  is the set of all continuously differentiable functions  $\gamma : [0, 1] \rightarrow \mathbb{R}^2$ , with  $\gamma(0) = a$  and  $\gamma(1) = b$ . Show that if
$$\delta(\gamma_1, \gamma_2) = \sup\{|\gamma_1(t) - \gamma_2(t)| : t \in [0, 1]\},$$
then  $\delta$  is a metric on  $X$ .
15. Show that the metric space of the previous exercise is not complete.
16. Let  $S$  be the surface of a sphere in  $\mathbb{R}^3$ . For  $x, y \in S$  let  $\delta(x, y)$  be the length of the shortest path on  $S$  joining  $x$  to  $y$ . Show that this is a metric on  $S$ .
17. Imagine a large building with many rooms. Let  $X$  be the set of rooms in this building and let  $\delta(x, y)$  be the length of the shortest path along the hallways and stairways of the building that leads from room  $x$  to room  $y$ . Show that  $\delta$  is a metric on  $X$ .

### 7.3 Open and Closed Sets

The open ball  $B_r(x_0)$  and closed ball  $\overline{B}_r(x_0)$ , centered at  $x_0 \in \mathbb{R}^d$ , with radius  $r > 0$ , are defined by

$$B_r(x_0) = \{x \in \mathbb{R}^d : \|x - x_0\| < r\} \quad \text{and} \quad \overline{B}_r(x_0) = \{x \in \mathbb{R}^d : \|x - x_0\| \leq r\}.$$

Of course, open and closed balls centered at a given point and with a given radius may be defined in any metric space – one simply uses the metric distance  $\delta(x, x_0)$  in place of the distance  $\|x - x_0\|$  defined by the norm in  $\mathbb{R}^d$ .

Open intervals and closed intervals on the real line play an important part in the calculus of one variable. Open and closed balls are the direct analogues in  $\mathbb{R}^d$  of open and closed intervals on the line. However, the geometry of  $\mathbb{R}^d$  is much more complicated than that of the line. We will need the concepts of open and closed for sets that are far more complicated than balls. This leads to the following definition.

**Definition 7.3.1.** If  $U$  is a subset of  $\mathbb{R}^d$ , we will say that  $U$  is *open* if, for each point  $x \in U$ , there is an open ball centered at  $x$  which is contained in  $U$ . We will say that a subset of  $\mathbb{R}^d$  is *closed* if its complement is open. A *neighborhood* of a point  $x \in \mathbb{R}^d$  is any open set which contains  $x$ .

It might seem obvious that open balls are open sets and closed balls are closed sets. However, that is only because we have chosen to call them *open* balls and *closed* balls. We actually have to prove that they satisfy the conditions of the preceding definition. We do this in the next theorem.

**Theorem 7.3.2.** In  $\mathbb{R}^d$ ,

- (a) the empty set  $\emptyset$  is both open and closed;
- (b) the whole space  $\mathbb{R}^d$  is both open and closed;
- (c) each open ball is open;
- (d) each closed ball is closed.

*Proof.* The empty set  $\emptyset$  is open because it has no points, and so the condition that a set be open, stated in Definition 7.3.1, is vacuously satisfied. The set  $\mathbb{R}^d$  is open because it contains any open ball centered at any of its points. Thus,  $\emptyset$  and  $\mathbb{R}^d$  are both open. Since they are complements of one another, they are also both closed.

To prove (c), we suppose  $B_r(x_0)$  is an open ball and  $y$  is one of its points. Then  $\|y - x_0\| < r$  and so, if we set  $s = r - \|y - x_0\|$ , then  $s > 0$ . Also, if  $x \in B_s(y)$ , then  $\|x - y\| < s$  and so

$$\|x - x_0\| \leq \|x - y\| + \|y - x_0\| < s + \|y - x_0\| = r,$$

which means  $x \in B_r(x_0)$  (see Figure 7.1). Thus, we have shown that, for each  $y \in B_r(x_0)$ , there is an open ball,  $B_s(y)$ , centered at  $y$ , which is contained in

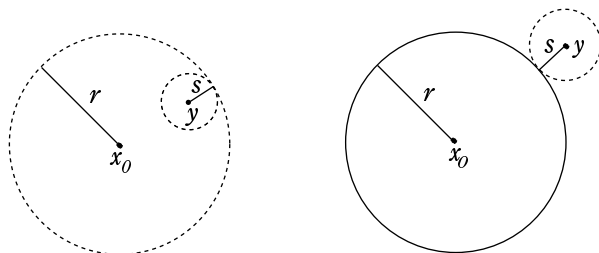


Figure 7.1: Proving Theorem 7.3.2 (c) and (d).

$B_r(x_0)$ . By definition, this means that  $B_r(x_0)$  is open. This completes the proof of (c).

To prove (d), we consider a closed ball  $\overline{B}_r(x_0)$ . To prove that it is a closed set, we must show its complement is open. Suppose  $y$  is a point in its complement. This means  $y \in \mathbb{R}^d$  but  $y \notin \overline{B}_r(x_0)$ , and so  $\|y - x_0\| > r$ . This time we set  $s = \|y - x_0\| - r$  and we claim that the open ball  $B_s(y)$  is contained in the complement of  $\overline{B}_r(x_0)$ . In fact, if  $x \in B_s(y)$ , then  $\|x - y\| < s$  and so, by the second form of the triangle inequality (Theorem 2.1.2 (b))

$$\|x - x_0\| \geq \|y - x_0\| - \|x - y\| > \|y - x_0\| - s = r,$$

which means  $x$  is in the complement of  $\overline{B}_r(x_0)$ . Thus, we have proved that each point of the complement of  $\overline{B}_r(x_0)$  is the center of an open ball contained in the complement of  $\overline{B}_r(x_0)$ . This proves that this complement is open, hence, that  $\overline{B}_r(x_0)$  is closed.  $\square$

The above theorem holds in any metric space and it has the same proof. The same thing is true of the next theorem. It tells us that the collection of all open subsets of  $\mathbb{R}^d$  forms what is called a *topology* for  $\mathbb{R}^d$ . A *topology* for a space  $X$  is a collection of sets which are declared to be the open sets of the space. This collection must contain the empty set and the space  $X$  and must have the property that it is closed under arbitrary unions and finite intersections. A space  $X$  with a specified topology is called a *topological space*.

**Theorem 7.3.3.** In  $\mathbb{R}^d$ ,

- (a) the union of an arbitrary collection of open sets is open;
- (b) the intersection of any finite collection of open sets is open;
- (c) the intersection of an arbitrary collection of closed sets is closed;
- (d) the union of any finite collection of closed sets is closed.

*Proof.* If  $\mathcal{V}$  is an arbitrary collection of open sets, and  $U = \bigcup \mathcal{V}$  is its union, then  $x$  is in  $U$  if and only if it is in at least one of the sets in  $\mathcal{V}$ . Suppose,  $x \in V$  with  $V$  in  $\mathcal{V}$ . Then, since  $V$  is open, there is a ball  $B_r(x)$ , centered at  $x$ , which

is contained in  $V$ . Since  $V \subset U$ , this ball is also contained in  $U$ . This proves that  $U$  is open and completes the proof of (a).

Now suppose  $\{V_1, V_2, \dots, V_n\}$  is a finite collection of open sets and

$$x \in U = V_1 \cap V_2 \cap \dots \cap V_n.$$

Then, since each  $V_k$  is open, there exists for each  $k$  a radius  $r_k$  such that  $B_{r_k}(x) \subset V_k$ . If  $r = \min\{r_1, r_2, \dots, r_n\}$ , then  $B_r(x) \subset V_k$  for every  $k$ , which implies that  $B_r(x) \subset U$ . It follows that  $U$  is open. This completes the proof of (b).

The proofs of the corresponding statements (c) and (d) for closed sets follow from those for open sets by taking complements. We leave the details to Exercise 7.3.5.  $\square$

**Remark 7.3.4.** An easy consequence of the above theorem is that if  $U$  is open and  $K$  is closed and if  $K \subset U$ , then the set theoretic difference  $U \setminus K$  is open. On the other hand, if  $U \subset K$ , then  $K \setminus U$  is closed (Exercise 7.3.6).

**Example 7.3.5.** If  $0 < r < R$ , prove that the annulus

$$A = \{x \in \mathbb{R}^2 : r < \|x\| < R\},$$

is open.

**Solution:** The ball  $B_R(0)$  is open, the ball  $\overline{B}_r(0)$  is closed, and  $A$  is the set theoretic difference  $B_R(0) \setminus \overline{B}_r(0)$ . Thus, by the previous remark,  $A$  is open.

A similar argument shows that an annulus of the form

$$\{x \in \mathbb{R}^2 : r \leq \|x\| \leq R\}.$$

is closed.

## Interior, Closure, and Boundary

If  $E$  is a subset of  $\mathbb{R}^d$ , then the union of all open subsets of  $E$  is open, by Theorem 7.3.3. By construction, it is a subset of  $E$  which contains all open subsets of  $E$ . Thus, every subset of  $\mathbb{R}^d$  contains a largest open subset – that is, an open subset which contains all other open subsets.

Similarly, the intersection of all closed sets containing  $E$  is a closed set containing  $E$  and it is contained in every closed set containing  $E$ . Thus, it is the smallest closed set containing  $E$ .

It is a consequence of this discussion that the following definition makes sense.

**Definition 7.3.6.** Let  $E$  be a subset of  $\mathbb{R}^d$ . Then:

- (a) the largest open subset of  $E$  is called the *interior* of  $E$  and is denoted  $E^\circ$ ;
- (b) the smallest closed set containing  $E$  is called the *closure* of  $E$  and is denoted  $\overline{E}$ ;

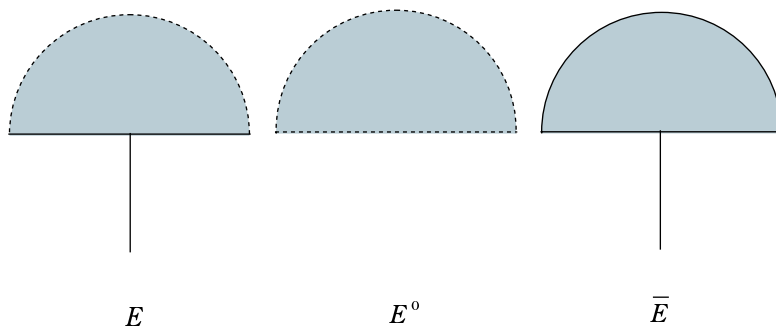


Figure 7.2: The Set  $E$  of Example 7.3.8, its Interior  $E^\circ$ , and Closure  $\bar{E}$ .

(c) the set  $\bar{E} \setminus E^\circ$  is called the boundary of  $E$  and is denoted  $\partial E$ .

Note that these concepts can be defined in exactly the same way in any topological space and, in particular, in any metric space.

Recall that a neighborhood of a point  $x \in \mathbb{R}^d$  is any open set containing  $x$ . The proof of the following theorem is elementary and is left to the exercises. This theorem also holds in any metric space.

**Theorem 7.3.7.** *Let  $E$  be a subset of  $\mathbb{R}^d$  and  $x$  an element of  $\mathbb{R}^d$ . Then:*

- (a)  $x \in E^\circ$  if and only if there is a neighborhood of  $x$  that is contained in  $E$ ;
- (b)  $x \in \bar{E}$  if and only if every neighborhood of  $x$  contains a point of  $E$ ;
- (c)  $x \in \partial E$  if and only if every neighborhood of  $x$  contains points of  $E$  and points of the complement of  $E$ .

**Example 7.3.8.** Find the interior, closure and boundary for the set

$$E = \{(x, y) \in \mathbb{R}^2 : \|(x, y)\| < 1, y \geq 0\} \cup \{(0, -y) : y \in [0, 1]\}.$$

**Solution:** It is immediate from the previous theorem that

$$E^\circ = \{(x, y) \in \mathbb{R}^2 : \|(x, y)\| < 1, y > 0, \}$$

$$\bar{E} = \{(x, y) \in \mathbb{R}^2 : \|(x, y)\| \leq 1, y \geq 0\} \cup \{(0, -y) : y \in [0, 1]\},$$

$$\partial E = \{(x, y) \in \mathbb{R}^2 : \|(x, y)\| = 1, y \geq 0\} \cup [-1, 1] \cup \{(0, -y) : y \in [0, 1]\}.$$

See Figure 7.2

## Sequences

The concepts of open and closed sets are intimately connected to the concept of convergence of a sequence.

**Theorem 7.3.9.** *A sequence  $\{x_n\}$  in  $\mathbb{R}^d$  converges to  $x \in \mathbb{R}^d$  if and only if, for every neighborhood  $U$  of  $x$ , there is a number  $N$  such that  $x_n \in U$  whenever  $n \geq N$ .*

*Proof.* If for every neighborhood  $U$  of  $x$  there is an  $N$  such that  $x_n \in U$  whenever  $n \geq N$ , then this is true, in particular, for each neighborhood of the form  $B_\epsilon(x)$  with  $\epsilon > 0$ . This means that for each  $\epsilon > 0$  there is an  $N$  such that  $\|x - x_n\| < \epsilon$  whenever  $n \geq N$ . That is,  $\lim x_n = x$ .

Conversely, if  $\lim x_n = x$  and  $U$  is any neighborhood of  $x$ , we may choose an  $\epsilon > 0$  such that the ball  $B_\epsilon(x)$  is contained in  $U$ . By the definition of limit, for this  $\epsilon$  there is an  $N$  such that  $\|x - x_n\| < \epsilon$  whenever  $n \geq N$ . Then  $x_n \in B_\epsilon(x) \subset U$  whenever  $n \geq N$ . This completes the proof.  $\square$

**Theorem 7.3.10.** *If  $A$  is a subset of  $\mathbb{R}^d$ , then  $\overline{A}$  is the set of all limits of convergent sequences in  $A$ . The set  $A$  is closed if and only if every convergent sequence in  $A$  converges to a point of  $A$ .*

*Proof.* If  $x \in \overline{A}$ , then each neighborhood of  $x$  contains a point of  $A$  by Theorem 7.3.7(b). In particular, each neighborhood of the form  $B_{1/n}(x)$ , for  $n \in \mathbb{N}$ , contains a point of  $A$ . We choose one and call it  $x_n$ . Since  $\|x - x_n\| < 1/n$ , the sequence  $\{x_n\}$  converges to  $x$ . Thus, each point in the closure of  $A$  is the limit of a sequence in  $A$ .

Conversely, suppose  $x = \lim x_n$  for some sequence  $\{x_n\}$  in  $A$ . By the previous theorem, each neighborhood of  $x$  contains points in this sequence. In particular, each neighborhood of  $x$  contains a point of  $A$ . Hence,  $x \in \overline{A}$  by Theorem 7.3.7(b).

Since a set is closed if and only if it is its own closure, it follows that  $A$  is closed if and only if it contains all limits of convergent sequences in  $A$ .  $\square$

### Exercise Set 7.3

1. Prove that the set  $\{(x, y) \in \mathbb{R}^2 : y > 0\}$  is an open subset of  $\mathbb{R}^2$ .
2. Prove that every finite subset of  $\mathbb{R}^d$  is closed.
3. Find the interior, closure, and boundary for the set

$$\{(x, y) \in \mathbb{R}^2 : 0 \leq x < 2, 0 \leq y < 1\}.$$

4. Find the interior, closure, and boundary for the set

$$\{(x, y) \in \mathbb{R}^2 : \|(x, y)\| < 1\} \cup \{(x, y) \in \mathbb{R}^2 : y = 0, -2 < x < 2\}.$$

5. Prove (c) and (d) of Theorem 7.3.3
6. Let  $A$  be an open set and  $B$  a closed set. If  $B \subset A$ , prove that  $A \setminus B$  is open. If  $A \subset B$ , prove that  $B \setminus A$  is closed.
7. Prove Theorem 7.3.7.

8. If  $E$  is a subset of  $\mathbb{R}^d$ , is the interior of the closure of  $E$  necessarily the same as the interior of  $E$ ? Justify your answer.
9. If  $A$  and  $B$  are subsets of  $\mathbb{R}^d$  show that  $\overline{A \cup B} = \overline{A} \cup \overline{B}$ . Is the analogous statement true for  $A \cap B$ ? Justify your answer.
10. If  $A$  and  $B$  are subsets of  $\mathbb{R}^d$ , prove that  $(A \cap B)^\circ = A^\circ \cap B^\circ$ . Is the analogous statement true for  $A \cup B$ ? Justify your answer.
11. Let  $\{x_n\}$  be a convergent sequence in  $\mathbb{R}^d$  with limit  $x$ . Set

$$A = \{x_1, x_2, x_3, \dots\} \cup \{x\},$$

that is,  $A$  is the set consisting of all the points occurring in the sequence together with the limit  $x$ . Show that  $A$  is a closed set.

12. Let  $\{x_n\}$  be any sequence in  $\mathbb{R}^d$  and let  $A$  be the set consisting of the points that occur in this sequence. Prove that the closure of  $A$  consists of  $A$  together with all limits of convergent subsequences of  $A$ .
13. Show that Theorem 7.3.10 remains true if  $\mathbb{R}^d$  is replaced by any metric space.
14. Find the interior and closure of the set  $Q$  of rationals in  $\mathbb{R}$ .
15. If  $E$  is a subset of  $\mathbb{R}^d$ , show that  $(\overline{E})^c = (E^c)^\circ$ .

## 7.4 Compact Sets

In this section and the next, we study two topological properties, compactness and connectedness, that a subset of  $\mathbb{R}^d$  may or may not have. A topological property of a set  $E$  is one that can be described using only knowledge of the open sets of  $\mathbb{R}^d$  and their relationship to  $E$ . Thus, they are properties that can be defined in any topological space. Compactness and connectedness are two such properties.

### Open Covers

An open cover of a set  $E \subset \mathbb{R}^d$  is a collection of open sets whose union contains  $E$ . An open cover of a set  $E$  may or may not have a finite subcover – that is, there may or may not be finitely many sets in the collection which also form a cover of  $E$ .

**Example 7.4.1.** The collection  $\mathcal{U}$  of all open intervals of length  $1/2$  and with rational endpoints is clearly an open cover of the interval  $[0, 1]$ . Show that it has a finite subcover.

**Solution:** The three intervals  $(-1/8, 3/8)$ ,  $(1/4, 3/4)$ , and  $(5/8, 9/8)$  belong to  $\mathcal{U}$  and they cover  $[0, 1]$ .

**Example 7.4.2.** The collection  $\{(1/n, 1) : n = 1, 2, \dots\}$  is a collection of open sets which covers  $(0, 1)$ . Does it have a finite subcover?

**Solution:** No. Since this collection of intervals is nested upward, any finite subcollection has a largest interval  $(1/m, 1)$ . Then the union of the sets in the subcollection is just  $(1/m, 1)$  and this does not contain  $(0, 1)$ .

## Compactness

The above discussion leads to the following definition:

**Definition 7.4.3.** A subset  $K$  of  $\mathbb{R}^d$  is called *compact* if every open cover of  $K$  has a finite subcover.

Note that Example 7.4.2 shows that the open interval  $(0, 1)$  is not compact, since it has an open cover with no finite subcover.

A subset  $E$  of  $\mathbb{R}^d$  is bounded if there is a number  $R$  such that  $\|x\| \leq R$  for every  $x \in E$  – that is, if  $E \subset \overline{B}_R(0)$  for some  $R$ .

**Theorem 7.4.4.** Every compact subset  $K$  of  $\mathbb{R}^d$  is bounded.

*Proof.* We have  $K \subset \mathbb{R}^d = \cup_n B_n(0)$ . This means that the open balls  $B_n(0)$  for  $n = 1, 2, \dots$  form an open cover of  $K$ . Since  $K$  is compact, finitely many of these balls must also form a cover of  $K$ . This implies  $K$  is contained in one these balls, say  $B_m(0)$ , since they form a sequence which is nested upward. Since  $K$  is contained in  $B_m(0) \subset \overline{B}_m(0)$ , it is bounded.  $\square$

**Theorem 7.4.5.** Every compact subset  $K$  of  $\mathbb{R}^d$  is closed.

*Proof.* We will prove this by showing that  $K = \overline{K}$ . If  $x \in \overline{K}$  and  $n$  is a positive integer, we let  $U_n$  be the complement in  $\mathbb{R}^d$  of  $\overline{B}_{1/n}(x)$ . The union of the nested sequence of open sets  $\{U_n\}$  is  $\mathbb{R}^d \setminus \{x\}$ .

If some finite subcollection of  $\{U_n\}$  covers  $K$  then some one of these sets, say  $U_m$ , contains  $K$ . This means that  $B_{1/m}(x) \cap K = \emptyset$ , which is impossible, since  $x \in \overline{K}$ . Because  $K$  is compact, this means that  $\{U_n\}$  cannot be an open cover of  $K$ . Since  $x$  is the only point of  $\mathbb{R}^d$  not covered by  $\{U_n\}$ ,  $x$  must be in  $K$ .

We conclude that  $K = \overline{K}$  and  $K$  is closed.  $\square$

## The Heine-Borel Theorem

The last two theorems show that a compact subset of  $\mathbb{R}^d$  is both closed and bounded. The Heine-Borel Theorem says the the converse is also true – every closed bounded subset of  $\mathbb{R}^d$  is compact. Before we prove this, we prove the following analogue of the Nested Interval Theorem (Theorem 2.5.1).

**Theorem 7.4.6.** If  $A_1 \supset A_2 \supset \dots \supset A_n \supset A_{n+1} \supset \dots$  is a nested sequence of non-empty bounded closed subsets of  $\mathbb{R}^d$ , then  $\cap_n A_n \neq \emptyset$ .

*Proof.* Since each  $A_n$  is non-empty, we may choose a point  $x_n \in A_n$  for each  $n$ . These points are all in  $A_1$ , which is bounded. Hence,  $\{x_n\}$  is a bounded sequence. By the Bolzano–Weierstrass Theorem (Theorem 7.2.14) this sequence has a convergent subsequence  $\{x_{n_k}\}$ . Let  $x$  be the limit of this subsequence.

Since  $A_1$  is closed and  $x_{n_k} \in A_1$  for every  $k$ , we have that  $x \in A_1$ . In fact, for each  $n$ ,  $n_k \geq n$  if  $k \geq n$ , and so, beginning with the  $n$ th term, each term of the sequence  $\{x_{n_k}\}$  belongs to  $A_n$ . Since  $A_n$  is closed, we have  $x \in A_n$ . We conclude that  $x \in \bigcap_n A_n$ . Hence,  $\bigcap_n A_n \neq \emptyset$ .  $\square$

In the proof of the following theorem, we will make use of the concept of an  $d$ -cube in  $\mathbb{R}^d$ . This is a set of the form  $C = I_1 \times I_2 \times \cdots \times I_d$ , where each  $I_j$  is a closed bounded interval in  $\mathbb{R}$  of length  $L$ . The intervals  $I_j$  are called the *edges* of  $C$  and the number  $L$  is called the *edge length* of  $C$ . Note that a 2-cube is just a square in  $\mathbb{R}^2$  with sides parallel to the coordinate axes, while a 3-cube is a cube in  $\mathbb{R}^3$  with edges parallel to the axes.

**Theorem 7.4.7. (Heine-Borel Theorem)** *A subset of  $\mathbb{R}^d$  is compact if and only if it is closed and bounded.*

*Proof.* We already know that every compact subset of  $\mathbb{R}^d$  is closed and bounded. Thus, to complete the proof we just need to show that every closed bounded subset of  $\mathbb{R}^d$  is compact.

Let  $K$  be a closed bounded subset of  $\mathbb{R}^d$  and  $\mathcal{V}$  an open cover of  $K$ . Suppose  $\mathcal{V}$  has no finite subcover. We will show that this leads to a contradiction.

Since  $K$  is bounded, it lies inside some  $d$ -cube  $C_1$ . Let  $L$  be the edge length of  $C_1$ . By partitioning each edge of  $C_1$  at its midpoint, we may partition  $C_1$  into  $2^d$   $d$ -cubes of edge length  $L/2$ . By intersecting each of these smaller cubes with  $K$ , we partition  $K$  into finitely many subsets. If each of these is covered by finitely many of the sets in  $\mathcal{V}$ , then  $K$  itself is also. Since it is not, we conclude that the intersection of  $K$  with at least one of these smaller  $d$ -cubes is not covered by finitely many sets in  $\mathcal{V}$ . Choose one and call it  $C_2$ .

By continuing in this way (actually, by induction), we may construct a nested sequence of  $d$ -cubes (see Figure 7.3)

$$C_1 \supset C_2 \supset \cdots \supset C_n \supset C_{n+1} \supset \cdots,$$

where, for each  $n$ ,  $C_n$  is a closed  $d$ -cube of edge length  $L/2^{n-1}$  and with the property that  $C_n \cap K$  cannot be covered by finitely many of the sets in  $\mathcal{V}$ .

The sets  $C_n \cap K$  form a sequence of closed, bounded sets, nested downward, as in the previous theorem. By that theorem  $\bigcap_n (C_n \cap K)$  is not empty. Let  $x$  be a point in this intersection. Then  $x \in K$  and, since  $\mathcal{V}$  is an open cover of  $K$ , there is some open set  $V$  in the collection  $\mathcal{V}$  such that  $x \in V$ . Since  $V$  is open, there is an open ball  $B_r(x)$ , centered at  $x$ , which is contained in  $V$ .

The diameter of  $C_n$  (maximum distance between two points of  $C_n$ ) is less than  $dL/2^{n-1}$ . Hence, for large enough  $n$ , the diameter of  $C_n$  is less than  $r$ . Then  $C_n$  must be contained in  $B_r(x)$  since it contains  $x$ . This implies that  $C_n \subset V$ . This is a contradiction, since  $C_n$  was chosen so that no finite

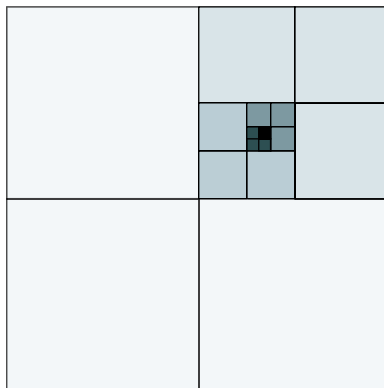


Figure 7.3: Nested Cubes of Theorem 7.4.7

subcollection of the sets in  $\mathcal{V}$  covers  $C_n \cap K$ . Thus, our assumption that  $K$  is not covered by any finite subcollection of  $\mathcal{V}$  has led to a contradiction.

We conclude that every open cover of  $K$  has a finite subcover and, hence, that  $K$  is compact.  $\square$

**Corollary 7.4.8.** *Each closed subset of a compact set in  $\mathbb{R}^d$  is also compact.*

*Proof.* If  $A$  is closed and contained in a compact set  $K$ , then  $A$  is bounded because  $K$  is bounded. Since  $A$  is closed and bounded, it is compact by the Heine-Borel Theorem.  $\square$

## Applications of Compactness

The next chapter will contain a large number of applications of compactness to function theory. The next example illustrates a technique that is often used in such applications.

**Example 7.4.9.** Let  $K$  be a compact subset of  $\mathbb{R}^d$  and let  $\rho$  be a function defined on  $K$  with  $\rho(x) > 0$  for each  $x \in K$ . Prove there exists a finite set of points  $\{x_1, x_2, \dots, x_m\}$  such that  $K$  is contained in the union of the open balls  $B_{\rho(x_i)}(x_i)$  for  $i = 1, 2, \dots, m$ .

**Solution:** The collection of open sets  $\{B_{\rho(x)}(x) : x \in K\}$  is an open cover of  $K$  (since, for each  $y \in K$ ,  $y \in B_{\rho(y)}(y) \subset \cup\{B_{\rho(x)}(x) : x \in K\}$ ). Since  $K$  is compact, there is a finite subcover  $\{B_{\rho(x_i)}(x_i) : i = 1, \dots, m\}$ . This means  $K$  is contained in the union of the  $B_{\rho(x_i)}(x_i)$  for  $i = 1, 2, \dots, m$ .

The next theorem is an application of this technique. It is a *separation theorem* which shows that a compact set is *separated* from the complement of any open set that contains it.

**Theorem 7.4.10.** *Suppose  $K$  is a compact subset and  $U$  an open subset of  $\mathbb{R}^d$  with  $K \subset U$ . Then there exists an open set  $V$  such that  $\bar{V}$  is compact and  $K \subset V \subset \bar{V} \subset U$ .*

*Proof.* Since  $U$  is open and contains  $K$ , for each  $x \in K$  there is an open ball centered at  $x$  which lies in  $U$ . Then the ball, centered at  $x$ , of half this radius has its closure contained in  $U$ . Let  $\rho(x)$  be the radius of this smaller ball. Then  $x \in B_{\rho(x)}(x) \subset \bar{B}_{\rho(x)}(x) \subset U$ . By the previous example, there are finitely many points  $x_1, \dots, x_m$  such that  $K$  is contained in the union  $V$  of the sets  $B_{\rho(x_i)}(x_i)$ . The closure of  $V$  is contained in the compact set which is the union of the sets  $\bar{B}_{\rho(x_i)}(x_i)$ , and this is contained in  $U$ . Thus,  $\bar{V}$  is compact, since it is a closed subset of a compact set, and  $K \subset V \subset \bar{V} \subset U$ .  $\square$

## Compact Metric Spaces

Since compactness is a topological property, it makes perfectly good sense in any metric space. The definition of a compact subset of a metric space  $X$  is exactly the same as Definition 7.4.3 except that  $\mathbb{R}^d$  is replaced by  $X$ . If the space  $X$  itself is compact, then  $X$  is called a *compact metric space*.

Any compact subset of  $\mathbb{R}^d$  is a compact metric space if it is considered a space by itself and is given the same metric it has as a subset of  $\mathbb{R}^d$ .

### Exercise Set 7.4

1. If  $K$  is a compact subset of  $\mathbb{R}^d$  and  $U_1 \subset U_2 \subset \dots \subset U_k \subset \dots$  is a nested upward sequence of open sets with  $K \subset \cup_k U_k$ , then prove that  $K$  is contained in one of the sets  $U_k$ .
2. Let  $K$  be a compact subset of  $\mathbb{R}^d$  and  $A_1 \supset A_2 \supset \dots \supset A_j \supset \dots$  a nested downward sequence of closed subsets of  $\mathbb{R}^d$ . Show that if  $A_k \cap K \neq \emptyset$  for each  $k$ , then  $(\cap_k A_k) \cap K \neq \emptyset$ .
3. Show that if  $K_1 \supset K_2 \supset \dots \supset K_j \supset \dots$  is a nested downward sequence of compact sets and  $U$  is an open set which contains  $\cap_j K_j$ , then  $U$  contains one of the sets  $K_j$ .
4. Prove that if  $K$  is a compact subset of  $\mathbb{R}^d$ , then  $K$  contains a point of maximal norm. That is, there is a point  $x_1 \in K$  such that

$$\|x\| \leq \|x_1\| \quad \text{for all } x \in K.$$

Hint: Set  $m = \sup\{\|x\| : x \in K\}$  and consider the open balls  $B_{m-1/n}(0)$ .

5. Prove that if  $K$  is a compact subset of  $\mathbb{R}^d$  and  $y$  is a point of  $\mathbb{R}^d$  which is not in  $K$ , then there is a closest point to  $y$  in  $K$ . That is, there is an  $x_0 \in K$  such that

$$\|x_0 - y\| \leq \|x - y\| \quad \text{for all } x \in K.$$

6. Prove that the conclusion of the previous exercise also holds if we only assume that  $K$  is a closed subset of  $\mathbb{R}^d$ . Hint: replace  $K$  by its intersection with a suitably large closed ball centered at  $y$ .
7. Prove that if  $K_1, K_2$  is a disjoint pair of compact sets, then there exists a disjoint pair of open sets  $V_1, V_2$  such that  $K_1 \subset V_1$  and  $K_2 \subset V_2$ . Hint: Use Theorem 7.4.10.
8. Prove that a set  $K \subset \mathbb{R}^d$  is compact if and only if every sequence in  $K$  has a subsequence which converges to an element of  $K$ . Hint: use the Bolzano – Weierstrass and Heine–Borel Theorems.
9. Show that it is true that the union of any finite collection of compact subsets of  $\mathbb{R}^d$  is compact, but it is not true that the union of an infinite collection of compact subsets is necessarily compact. Show the latter statement by finding an example of an infinite union of compact sets which is not compact.
10. Prove that if  $A$  and  $B$  are compact subsets of a metric space, then  $A \cup B$  and  $A \cap B$  are also compact.
11. Prove that if  $X$  is a compact metric space, then every sequence in  $X$  has a convergent subsequence.
12. Prove that if  $X$  is a compact metric space, then every closed subset of  $X$  is also compact.
13. Prove that a compact metric space is complete (that is, every Cauchy sequence converges).
14. We will say a metric space  $X$  is *bounded* if, for some  $M > 0$  and  $x \in X$ , the entire space  $X$  is contained in  $B_M(x) = \{y \in X : \delta(x, y) \leq M\}$ . Show that a compact metric space is bounded.
15. Consider the metric space of Exercise 7.2.12. Show that it is complete and bounded, but not compact. Thus, the analogue of the Heine–Borel Theorem does not hold in general metric spaces.

## 7.5 Connected Sets

Consider the three sets  $A, B, C$  described in Figure 7.4. Each of these sets is the union of two closed discs of radius one in  $\mathbb{R}^2$ . In  $A$  the distance between the centers of the two discs is greater than 2; in  $B$  it is less than 2 and in  $C$  it is exactly 2. The point about these three sets that we wish to discuss is this: set  $A$  is *disconnected* – one cannot pass from one of the discs making up this set to the other without leaving the set. On the other hand,  $B$  and  $C$  are *connected* – one can pass from any point in the set to any other point in the set without leaving the set. As stated so far, these are not very precise ideas. The precise definition of connectedness is as follows.

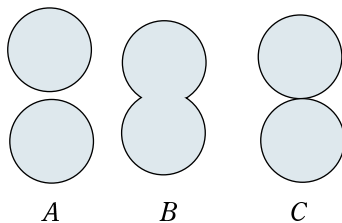


Figure 7.4: Disconnected and Connected Sets

**Definition 7.5.1.** A subset  $E$  of  $\mathbb{R}^d$  is said to be *separated* by a pair of open sets  $U$  and  $V$  in  $\mathbb{R}^d$  if

- (a)  $E \subset U \cup V$ ;
- (b)  $(E \cap U) \cap (E \cap V) = \emptyset$ ;
- (c)  $E \cap U \neq \emptyset$ , and  $E \cap V \neq \emptyset$ .

If no pair of open subsets of  $\mathbb{R}^d$  separates  $E$ , then we will say that  $E$  is *connected*.

The above definition becomes somewhat simpler to state if we give a special name to subsets of  $E$  of the form  $E \cap U$  where  $U$  is an open set.

**Definition 7.5.2.** Let  $E$  be a subset of  $\mathbb{R}^d$ . A subset  $A$  of  $E$  is said to be *relatively open* (in  $E$ ) if it has the form  $A = E \cap U$  for some open subset  $U$  of  $\mathbb{R}^d$ . Similarly, a subset  $B$  is said to be *relatively closed* (in  $E$ ) if it has the form  $E \cap C$  for some closed subset  $C$  of  $\mathbb{R}^d$ .

Using these concepts, the definition of connectedness can be rephrased as follows.

**Remark 7.5.3.** A subset  $E$  of  $\mathbb{R}^d$  is connected if and only if it is not the disjoint union of two non-empty relatively open subsets.

### Connected Subsets of $\mathbb{R}$

The connected subsets of  $\mathbb{R}$  are easily characterized.

**Theorem 7.5.4.** A non-empty subset of  $\mathbb{R}$  is connected if and only if it is an interval.

*Proof.* Suppose  $E$  is a non-empty subset of  $\mathbb{R}$ . Let

$$a = \inf E \quad \text{and} \quad b = \sup E.$$

Now  $a$  and  $b$  may not be finite, but  $E$  is certainly contained in the interval consisting of  $(a, b)$  together with  $\{a\}$  if  $a$  is finite and  $\{b\}$  if  $b$  is finite. The set  $E$  will be an interval if and only if it contains  $(a, b)$ .

Suppose  $E$  is not an interval. Then there is an  $x \in (a, b)$  such that  $x \notin E$ . Then  $E$  is contained in the set  $(-\infty, x) \cup (x, \infty)$ . Furthermore, since  $a = \inf E$  and  $a < x$ , there must be points of  $E$  which are less than  $x$  – that is,  $E \cap (-\infty, x) \neq \emptyset$ . Similarly, since  $b = \sup E$  and  $x < b$ ,  $E \cap (x, \infty) \neq \emptyset$ . Thus, by Definition 7.5.1, the set  $E$  is separated by the pair of open sets  $(-\infty, x)$  and  $(x, \infty)$  and, hence, is not connected. Thus, if  $E$  is connected, it must be an interval.

Conversely, suppose  $E$  is an interval. Then  $E$  is  $(a, b)$  possibly together with one or more of its endpoints. Suppose  $U$  and  $V$  are open subsets of  $\mathbb{R}$  with  $(U \cap E) \cap (V \cap E) = \emptyset$  and  $E \subset U \cup V$ . We define a function  $f$  on  $E$  by  $f(x) = 0$  if  $x \in E \cap U$  and  $f(x) = 1$  if  $x \in E \cap V$ .

We claim  $f$  is a continuous function on the interval  $E$ . If  $x \in E$  and  $\epsilon > 0$ , then  $x$  is in one of the sets  $U$  or  $V$ . Since they are both open, there is an interval  $(x - \delta, x + \delta)$  which is also contained in whichever of these sets contains  $x$ . Then  $f$  has the same value at any  $y \in E \cap (x - \delta, x + \delta)$  that it has at  $x$ . Thus,

$$|f(x) - f(y)| = 0 < \epsilon \quad \text{whenever } y \in E \quad \text{and} \quad |x - y| < \delta.$$

This proves that  $f$  is continuous on  $E$ . However, its only possible values are 0 and 1. By the Intermediate Value Theorem (Theorem 3.2.3) it cannot take on both these values, since it would then have to take on every value in between. This means one of the sets  $E \cap U$ ,  $E \cap V$  is empty. Hence,  $E$  is not separated by  $U$  and  $V$ . We conclude that no pair of open sets separates  $E$  and, hence,  $E$  is connected.  $\square$

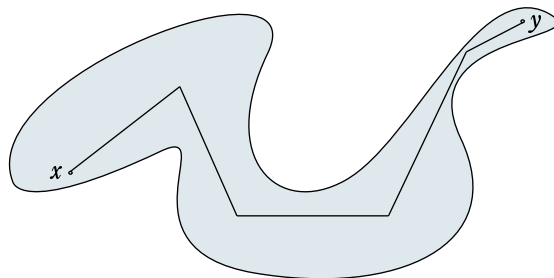
If  $L$  is a straight line in  $\mathbb{R}^d$ , then the intersection of an open ball in  $\mathbb{R}^d$  with  $L$  is an open interval in  $L$  (or is empty). It follows that the relatively open subsets of  $L$  are exactly the open subsets of  $L$  considered as a copy of  $\mathbb{R}$ . It follows from the above theorem that intervals in  $L$  are connected subsets of  $\mathbb{R}^d$ . Thus, the line segment joining two points in  $\mathbb{R}^d$  is a connected set.

## Connected Components

**Theorem 7.5.5.** *If  $A$  and  $B$  are connected subsets of  $\mathbb{R}^d$  and  $A \cap B \neq \emptyset$ , then  $A \cup B$  is also connected.*

*Proof.* Suppose  $U$  and  $V$  are disjoint relatively open subsets of  $A \cup B$  such that  $A \cup B = U \cup V$ . Then  $U \cap A$  and  $V \cap A$  are disjoint relatively open subsets of  $A$ . Since  $A$  is connected,  $U$  and  $V$  cannot both have non-empty intersection with  $A$ . Since  $A$  is contained in their union and can't meet both of them,  $A$  must be contained in either  $U$  or  $V$ . Similarly,  $B$  must be contained in either  $U$  or  $V$ . Since  $U$  and  $V$  are disjoint and  $A$  and  $B$  are not,  $A$  and  $B$  must be contained in the same one of the sets  $U$ ,  $V$  and must both be disjoint from the other. Since  $U \cup V = A \cup B$  one of the sets  $U$ ,  $V$  is empty. This shows that  $U$  and  $V$  do not separate  $A \cup B$ . Hence,  $A \cup B$  is connected.  $\square$

Basically the same argument shows that the union of any collection of connected sets with at least one point in common is also connected (Exercise 7.5.6).

Figure 7.5: A piecewise linear path in  $E$ 

In particular, if  $x \in E$  where  $E$  is some subset of  $\mathbb{R}^d$ , then the union of all connected subsets of  $E$  containing  $x$  is itself connected. Thus, for each point  $x \in E$  there is a connected subset of  $E$  which contains all connected subsets containing  $x$  – that is, a *maximal* connected subset containing  $x$ .

**Definition 7.5.6.** If  $E$  is a subset of  $\mathbb{R}^d$  and  $x \in E$ , then the union of all connected subsets of  $E$  containing  $x$  is called the *connected component* of  $E$  containing  $x$ .

Clearly, the connected components of  $E$  are the maximal connected subsets of  $E$ . Any two distinct components are disjoint since, otherwise, their union would be a connected set larger than at least one of them. Two points  $x$  and  $y$  of  $E$  are in the same component of  $E$  if and only if there is some connected subset of  $E$  that contains both  $x$  and  $y$ . In particular, if the line segment joining two points  $x$  and  $y$  of  $E$  also lies in  $E$ , then  $x$  and  $y$  are in the same connected component of  $E$ .

Since every point in an open or closed ball is joined by a line segment to the center of the ball, we have:

**Theorem 7.5.7.** *Every open or closed ball in  $\mathbb{R}^d$  is a connected set.*

More generally, a piecewise linear path joining  $x$  and  $y$  in  $E$  is a finite set of line segments  $\{[x_{i-1}, x_i]\}_{i=1}^m$ , each contained in  $E$ , with each line segment beginning where the preceding one ends, and with  $x_0 = x$  and  $x_m = y$ . One easily proves by induction that the union of the line segments in such a path is a connected set (see Figure 7.5). It follows that:

**Theorem 7.5.8.** *If  $E$  is a subset of  $\mathbb{R}^d$  and  $x$  and  $y$  are points of  $E$  that may be joined by a piecewise linear path in  $E$ , then  $x$  and  $y$  are in the same component of  $E$ . If every pair of points in  $E$  can be joined by a piecewise linear path in  $E$ , then  $E$  is connected.*

**Example 7.5.9.** Find a subset of  $\mathbb{R}^2$  with infinitely many components.

**Solution:** This is easy. The set of integers on the  $x$ -axis is such a set. Since the only connected subsets of this set are the single point subsets, each

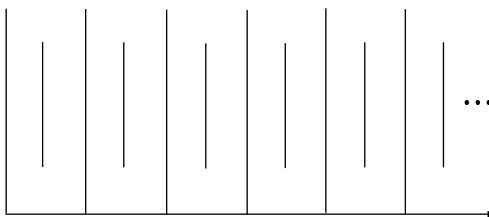


Figure 7.6: A set with infinitely many components

point is a component. A more complicated example is illustrated in Figure 7.6. The vertical lines that touch the bottom horizontal line together with this horizontal line form one component, while each of the shorter vertical lines is itself a component.

### Components of an Open Set

**Theorem 7.5.10.** *If  $U$  is an open subset of  $\mathbb{R}^d$ , then each of its connected components is also open.*

*Proof.* Let  $V$  be a connected component of the open set  $U$  and let  $x$  be a point of  $V$ . Since  $U$  is open, there is an open ball  $B_r(x)$ , centered at  $x$ , such that  $B_r(x) \subset U$ . Since  $V$  is the union of all connected subsets of  $U$  containing  $x$  and  $B_r(x)$  is connected, it must be true that  $B_r(x) \subset V$ . Since every point of  $V$  is the center of an open ball contained in  $V$ , the set  $V$  is open.  $\square$

The components of an open set  $U$  form a pairwise disjoint family of open connected subsets of  $U$  with union  $U$ . Conversely:

**Theorem 7.5.11.** *If an open set  $U$  can be written as the union of a pairwise disjoint family  $\mathcal{V}$  of open connected subsets, then these subsets must be the components of  $U$ .*

*Proof.* If  $V$  is one of the open sets in  $\mathcal{V}$ , then  $V$  must have non-empty intersection with at least one component of  $U$ , call it  $C$ . Then  $V \subset C$  since  $V$  is a connected set containing a point of the component  $C$ .

We must also have  $C \subset V$ , since, otherwise,  $V$  and the union of all the sets in  $\mathcal{V}$  other than  $V$  would be two open sets which separate  $C$ . Thus,  $V = C$ .

We now have that every set in  $\mathcal{V}$  is a component of  $U$ . Since the union of the sets in  $\mathcal{V}$  is  $U$ , every component of  $U$  must occur in  $\mathcal{V}$ . This completes the proof.  $\square$

**Example 7.5.12.** What are the components of the complement of the set  $D \cup E$  where

$$D = \{(x, y) \in \mathbb{R}^2 : \|(x + 1, y)\| = 1\} \quad \text{and} \quad E = \{x \in \mathbb{R}^2 : \|(x - 1, y)\| = 1\}.$$

**Solution:** The complement of  $D \cup E$  is the union of the open sets

$$\begin{aligned} A &= \{(x, y) \in \mathbb{R}^2 : \|(x + 1, y)\| < 1\}, \\ B &= \{(x, y) \in \mathbb{R}^2 : \|(x - 1, y)\| < 1\}, \quad \text{and} \\ C &= \{(x, y) \in \mathbb{R}^2 : \|(x + 1, y)\| > 1 \text{ and } \|(x - 1, y)\| > 1\}. \end{aligned} \tag{7.5.1}$$

These three sets are pairwise disjoint and each of them is connected. Hence, they must be the components of the complement of  $D \cup E$ , by the previous theorem.

### Exercise Set 7.5

In the first four exercises below, tell whether or not the set  $A$  is connected. If  $A$  is not connected, describe its connected components. Justify your answers.

- $A = \{(x, y) \in \mathbb{R}^2 : \|(x, y)\| < 1\} \cup \{(x, y) \in \mathbb{R}^2 : 1 \leq x \leq 2, y = 0\}$ .
- $A = \{(x, y) \in \mathbb{R}^2 : \|(x, y)\| < 1\} \cup \{(x, y) \in \mathbb{R}^2 : 1 < x \leq 2, y = 0\}$ .
- $A = \{(x, y) \in \mathbb{R}^2 : 1 < \|(x, y)\| < 2\}$ .
- $A = \{(x, y) \in \mathbb{R}^2 : 1 < \|(x, y)\| < 2\} \cup \{(x, y) \in \mathbb{R}^2 : \|(x, y)\| < 1\}$ .
- What are the connected components of the complement of the set of integers in  $\mathbb{R}$ ?
- Prove that the union of a collection of connected subsets of  $\mathbb{R}^d$  with a point in common is also connected.
- Which subsets of  $\mathbb{R}$  are both compact and connected? Justify your answer.
- Give an example of two connected subsets of  $\mathbb{R}^2$  whose intersection is not connected.
- Prove that if  $E$  is an open connected subset of  $\mathbb{R}^d$ , then each pair of points in  $E$  can be connected by a piecewise linear path in  $E$ . Hint: fix a point  $x_0 \in E$  and consider two sets: (1) the set  $U$  of all points in  $E$  that can be connected to  $x_0$  by a piecewise linear path in  $E$ , and (2) the set  $V$  of points in  $E$  that cannot be connected to  $x_0$  by a piecewise connected path in  $E$ .
- Prove that the closure of a connected set is connected.
- Is the interior of a connected set necessarily connected? Justify your answer.
- Are the components of a closed set necessarily closed? Justify your answer.
- Connected sets in a metric space (or any topological space) are defined in the same way as they are in  $\mathbb{R}^d$ . Is it true in general for metric spaces that open balls are connected?

14. A subset of a metric space is said to be *totally disconnected* if its components are all single points. Find a compact, totally disconnected subset of  $\mathbb{R}$  which is not a finite set.
15. Find a compact, totally disconnected subset of  $\mathbb{R}$  (see the previous exercise) which has no isolated points (a point  $x \in E$  is an isolated point of  $E$  if  $\{x\}$  is relatively open in  $E$  – that is, if there is an open set  $U$  such that  $U \cap E = \{x\}$ ).