

Chapter 6

Infinite Series

Infinite series play a fundamental role in mathematics. They are used to approximate complicated or uncomputable quantities or functions by simpler quantities or functions. They are widely used by engineers and scientists in real world applications of mathematics.

6.1 Convergence of Infinite Series

An infinite series of numbers is a formal sum

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots + a_k + \cdots \quad (6.1.1)$$

of an infinite sequence of numbers a_k called the *terms* of the series. We say *formal sum*, because the actual sum may or may not exist. What does it mean for the actual sum to exist? To answer this, we proceed in much the same way that we did in defining improper integrals. We cut off the sum after some finite number n of terms and then take the limit as $n \rightarrow \infty$. That is, we set

$$s_n = \sum_{k=1}^n a_k = a_1 + a_2 + a_3 + \cdots + a_n. \quad (6.1.2)$$

The number s_n is called the n th *partial sum* of the series.

Definition 6.1.1. The series (6.1.1) is said to converge to the number s if $\lim s_n = s$. In this case we write

$$\sum_{k=1}^{\infty} a_k = s.$$

The number s is called the *sum* of the series. If the sequence $\{s_n\}$ diverges, then we say the series (6.1.1) diverges.

It is important to keep firmly in mind the difference between a sequence and a series. A *series* is a formal sum of a *sequence* of numbers. Each series

$$a_1 + a_2 + a_3 + \cdots + a_k + \cdots$$

has two sequences associated to it: the sequence of terms $\{a_k\}$ and the sequence of partial sums $\{s_n\}$, where $s_n = a_1 + a_2 + \cdots + a_n$.

A series (6.1.1) converges if and only if its sequence of partial sums converges. What about the sequence of terms $\{a_n\}$? What is the relationship between convergence of the series and convergence of its sequence of terms? The following theorem gives a partial answer.

Theorem 6.1.2. (Term Test) *If a series $a_1 + a_2 + a_3 + \cdots + a_k + \cdots$ converges, then $\lim a_n = 0$.*

Proof. If the series converges to s , then $\lim s_n = s$, where $\{s_n\}$ is the sequence of partial sums (6.1.2). However, $a_n = s_n - s_{n-1}$ if $n > 1$, and so

$$\lim a_n = \lim s_n - \lim s_{n-1} = s - s = 0.$$

□

The above theorem is called the term test because it provides a test that the terms of a series must pass if the series converges. If the series fails this test – that is, if $\lim a_n$ either fails to exist or is not 0 if it does exist, then the series diverges. However, this test can never be used to prove that a series converges, since it does *not* say that if $\lim a_n = 0$ then the series converges. In fact, the series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} + \cdots$$

has a sequence of terms $\{1/k\}$ which converges to 0, but the series itself does not converge. This series is called the *harmonic series*. To see that it diverges, group the terms in the following way:

$$(1) + \left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots$$

Each group in parentheses is a sum of 2^n terms each of which is at least as big as $1/2^{n+1}$. Thus, each group in parentheses sums to a number greater than or equal to $1/2$. It follows that the 2^n th partial sum of the harmonic series is at least $n/2$. Thus, the sequence of partial sums has limit $+\infty$, and so the series diverges.

Example 6.1.3. Does the series $\sum_{k=1}^{\infty} \frac{k}{2k+1}$ converge?

Solution: No. Its sequence of terms is $\left\{\frac{k}{2k+1}\right\}$ and this sequence has limit $1/2$ as $k \rightarrow \infty$. Since the sequence of terms does not converge to 0, the series fails the term test, and so it diverges.

Example 6.1.4. Does the term test tell us whether $\sum_{k=1}^{\infty} \frac{k}{k^2+1}$ converges?

Solution: If we apply the term test, the result is

$$\lim_{k \rightarrow \infty} \frac{k}{k^2+1} = \lim_{k \rightarrow \infty} \frac{1/k}{1+1/k^2} = 0.$$

The fact that this limit is 0 tells us nothing. The series may or may not converge (in fact, in Example 6.1.14 we will prove that it diverges).

Remark 6.1.5. Although, in our discussion so far, we have assumed that the index of summation k for a series runs from 1 to ∞ , there is really no reason to start the summation at $k = 1$. It could just as easily start at $k = 0$, $k = 2$, or $k = 100$. Our discussion of convergence for series is not effected by where the summation begins, since the only effect on the partial sums s_n of changing the starting point will be to add the same constant to each of them.

Geometric Series

The simplest meaningful series is also one of the most useful. This is the *geometric* series

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + \cdots + ar^k + \cdots . \quad (6.1.3)$$

Here a and r are any two real numbers. The number a is the initial term of the series, while the number r is called the *ratio* for the geometric series, since, for $k > 1$, it is the ratio of the k th term ar^k to the previous term ar^{k-1} . It is the fact that this ratio is independent of k that characterizes the geometric series.

Theorem 6.1.6. *If $a \neq 0$, the geometric series (6.1.3) converges to $\frac{a}{1-r}$ if $|r| < 1$ and diverges if $|r| \geq 1$.*

Proof. The series fails the term test if $|r| \geq 1$, since $\lim ar^k \neq 0$ in this case. Thus, the geometric series diverges if $|r| \geq 1$.

Assume $|r| < 1$. If $s_n = a + ar + ar^2 + \cdots + ar^n$ is the n th partial sum of the series, then

$$rs_n = ar + ar^2 + ar^3 + \cdots + ar^{n+1}$$

and so

$$(1-r)s_n = s_n - rs_n = a - ar^{n+1}.$$

Thus, since $r \neq 1$, we may divide by $1-r$ to obtain

$$s_n = \frac{a - ar^{n+1}}{1-r}.$$

This sequence converges to $\frac{a}{1-r}$ since $\lim r^{n+1} = 0$. □

Example 6.1.7. Does the series $1/2 + 1/4 + 1/8 + \cdots + 1/2^n + \cdots$ converge? If so what does it converge to?

Solution: This is a geometric series with ratio $r = 1/2$ and initial term $a = 1/2$. Thus, it converges to $\frac{1/2}{1 - 1/2} = 1$, by the previous theorem.

Series with Non-Negative Terms

Let $a_1 + a_2 + \cdots + a_k + \cdots$ be a series with $a_k \geq 0$ for all k . Then, its sequence $\{s_n\}$ of partial sums satisfies

$$s_{n+1} = s_n + a_{n+1} \geq s_n.$$

That is, it is a non-decreasing sequence. If such a sequence is bounded above, then it converges by Theorem 2.4.1. If it is not bounded above, then it has limit $+\infty$. This proves the following theorem.

Theorem 6.1.8. *An infinite series of non-negative terms converges if and only if its sequence of partial sums is bounded above.*

Comparison Test

The comparison test stated in most calculus texts follows easily from the preceding theorem (see Exercise 6.1.11). With a little more work, the following, more general, version of the comparison test can also be proved this way. We give a different proof, based on Cauchy's criterion for convergence.

Theorem 6.1.9. (Comparison Test) *Suppose $a_1 + a_2 + \cdots + a_k + \cdots$ and $b_1 + b_2 + \cdots + b_k + \cdots$ are series, with $b_k \geq 0$ for all k , and suppose there are positive constants K and M such that*

$$|a_k| \leq M b_k \quad \text{for all } n \geq K. \quad (6.1.4)$$

Then if $b_1 + b_2 + \cdots + b_k + \cdots$ converges, so does $a_1 + a_2 + \cdots + a_k + \cdots$.

Proof. Let $s_n = \sum_{k=1}^n a_k$ and $t_n = \sum_{k=1}^n b_k$ be the n th partial sums for the two series. If the series with terms b_k converges, then the sequence $\{t_n\}$ converges and, hence, is Cauchy. This implies that, given $\epsilon > 0$, there is an N such that

$$\sum_{k=n+1}^m b_k = |t_m - t_n| \leq \frac{\epsilon}{M} \quad \text{whenever } m \geq n > N.$$

Then (6.1.4) implies that

$$|s_m - s_n| = \left| \sum_{k=n+1}^m a_k \right| \leq \sum_{k=n+1}^m |a_k| \leq M \sum_{k=n+1}^m b_k < \epsilon$$

whenever $m \geq n > \max(N, K)$. This implies that $\{s_n\}$ is a Cauchy sequence and, hence, converges. It follows that the series $\sum_{k=1}^{\infty} a_k$ converges. \square

Suppose $\sum_{k=1}^{\infty} a_k$ is an arbitrary series. If we set $b_k = |a_k|$, then the condition $|a_k| \leq Mb_k$ of the previous theorem is satisfied with $M = 1$ and $K = 1$. This observation yields the following corollary.

Corollary 6.1.10. *If $\sum_{k=1}^{\infty} |a_k|$ converges, then so does $\sum_{k=1}^{\infty} a_k$.*

This leads to the following definition.

Definition 6.1.11. A series $\sum_{k=1}^{\infty} a_k$ is said to *converge absolutely* if the series $\sum_{k=1}^{\infty} |a_k|$ converges.

Thus, Corollary 6.1.10 asserts that if a series converges absolutely, then it converges.

Example 6.1.12. Does the series $\sum_{k=1}^{\infty} \frac{k}{2^k}$ converge? Why?

Solution: Since $\lim_{k \rightarrow \infty} \frac{k}{2^{k/2}} = 0$ (l'Hôpital's Rule), there is an N such that

$$\frac{k}{2^{k/2}} < 1 \quad \text{whenever } k > N.$$

Then

$$\frac{k}{2^k} < \frac{1}{2^{k/2}} = \frac{1}{(\sqrt{2})^k} \quad \text{whenever } k > N.$$

Since the series $\sum_{k=1}^{\infty} \frac{1}{(\sqrt{2})^k}$ is a convergent geometric series, the series $\sum_{k=1}^{\infty} \frac{k}{2^k}$ converges by the comparison test.

Example 6.1.13. Does the series $\sum_{k=1}^{\infty} (-1)^k \frac{k}{2^k}$ converge? Why?

Solution: By the previous exercise, the series $\sum_{k=1}^{\infty} \frac{k}{2^k}$ converges and this means that $\sum_{k=1}^{\infty} (-1)^k \frac{k}{2^k}$ converges absolutely and, hence, converges by Corollary 6.1.10.

The comparison test can also be used to prove that a series diverges.

Example 6.1.14. Prove that the series $\sum_{k=1}^{\infty} \frac{k}{k^2+1}$ diverges.

Solution: We compare with the harmonic series. Since $k^2+1 \leq 2k^2$ for $k \in \mathbb{N}$, we have

$$\frac{1}{k} \leq 2 \frac{k}{k^2+1} \quad \text{for all } k \in \mathbb{N}.$$

If the series $\sum_{k=1}^{\infty} \frac{k}{k^2+1}$ converges, then so does $\sum_{k=1}^{\infty} \frac{1}{k}$ by the comparison test.

However, the harmonic series diverges. Therefore $\sum_{k=1}^{\infty} \frac{k}{k^2+1}$ also diverges.

Exercise Set 6.1

In each of the following six exercises, determine whether the indicated series converges. Justify your answer.

$$1. \sum_{k=2}^{\infty} \frac{k-1}{2k+1}.$$

$$2. \sum_{k=1}^{\infty} \frac{1}{2^k + k - 1}.$$

$$3. \sum_{k=0}^{\infty} \frac{2^{k+1}}{3^k}.$$

$$4. \sum_{k=1}^{\infty} \frac{k^2 - 3k + 1}{3k^2 + k - 2}.$$

$$5. \sum_{k=1}^{\infty} \frac{k^2}{4^k}.$$

$$6. \sum_{k=1}^{\infty} \frac{k}{k^2 - k + 2}.$$

In each of the next four exercises, determine whether the indicated series converges absolutely. Justify your answer.

$$7. \sum_{k=0}^{\infty} (-2/3)^k.$$

$$8. \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}}.$$

$$9. \sum_{k=1}^{\infty} \frac{\sin k}{2^k}.$$

$$10. \sum_{k=1}^{\infty} \frac{(-1)^k}{\ln(1+k)}.$$

11. Prove the following weak version of the comparison test using Theorem 6.1.8: If $a_1 + a_2 + \cdots + a_k + \cdots$ and $b_1 + b_2 + \cdots + b_k + \cdots$ are series of non-negative terms with $a_k \leq b_k$ for all k , then if $b_1 + b_2 + \cdots + b_k + \cdots$ converges, so does $a_1 + a_2 + \cdots + a_k + \cdots$.
12. Consider the decimal expansion $.d_1d_2d_3d_4 \cdots$ of a real number between 0 and 1, where $\{d_k\}$ is a sequence of integers between 0 and 9. This decimal expansion represents the sum of a certain infinite series. What series is it and why does it converge?
13. Show that every real number in the interval $[0, 1]$ has a decimal expansion as described in the previous exercise.

6.2 Tests for Convergence

In this section we will develop the standard tests for convergence of infinite series. Most of these are based on Theorem 6.1.8 or Theorem 6.1.9.

Integral Test

Theorem 6.2.1. *Suppose f is a positive, non-increasing function on $[1, \infty)$ and $a_k = f(k)$ for each $k \in \mathbb{N}$. Then the series $\sum_{k=1}^{\infty} a_k$ converges if and only if the improper integral $\int_1^{\infty} f(x) dx$ converges.*

Proof. Consider the function $g(x)$ on $[1, \infty)$ which, for each $k \in \mathbb{N}$, is constant on the interval $[k, k+1)$ and equal to $f(k)$ at k . That is,

$$g(x) = f(k) = a_k \quad \text{if } k \leq x < k+1, \quad k \in \mathbb{N}.$$

This is a piecewise continuous function, hence integrable on any finite interval $[1, b)$. Also, since f is non-increasing, it follows that

$$g(x+1) \leq f(x) \leq g(x) \quad \text{for all } x \in [1, \infty).$$

(see Figure 6.1). On integrating from 1 to n , this yields

$$\int_1^n g(x+1) dx \leq \int_1^n f(x) dx \leq \int_1^n g(x) dx.$$

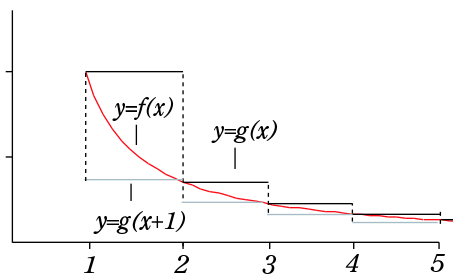


Figure 6.1: Setup for Proof of the Integral Test.

However, by Exercise 6.2.9,

$$\int_1^n g(x+1) dx = \sum_{k=2}^n a_k \quad \text{and} \quad \int_1^n g(x) dx = \sum_{k=1}^{n-1} a_k. \quad (6.2.1)$$

If $s_n = \sum_{k=1}^n a_k$, then this implies that

$$s_n - a_1 \leq \int_1^n f(x) dx \leq s_{n-1}.$$

It follows that the sequence of partial sums $\{s_n\}$ is bounded above if and only if the increasing function of b , $\int_1^b f(x) dx$, is bounded above. A non-decreasing sequence converges if and only if it is bounded above and a non-decreasing function on $[1, \infty)$ has a finite limit at ∞ if and only if it is bounded above. Thus, the series converges if and only if the improper integral converges. \square

Example 6.2.2. A p -series is a series of the form $\sum_{k=1}^{\infty} \frac{1}{k^p}$, where $p > 0$. Prove that a p -series converges if and only if $p > 1$.

Solution We apply the integral test for the function $f(x) = 1/x^p$. Note that this is a positive, decreasing function on $[1, \infty)$ and $f(k) = 1/k^p$ for $k \in \mathbb{N}$. If $p \neq 1$ we have

$$\int_1^b \frac{1}{x^p} dx = \frac{b^{1-p} - 1}{1-p}.$$

As $b \rightarrow \infty$, this has limit $\frac{1}{p-1}$ if $p > 1$ and $+\infty$ if $p < 1$. Thus, the p -series converges for $p > 1$ and diverges for $p < 1$ by the Integral Test.

For $p = 1$, the p -series is the harmonic series and we already know it diverges. However, it is instructive to see how this follows from the Integral Test.

In the case $p = 1$, the function f is $f(x) = 1/x$. We have

$$\int_1^b \frac{1}{x} dx = \ln b,$$

and this has limit $+\infty$ as $b \rightarrow \infty$. Thus, applying the Integral Test gives another proof that the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges.

Example 6.2.3. Does the series $\sum_{k=1}^{\infty} \frac{3\sqrt{k}}{2k^2 - 1}$ converge or diverge. Justify your answer.

Solution: For large k , $\frac{3\sqrt{k}}{2k^2 - 1}$ is close to $\frac{3}{k^{3/2}}$. This suggests we do a comparison with the p -series $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$.

We have $2k^2 - 1 \geq k^2$ for all $k \geq 1$ and so

$$\frac{3\sqrt{k}}{2k^2 - 1} \leq \frac{3\sqrt{k}}{k^2} = \frac{3}{k^{3/2}}.$$

Since the p -series with $p = 3/2$ converges, so does our series, by the comparison test.

Root Test

This test is particularly important in the study of power series.

Theorem 6.2.4. Given an infinite series $\sum_{k=1}^{\infty} a_k$, let

$$\rho = \limsup |a_k|^{1/k}.$$

Then the series converges absolutely if $\rho < 1$ and diverges if $\rho > 1$.

Proof. Recall that

$$\limsup |a_k|^{1/k} = \lim t_n \quad \text{where} \quad t_n = \sup\{|a_k|^{1/k} : k \geq n\}.$$

Also recall that $\{t_n\}$ is a non-increasing sequence. Thus, if $\rho > 1$, then

$$t_n = \sup\{|a_k|^{1/k} : k \geq n\} > 1 \quad \text{for all} \quad n \in \mathbb{N}.$$

This means that, for every $n \in \mathbb{N}$, there is an $k \geq n$ such that $|a_k|^{1/k} > 1$. Then $|a_k| > 1$ also. It follows that the sequence of terms $\{a_k\}$ does not have limit 0. Hence, the series fails the term test and must diverge in this case.

If $\rho < 1$, we can choose r such that $\rho < r < 1$. Then there is an N such that

$$t_n < r \quad \text{whenever} \quad n > N$$

and this implies that

$$|a_k|^{1/k} < r \quad \text{whenever } k > N.$$

This, in turn, implies that

$$|a_k| < r^k \quad \text{whenever } k > N.$$

Thus, the series $\sum_{k=1}^{\infty} |a_k|$ converges in this case, by comparison with the geometric series with ratio $r < 1$. Therefore, the original series converges absolutely. \square

Note that the root test tells us nothing about convergence if the number ρ turns out to be 1.

Example 6.2.5. Does the series $\sum_{k=1}^{\infty} k(9/10)^k$ converge? Why?

Solution: We apply the root test. In this case, the lim sup of Theorem 6.2.4 is actually a limit, since the limit exists. In fact,

$$\rho = \lim k^{1/k}(9/10) = (9/10) \lim k^{1/k} = 9/10 < 1,$$

since $\lim k^{1/k} = 1$ by Exercise 2.3.12. By the root test, the series converges.

Ratio Test

Theorem 6.2.6. Given a series $\sum_{k=1}^{\infty} a_k$, let

$$r = \lim \frac{|a_{k+1}|}{|a_k|} \tag{6.2.2}$$

provided this limit exists. Then the series converges absolutely if $r < 1$ and diverges if $r > 1$.

Proof. Observe first that, for the limit defining r to exist, the numbers a_k must eventually be all non-zero – otherwise, the ratio $|a_{k+1}|/|a_k|$ would be undefined or $+\infty$ for infinitely many k .

If $r > 1$, then there is an N such that

$$|a_k| > 0 \quad \text{and} \quad \frac{|a_{k+1}|}{|a_k|} > 1 \quad \text{for all } k \geq N.$$

Then, for $k > N$

$$|a_k| = \frac{|a_k|}{|a_{k-1}|} \frac{|a_{k-1}|}{|a_{k-2}|} \cdots \frac{|a_{N+2}|}{|a_{N+1}|} \frac{|a_{N+1}|}{|a_N|} |a_N| > |a_N|.$$

This implies the sequence of terms $\{a_k\}$ fails to have limit 0, and the sequence diverges by the term test.

If $r < 1$ we choose a t such that $r < t < 1$. Since (6.2.2) holds, there is an N such that

$$\frac{|a_{k+1}|}{|a_k|} < t \quad \text{whenever } n \geq N.$$

Then, for $k > N$,

$$|a_k| = \frac{|a_k|}{|a_{k-1}|} \frac{|a_{k-1}|}{|a_{k-2}|} \cdots \frac{|a_{N+2}|}{|a_{N+1}|} \frac{|a_{N+1}|}{|a_N|} |a_N| < t^{k-N} |a_N|.$$

Thus, $|a_k| < t^k \frac{|a_N|}{t^N}$ whenever $k > N$. By comparison with the geometric series with ratio t , the series converges. \square

The ratio test tends to work well on series where the terms a_k involve products of an increasing number of factors – things like factorials. These are generally more difficult to attack with the root test than with the ratio test.

Example 6.2.7. Does the series $\sum_{k=1}^{\infty} \frac{k!}{k^k}$ converge? Why?

Solution: We apply the ratio test.

$$\begin{aligned} r &= \lim \frac{(k+1)!}{(k+1)^{k+1}} \div \frac{k!}{k^k} = \lim \frac{(k+1)!k^k}{(k+1)^{k+1}k!} \\ &= \lim \left(\frac{k}{k+1} \right)^k = \lim \frac{1}{(1+1/k)^k} = \frac{1}{e} < 1. \end{aligned}$$

Hence, the series converges by the ratio test.

For many series, the ratio test and the root test work equally well. However, the ratio test is not applicable in many situations where the root test works well.

Example 6.2.8. Prove that the series $1/3 + 1/2^2 + 1/3^3 + 1/2^4 + 1/3^5 + \cdots$ converges.

Solution: This one can easily be done using the comparison test. However, it is instructive to see how attempts to use the ratio test and root test work out. The ratio test doesn't work, because the successive ratios are

$$3/4, 4/27, 27/16, 16/243, 243/64 \cdots,$$

and this sequence of numbers has no limit.

On the other hand, the root test yields that ρ is the lim sup of the sequence

$$1/3, 1/2, 1/3, 1/2, 1/3, \cdots.$$

That is, $\rho = 1/2$. Therefore, the series converges by the root test.

Exercise Set 6.2

In each of the following eight exercises, determine whether the indicated series converges. Justify your answer by indicating what test to use and then carrying out the details of the application of that test.

$$1. \sum_{k=2}^{\infty} \frac{1}{k \ln k}.$$

$$2. \sum_{k=1}^{\infty} \frac{\ln k}{k^2}$$

$$3. \sum_{k=1}^{\infty} \frac{k2^k}{3^k}.$$

$$4. \sum_{k=0}^{\infty} \frac{5^k}{k!}.$$

$$5. \sum_{k=1}^{\infty} \frac{k}{(3 + (-1)^k)^k}.$$

$$6. \sum_{k=1}^{\infty} \frac{k!}{4^k}.$$

$$7. \sum_{k=1}^{\infty} \frac{\sqrt{k}}{k^2 - k + 2}.$$

$$8. \sum_{k=1}^{\infty} k e^{-\sqrt{k}}.$$

9. Verify the integral formulas (6.2.1) used in the proof of the Integral Test.

10. Prove that if $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are convergent series and c is a constant, then $\sum_{k=1}^{\infty} ca_k$ and $\sum_{k=1}^{\infty} (a_k + b_k)$ are also convergent. Furthermore,

$$\sum_{k=1}^{\infty} ca_k = c \sum_{k=1}^{\infty} a_k, \quad \text{and}$$

$$\sum_{k=1}^{\infty} (a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k.$$

11. Prove that if $\sum_{k=1}^{\infty} a_k$ converges absolutely and $\{b_k\}$ is a bounded sequence, then $\sum_{k=1}^{\infty} a_k b_k$ also converges absolutely.
12. Prove that if $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are series and $a_k = b_k$ except for finitely many values of k , then the two series either both converge or they both diverge.

6.3 Absolute and Conditional Convergence

By Corollary 6.1.10, if a series converges absolutely, then it converges. The converse is not true. As we shall see, it is possible for a series to converge even though the corresponding series of absolute values does not converge.

Definition 6.3.1. A series which converges, but does not converge absolutely is said to converge *conditionally*.

Thus, a conditionally convergent series is one which converges, but its corresponding series of absolute values does not converge. For examples of conditionally convergent series, we turn to alternating series.

Alternating Series

An alternating series is one in which the terms alternate in sign – each positive term is followed by a negative term and vice-versa. Under reasonable additional conditions, such a series will converge.

Theorem 6.3.2. (Alternating Series Test) *Let $\{a_k\}$ be a non-increasing sequence of non-negative numbers which converges to 0. Then the series*

$$\sum_{k=1}^{\infty} (-1)^{k+1} a_k = a_1 - a_2 + a_3 - a_4 + \cdots$$

converges. In fact, if s_n is the n th partial sum of this series and $s = \lim s_n$, then

$$|s - s_n| \leq a_{n+1} \quad \text{for all } n.$$

Proof. Since $\{a_k\}$ is a non-increasing sequence of non-negative numbers, we have $a_k - a_{k+1} \geq 0$ for all k . For n odd, this means

$$s_{n+1} \leq s_{n+1} + a_{n+2} = s_{n+2} = s_n - (a_{n+1} - a_{n+2}) \leq s_n.$$

That is,

$$s_{n+1} \leq s_{n+2} \leq s_n \quad \text{for odd } n.$$

Similarly,

$$s_n \leq s_{n+2} \leq s_{n+1} \quad \text{for even } n.$$

Thus, $s_2 \leq s_3 \leq s_1$ and, after that, each term of the sequence $\{s_n\}$ lies between the previous two terms. It follows that

$$s_2 \leq s_4 \leq s_6 \leq \cdots \leq s_{2n} \leq s_{2n+1} \leq \cdots \leq s_5 \leq s_3 \leq s_1.$$

Hence, the subsequence of $\{s_n\}$ consisting of terms with odd index n forms a non-increasing sequence which is bounded below, while the subsequence of terms with even index n forms a non-decreasing sequence which is bounded above. These two monotone, bounded sequences converge, and they must converge to the same limit s because

$$|s_{n+1} - s_n| = a_{n+1}$$

and the sequence $\{a_n\}$ converges to 0. Since s is between s_n and s_{n+1} for each n , this also shows that

$$|s - s_n| \leq a_{n+1},$$

as claimed. □

An alternating p -series is a series of the form

$$1 - \frac{1}{2^p} + \frac{1}{3^p} - \cdots + (-1)^{k-1} \frac{1}{k^p} + \cdots$$

where $p > 0$.

Example 6.3.3. Show that each alternating p -series with $0 < p \leq 1$ converges conditionally.

Solution: The alternating p -series satisfies the conditions of the alternating series test, since $\{1/k^p\}$ is a decreasing sequence which converges to 0. Thus, the alternating p -series converges for all $p > 0$. However, the ordinary p -series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ diverges if $p \leq 1$ (Example 6.2.2). Thus, the alternating p -series converges conditionally for $0 < p \leq 1$.

In particular, the alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{k-1} \frac{1}{k} + \cdots$$

converges conditionally.

Absolute versus Conditional Convergence

Absolute convergence is a much stronger condition than conditional convergence. The importance of the concept of absolute convergence stems from the fact that, if the terms of an absolutely convergent series are rearranged to form a new series, then the new series converges to the same number as the original series (Theorem 6.3.5 below). This is not true of conditionally convergent

series – in fact, it fails spectacularly. A conditionally convergent series can be rearranged so as to diverge to ∞ or $-\infty$ or to converge to any given number (Theorem 6.3.4 below).

By a *rearrangement* of a series $\sum_{k=1}^{\infty} a_k$ we mean a series of the form $\sum_{j=1}^{\infty} a_{k(j)}$,

where $k(j)$ is a one-to-one function from \mathbb{N} onto \mathbb{N} . In other words, the rearranged series has exactly the same terms as the original series, but arranged in a different order.

Theorem 6.3.4. *A conditionally convergent series has, for each extended real number L , a rearrangement that converges to L .*

Proof. If $\sum_{k=1}^{\infty} a_k$ is a conditionally convergent series, then by Exercise 6.3.7, the series of positive terms of this series diverges, as does the series of negative terms. Since the series of positive terms diverges, its sequence of partial sums is unbounded and, hence, has limit ∞ . Similarly, for the series of negative terms, the partial sums have limit $-\infty$.

We will prove the theorem in the case where L is a real number. The cases where L is ∞ or $-\infty$ are left to the exercises.

Given a number L , we will define a sequence $\{b_j\}$ inductively in the following way: We let b_1 be the first positive term in $\{a_k\}$ if $0 < L$ and the first non-positive term in $\{a_k\}$ if $L \leq 0$. Suppose b_1, b_2, \dots, b_n have been chosen. We set

$$s_n = \sum_{j=1}^n b_j$$

and choose b_{n+1} according to the following rule: If $s_n < L$ we choose b_{n+1} to be the first positive term in $\{a_k\}$ that has not already been used. If $L \leq s_n$ we choose b_{n+1} to be the first non-positive term in $\{a_k\}$ that has not already been used. This defines the sequence $\{b_j\}$ inductively. The series $\sum_{j=1}^{\infty} b_j$ defined in

this way has the following properties:

(1) Each successive partial sum s_n is either as close or closer to L than its predecessor s_{n-1} , or one of them is less than L and the other is greater than or equal to L . In the latter case, the distance from s_n to L is less than $|s_n - s_{n-1}| = |b_n|$. We call n a *crossing* integer in this case.

(2) There are infinitely many crossing integers. Our description of $\sum_{j=1}^{\infty} b_j$

involves adding successive positive terms until we reach or exceed L and then adding successive non-positive terms until we fall below L . Since the series of positive terms and the series of negative terms both diverge, no matter where a given partial sum lies we will always be able to add enough of the remaining positive terms to reach or exceed L or add enough of the remaining non-positive terms to fall below L . Thus, crossing L will occur infinitely often.

(3) All the terms of $\{a_k\}$ are used in constructing the sequence $\{b_j\}$, since at each step we are selecting the first positive term not already chosen or the first non-positive term not already chosen and both cases occur infinitely often. Thus, each a_k will be chosen eventually. Also, at each stage we only choose from the terms not already chosen, and so each a_k will be used just once. This means that the sequence $\{b_j\}$ is a rearrangement of the sequence $\{a_k\}$.

(4) Since $\sum_{k=1}^{\infty} a_k$ converges, we have $\lim a_k = 0$, and this implies $\lim b_j = 0$

also. This is proved as follows: If $\epsilon > 0$, there is an N such that $|a_k| < \epsilon$ whenever $k > N$. However, if we choose M to be an integer such that, by stage M in our construction all the terms a_1, a_2, \dots, a_N have been chosen, then $j > M$ implies that b_j is not one of these terms and, hence, is a term a_k with $k > N$. This, in turn, implies that $|b_j| < \epsilon$.

Now (1) and (2) and (4) imply that $\lim s_n = L$. That is, the crossing integers define a subsequence of $\{s_n\}$ (by (2)) that is converging to L (by (1) and (4)) and, between two successive crossing integers, the sequence $\{s_n\}$ stays at least as close to L as it was at the first crossing integer of the pair (by (1)).

Thus, $\sum_{k=1}^{\infty} b_k$ is a rearrangement of $\sum_{k=1}^{\infty} a_k$ which converges to L . \square

The above theorem illustrates that a conditionally convergent series is a rather unstable object, since its sum is dependent on the order in which the terms are added. On the other hand, an absolutely convergent series is quite stable in the sense that the sum is always the same regardless of the order in which the terms are summed. That is the content of the next theorem.

Theorem 6.3.5. *Each rearrangement of an absolutely convergent series converges to the same number as the original series.*

Proof. Let $\sum_{k=1}^{\infty} a_k$ be an absolutely convergent series which converges to the

number s . Since this series is absolutely convergent, the series $\sum_{k=1}^{\infty} |a_k|$ also converges to a number t . The difference between t and the n th partial sum of this series is

$$\sum_{k=n+1}^{\infty} |a_k|.$$

Because the partial sums converge to t , given $\epsilon > 0$, there is an N such that

$$\sum_{k=n+1}^{\infty} |a_k| < \epsilon/2 \quad \text{for all } n > N. \quad (6.3.1)$$

We fix one such n , and we also choose it to be large enough so that

$$\left| s - \sum_{k=1}^n a_k \right| < \epsilon/2. \quad (6.3.2)$$

Now suppose $\sum_{j=1}^{\infty} b_j$ is a rearrangement of $\sum_{k=1}^{\infty} a_k$. Then $b_j = a_{k(j)}$ for some one-to-one function $k(j)$ of \mathbb{N} onto \mathbb{N} . Let J be the largest value of j for which $k(j) \leq n$. Then the terms a_1, a_2, \dots, a_n of the original series all appear as terms in the partial sum $\sum_{j=1}^m b_j$ as long as $m \geq J$. For such an m , the expression

$$\sum_{j=1}^m b_j - \sum_{k=1}^n a_k$$

is a finite sum of terms a_k with $k > n$. By (6.3.1) and the triangle inequality, such a sum must have absolute value less than $\epsilon/2$. This, combined with (6.3.2), implies that

$$\left| s - \sum_{j=1}^m b_j \right| < \epsilon \quad \text{whenever } m \geq J.$$

Hence, the series $\sum_{j=1}^{\infty} b_j$ also converges to s . \square

Products of Series

In calculus we are taught how to multiply two power series. The formula for doing this relies on the following result, which requires that the two series be absolutely convergent (see Exercise 6.3.12).

Theorem 6.3.6. Let $\sum_{k=0}^{\infty} a_k$ and $\sum_{j=0}^{\infty} b_j$ be two absolutely convergent series.

Then

$$\left(\sum_{k=0}^{\infty} a_k \right) \left(\sum_{j=0}^{\infty} b_j \right) = \sum_{n=0}^{\infty} \sum_{k=0}^n a_k b_{n-k}, \quad (6.3.3)$$

where the series on the right also converges absolutely.

Proof. Consider the set $S = \{a_k b_j : j, k \in \mathbb{N}\}$. The numbers in this set can be displayed in an infinite array or *matrix* as follows:

$$\begin{array}{cccccc} a_0 b_0 & a_1 b_0 & a_2 b_0 & \cdots & a_n b_0 & \cdots \\ a_0 b_1 & a_1 b_1 & a_2 b_1 & \cdots & a_n b_1 & \cdots \\ a_0 b_2 & a_1 b_2 & a_2 b_2 & \cdots & a_n b_2 & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \cdots \\ a_0 b_n & a_1 b_n & a_2 b_n & \cdots & a_n b_n & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \cdots \end{array} \quad (6.3.4)$$

The sum of the absolute values of the members of any finite subset of this set is less than

$$M = \left(\sum_{k=0}^{\infty} |a_k| \right) \left(\sum_{j=0}^{\infty} |b_j| \right) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |a_k b_j|.$$

Since M is finite, this means that, given any series formed by summing the elements of S in some order, the corresponding series of absolute values will have partial sums bounded above by M . Such a series must converge. Thus, each series formed by summing the elements of S in some order will be absolutely convergent, and all such series will converge to the same number by the previous theorem.

One series formed by summing the elements of S is

$$a_0 b_0 + a_0 b_1 + a_1 b_1 + a_1 b_0 + a_0 b_2 + a_1 b_2 + a_2 b_2 + a_2 b_1 + a_2 b_0 + \cdots .$$

That is, in the array (6.3.4), for successive values of n , we sum from left to right along the n th row to the main diagonal and then along n th column from the main diagonal back to the top row. The n^2 partial sum of this sequence is

$$\left(\sum_{k=0}^n a_k \right) \left(\sum_{j=0}^n b_j \right) = \sum_{j=0}^n \sum_{k=0}^n a_k b_j.$$

This sequence of numbers converges to the left side of equation (6.3.3).

Another way of summing the numbers in the set S yields the series

$$\sum_{n=0}^{\infty} \sum_{k=0}^n a_k b_{n-k}.$$

This is obtained by summing the array (6.3.4) along diagonals of the form $k + j = n$ for successive values of n . The resulting sum is the right side of equation (6.3.3). Since these two series must sum to the same number by the previous theorem, Equation (6.3.3) is true. \square

Exercise Set 6.3

In each of the next five exercises, determine whether the given series converges absolutely, converges conditionally, or diverges. Justify your answer.

- $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{1/3}}.$

- $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}.$

3.
$$\sum_{k=2}^{\infty} \frac{(-1)^k}{\ln(k)}.$$

4.
$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{2^k}{2^k + k^2}.$$

5.
$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^{2+(-1)^k}}.$$

6. Give an example of two convergent series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ such that the series $\sum_{k=1}^{\infty} a_k b_k$ diverges.

7. If $\sum_{k=1}^{\infty} a_k$ is a series, we set $a_k^+ = a_k$ if $a_k > 0$, $a_k^+ = 0$ if $a_k \leq 0$ and $a_k^- = a_k$ if $a_k < 0$, $a_k^- = 0$ if $a_k > 0$. Prove that if the series is conditionally convergent, then both $\sum_{k=1}^{\infty} a_k^-$ and $\sum_{k=1}^{\infty} a_k^+$ diverge.

8. Approximate the sum of the alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{n-1} \frac{1}{n} + \cdots$$

to within .01 by computing an appropriate partial sum. You will need a calculator or computer.

9. For the alternating harmonic series of the preceding exercise, follow the procedure used in the proof of Theorem 6.3.4 to rearrange the series so that it converges to $\sqrt{2}$. Carry out this procedure until the partial sum of your new series is within .02 of $\sqrt{2}$. You will need a calculator or a computer.

10. Show how to modify the proof of Theorem 6.3.4 to cover the cases $L = \infty$ and $L = -\infty$.

11. The geometric series $\sum_{k=0}^{\infty} 2^{-k}$ converges to 2. Use the product formula of

Theorem 6.3.6 to show that the series $\sum_{k=0}^{\infty} (k+1)2^{-k}$ converges to 4.

12. Show that the product formula in Theorem 6.3.6 may fail to be true if the series involved are not absolutely convergent. Hint: consider the case

where both series are $\sum_{k=0}^{\infty} \frac{(-1)^k}{\sqrt{k+1}}$.

6.4 Power Series

One of the most useful and widely used techniques of modern mathematics is that of expressing a complicated function as the sum of a series of simple functions. Examples include power series, Fourier series, and various eigenfunction expansions for differential equations. All involve series whose terms are functions rather than numbers.

Series of Functions

Consider a series of the form

$$\sum_{k=1}^{\infty} f_k(x) = f_1(x) + f_2(x) + f_3(x) + \cdots + f_k(x) + \cdots, \quad (6.4.1)$$

where I is an interval in \mathbb{R} and each of the functions $f_k(x)$ is a function defined on I . For each fixed value of $x \in I$, this is just an ordinary series of numbers and it may or may not converge. The series may converge for some values of x and not for others. On the subset of I for which the series does converge, it defines a new function

$$g(x) = \sum_{k=1}^{\infty} f_k(x).$$

This function is the limit of the sequence of functions

$$g_n(x) = \sum_{k=1}^n f_k(x)$$

obtained by taking the partial sums of the series.

There are many questions one can ask about this situation: if the functions $f_k(x)$ are continuous or differentiable, is the same thing true of the function g that the series converges to? Can we integrate the function g over a subinterval of I by integrating the series term by term? When can we differentiate g by differentiating the series term by term? We can give satisfactory answers to a couple of these questions right away.

Definition 6.4.1. We say a series of functions (6.4.1) converges *uniformly* to g on I if its sequence of partial sums $\{g_n\}$ converges uniformly to g .

Theorem 6.4.2. *If each f_k is a continuous function on I and the series (6.4.1) converges uniformly to g on I , then g is also continuous on I .*

Proof. If the series (6.4.1) converges uniformly to g on I , then its sequence of partial sums $\{g_n\}$ converges uniformly to g on I . Each g_n is a finite sum of functions f_k which are continuous on I and, hence, is also continuous on I . Since the limit of a uniformly convergent sequence of continuous functions is continuous (Theorem 3.4.4), we conclude that g is continuous on I . \square

The proof of the next theorem is very similar – the theorem follows directly from the analogous result about integrating the uniform limit of a sequence of functions (Exercise 5.2.13). We leave the details to the exercises.

Theorem 6.4.3. *If each f_k is continuous on $[a, b]$ and the series (6.4.1) converges uniformly to g on $[a, b]$, then*

$$\int_a^b g(x) dx = \sum_{k=1}^{\infty} \int_a^b f_k(x) dx.$$

This means, in particular, that the series on the right converges.

Weierstrass M-test

The following is a test for uniform convergence of a series. It follows from an analogous test for uniform convergence of sequences (Theorem 3.4.6).

Theorem 6.4.4. (Weierstrass M-test) *A series of functions (6.4.1) on an interval I converges uniformly on I if there is a convergent series of positive terms*

$$\sum_{k=1}^{\infty} M_k$$

such that $|f_k(x)| \leq M_k$ for all $x \in I$ and all $k \in \mathbb{N}$.

Proof. By the comparison test, at each x the series (6.4.1) converges to a number $g(x)$. If

$$g_n(x) = \sum_{k=1}^n f_k(x)$$

then

$$\begin{aligned} |g(x) - g_n(x)| &= \left| \sum_{k=n+1}^{\infty} f_k(x) \right| \leq \sum_{k=n+1}^{\infty} |f_k(x)| \\ &\leq \sum_{k=n+1}^{\infty} M_k = S - S_n \end{aligned}$$

where S and S_n are the sum and n th partial sum of the series $\sum_{k=1}^{\infty} M_k$. Since this series converges, $\lim(S - S_n) = 0$. The theorem now follows from Theorem 3.4.6. \square

Example 6.4.5. Analyze the Fourier Series

$$\sum_{k=1}^{\infty} \frac{\cos kx}{k^2},$$

using the preceding three theorems.

Solution: We have $\left| \frac{\cos kx}{k^2} \right| \leq \frac{1}{k^2}$ for all $x \in \mathbb{R}$. The series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, since it is a p -series with $p > 1$. Thus, it follows from the Weierstrass M -test that the series $\sum_{k=1}^{\infty} \frac{\cos kx}{k^2}$ converges uniformly on \mathbb{R} . The function g that it converges to is continuous on \mathbb{R} by Theorem 6.4.2. On every bounded interval $[a, b]$, we have

$$\int_a^b g(x) dx = \sum_{k=1}^{\infty} \frac{1}{k^2} \int_a^b \cos kx dx = \sum_{k=1}^{\infty} \frac{1}{k^3} (\sin kb - \sin ka),$$

also by Theorem 6.4.2.

Power Series

A *power series* centered at a is a series of the form

$$\sum_{k=0}^{\infty} c_k (x - a)^k \tag{6.4.2}$$

This is a series with terms $c_k(x - a)^k$ which are very simple – they are simple monomials in $(x - a)$ and, hence, each is defined on all of \mathbb{R} , is continuous and, in fact, has derivatives of all orders. The partial sums of a power series are polynomials.

A power series may converge for some values of x and not for others. The next theorem tells us a great deal about this question.

Theorem 6.4.6. *Given a power series (6.4.2), let*

$$R = \frac{1}{\limsup |c_k|^{1/k}},$$

where we interpret R to be ∞ (resp. 0) if $\limsup |c_k|^{1/k}$ is 0 (resp. ∞).

If $R > 0$, then the series (6.4.2) converges for each x with $|x - a| < R$ and diverges for each x with $|x - a| > R$. Furthermore, the series converges uniformly on every interval of the form $[a - r, a + r]$ with $0 < r < R$. If $R = 0$, then the series converges only when $x = a$.

Proof. We first suppose $R > 0$. Given any $r > 0$, we have

$$\limsup |c_k r^k|^{1/k} = r \limsup |c_k|^{1/k} = \frac{r}{R}. \tag{6.4.3}$$

Now suppose $|x - a| = r > R$. Then $|c_k(x - a)^k| = |c_k| r^k$ and the series (6.4.2) diverges, by (6.4.3) and the root test.

On the other hand, if $r < R$ and $|x - a| \leq r$, then $|c_k(x - a)^k| \leq |c_k| r^k$. In this case $\sum_{k=1}^{\infty} |c_k| r^k$ converges, by the root test and (6.4.3). Then the Weierstrass

M-test implies that the series (6.4.2) converges uniformly on the closed interval $[a - r, a + r] = \{x : |x - a| \leq r\}$.

The uniform convergence of (6.4.2) on $[a - r, a + r]$ for every $r < R$ implies that the series converges on $(a - R, a + R)$, since for every x in this interval, there is an $r < R$ such that x is also in the interval $[a - r, a + r]$.

If $R = 0$ – that is, if $\limsup |c_k|^{1/k} = \infty$ – then the only value of x that will lead to $\limsup |c_k(x - a)^k|^{1/k} < 1$ is $x = a$. Thus, the power series converges only at $x = a$ in this case. □

The above theorem tells us that the convergence set for a power series (6.4.2) is an interval of radius $R = (\limsup |c_k|^{1/k})^{-1}$, centered at a . The number R is called the *radius of convergence* of the series. Since the theorem says nothing when $|x - a| = R$, it does not tell us whether this interval is open, closed, or half-open, half-closed. Each of these possibilities occurs.

Example 6.4.7. Give examples where the three possibilities mentioned in the previous paragraph occur.

Solution The examples are

$$(a) \sum_{k=0}^{\infty} x^k \quad (b) \sum_{k=0}^{\infty} \frac{x^k}{k} \quad (c) \sum_{k=0}^{\infty} \frac{x^k}{k^2}.$$

In each case, the radius of convergence R is 1, since

$$1 = \lim k^{1/k} = (\lim k^{1/k})^2 = \lim (k^2)^{1/k}.$$

When $x = \pm 1$, series (a) diverges by the term test, since its terms are all ± 1 ; thus, its interval of convergence is $(-1, 1)$.

Series (b) is the harmonic series when $x = 1$ and the alternating harmonic series when $x = -1$; thus, its interval of convergence is $[-1, 1)$.

Series (c) is the p -series with $p = 2$ at $x = 1$ and the alternating p -series with $p = 2$ when $x = -1$. Both series are convergent and so the interval of convergence for (c) is $[-1, 1]$.

Remark 6.4.8. Although the expression for the radius of convergence R , given in the previous theorem, is useful because it makes sense regardless of the series, it is often the case that the ratio test provides a more practical method for calculating the radius of convergence of a power series.

Example 6.4.9. Find the radius of convergence of the power series $\sum_{k=1}^{\infty} \frac{x^k}{k!}$.

Solution: We apply the ratio test. We have

$$\lim \left| \frac{x^{k+1}}{(k+1)!} \right| \div \left| \frac{x^k}{k!} \right| = \lim \frac{|x|}{k+1} = 0$$

for all x . Thus, the series converges for all x and its radius of convergence is $+\infty$.

Integration of Power Series

Since a power series centered at a , with radius of convergence R , converges uniformly on each interval of the form $[a - r, a + r]$ with $0 < r < R$, our earlier theorems concerning continuity (Theorem 6.4.2) and term by term integration (Theorem 6.4.3) apply. They lead to the following theorem.

Theorem 6.4.10. *If $f(x) = \sum_{k=0}^{\infty} c_k(x-a)^k$ on $(a-R, a+R)$, where R is the radius of convergence of this series, then f is continuous on $(a-R, a+R)$ and*

$$\int_a^x f(t) dt = \sum_{k=0}^{\infty} \frac{c_k}{k+1} (x-a)^{k+1}, \quad (6.4.4)$$

if $x \in (a-R, a+R)$. The latter series also has radius of convergence R .

Proof. The continuity of f is a direct consequence of Theorem 6.4.2, while the integral formula follows directly from Theorem 6.4.3 and the fact that

$$\int_a^x (t-a)^k dt = \frac{(x-a)^{k+1}}{k+1}$$

The statement about radius of convergence is proved as follows: If we factor $(x-a)$ out of the series in (6.4.4), the remaining factor is

$$\sum_{k=0}^{\infty} \frac{c_k}{k+1} (x-a)^k,$$

which clearly has the same convergence set and radius of convergence. By Theorem 6.4.6, its radius of convergence is the inverse of

$$\limsup \left(\frac{|c_k|}{k+1} \right)^{1/k} = \limsup |c_k|^{1/k} \lim \frac{1}{(k+1)^{1/k}} = \limsup |c_k|^{1/k},$$

which is the radius of convergence of the original series. Here, the first equality follows from Exercise 2.6.8, while the second equality follows from the fact that $\lim(1+k)^{1/k} = 1$ (a simple consequence of l'Hôpital's Rule). Thus, the series in (6.4.4) has the same radius of convergence as the original series. \square

Example 6.4.11. Find a power series in x which converges to $\ln(1+x)$ in an open interval centered at 0. What is the largest such open interval?

Solution: If $|x| < 1$, the geometric series $\sum_{k=0}^{\infty} x^k$ converges to $\frac{1}{1-x}$. If we replace x by $-t$ in this series, the result is

$$\frac{1}{1+t} = \sum_{k=0}^{\infty} (-t)^k \quad \text{for } |t| < 1.$$

If we integrate with respect to t from 0 to x , then it follows from the previous theorem that

$$\ln(1+x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1} = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k}.$$

for $|x| < 1$. The radius of convergence of this series is $(\limsup(1/k)^{1/k})^{-1} = 1$ and so $(-1, 1)$ is the largest open interval on which this series converges to $\ln(1+x)$.

Differentiation of Power Series

We may also differentiate power series term by term.

Theorem 6.4.12. *If $f(x) = \sum_{k=0}^{\infty} c_k(x-a)^k$ on $(a-R, a+R)$, where R is the radius of convergence of this series, then f is differentiable on $(a-R, a+R)$ and, on this interval,*

$$f'(x) = \sum_{k=1}^{\infty} k c_k (x-a)^{k-1}. \quad (6.4.5)$$

This series also has radius of convergence R .

Proof. We set

$$g(x) = \sum_{k=1}^{\infty} k c_k (x-a)^{k-1}.$$

This series has the same radius of convergence as the series

$$\sum_{k=1}^{\infty} k c_k (x-a)^k = (x-a) \sum_{k=1}^{\infty} k c_k (x-a)^{k-1},$$

and that is

$$(\limsup |k c_k|^{1/k})^{-1} = (\lim k^{1/k} \limsup |c_k|^{1/k})^{-1} = R,$$

since $\lim k^{1/k} = 1$.

To complete the proof, we just need to show that g is the derivative of f . However, by the previous theorem,

$$\int_a^x g(t) dt = \sum_{k=1}^{\infty} c_k (x-a)^k = f(x) - c_0.$$

By the Second Fundamental Theorem, $f'(x) = g(x)$. □

Example 6.4.13. Find a power series in x which converges to $\frac{1}{(1-x)^2}$ on an open interval centered at 0. What is the largest open interval on which this power series expansion is valid?

Solution: As in the last example, we begin with the power series expansion of $\frac{1}{1-x}$ on $(-1, 1)$,

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k.$$

If we differentiate, using the previous theorem, the result is

$$\frac{1}{(1-x)^2} = \sum_{k=1}^{\infty} kx^{k-1} = \sum_{k=0}^{\infty} (k+1)x^k.$$

on $(-1, 1)$. By the theorem, this series has radius of convergence 1. Thus, $(-1, 1)$ is the largest open interval on which this expansion is valid.

Exercise Set 6.4

1. Prove that the function $f(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^2}$ is continuous on the interval $[-1, 1]$.
2. Prove that the function $f(x) = \sum_{k=1}^{\infty} \frac{\sin kx}{2^k}$ is continuous on the entire real line.
3. Let $\{f_k\}$ be a sequence of differentiable functions on (a, b) and suppose there is a point $c \in (a, b)$ such that the series $\sum_{k=1}^{\infty} f_k(c)$ converges. Suppose also that the sequence of derivatives $\{f'_k\}$ satisfies $|f'_k(x)| \leq M_k$ on (a, b) and the series $\sum_{k=1}^{\infty} M_k$ converges. Then prove that the series defining

$$f(x) = \sum_{k=1}^{\infty} f_k(x) \quad \text{and} \quad g(x) = \sum_{k=1}^{\infty} f'_k(x)$$

converge on (a, b) and f is differentiable with derivative g on (a, b) .

In each of the next five exercises, find the radius of convergence of the indicated power series.

4. $\sum_{k=1}^{\infty} \frac{1}{k3^k} x^k$.

$$5. \sum_{k=0}^{\infty} \frac{(-1)^{k-1}}{k+1} (x+2)^k.$$

$$6. \sum_{k=1}^{\infty} \frac{1}{k\sqrt{k}} x^k.$$

$$7. \sum_{k=0}^{\infty} k!(x-5)^k.$$

$$8. \sum_{k=0}^{\infty} 2^k x^{2k}.$$

9. Beginning with the geometric series which converges to $\frac{1}{1-x}$ on $(-1, 1)$, find power series which converge to $\frac{1}{1+x^2}$ and to $\arctan x$ on this same interval.
10. Prove that if $f(x)$ is the sum of a power series centered at a and with radius of convergence R , then f is infinitely differentiable on $(a-R, a+R)$ – that is, its derivative of order m exists on this interval for all $m \in \mathbb{N}$.
11. Suppose functions g and h are defined by

$$g(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2n+1}}{(2n+1)!} + \cdots$$

$$h(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2n}}{(2n)!} + \cdots$$

Find the interval of convergence for each of these functions.

12. Prove that the functions in the previous exercise satisfy $g' = h$ and $h' = g$.
13. Prove Theorem 6.4.3.

6.5 Taylor's Formula

Definition 6.5.1. Suppose f is a function defined in an open interval containing a . If there is a power series, centered at a , which converges to f in some open interval centered at a , then we will say that f is *analytic* at a . If f is analytic at every point of an open interval I , then we will say that f is analytic on I .

When can we expect that f is analytic at a ? According to Exercise 6.4.10, if f is the sum of a power series in some interval centered at a , then f is infinitely differentiable in this interval (meaning its n th derivative exists for every $n \in \mathbb{N}$). Thus, in order for a function to be analytic at a it must be infinitely differentiable in some interval centered at a . However, this is not enough. In fact Exercise 6.5.13 shows that there is a function which is infinitely differentiable in an open interval centered at 0, but is not the sum of a power series centered at 0.

Power Series Coefficients

If a function is analytic at a – that is, it has a power series expansion centered at a , then it is easy to see what the coefficients of the power series expansion must be.

Theorem 6.5.2. Suppose $f(x) = \sum_{k=0}^{\infty} c_k(x-a)^k$, where this series converges to $f(x)$ on an open interval containing a . Then $c_n = \frac{f^{(n)}(a)}{n!}$ for each n .

Proof. We prove by induction that the n th derivative of f is

$$f^{(n)}(x) = \sum_{k=n}^{\infty} \frac{k!}{(k-n)!} c_k (x-a)^{k-n}. \quad (6.5.1)$$

When $n = 1$, this just says that

$$f'(x) = \sum_{k=1}^{\infty} k c_k (x-a)^{k-1},$$

which is true by Theorem 6.4.12.

If we assume that (6.5.1) is true for a given n , then by differentiating it and again using Theorem 6.4.12, we obtain

$$\begin{aligned} f^{(n+1)}(x) &= \sum_{k=n}^{\infty} \frac{k!}{(k-n)!} (k-n) c_k (x-a)^{k-n-1} \\ &= \sum_{k=n+1}^{\infty} \frac{k!}{(k-n-1)!} c_k (x-a)^{k-n-1}. \end{aligned}$$

Since this is (6.5.1) with n replaced by $n+1$, the induction is complete and we conclude that (6.5.1) is true for all $n \in \mathbb{N}$.

If we set $x = a$ in (6.5.1), all terms vanish except for the first one (the one where $k = n$). Thus,

$$f^{(n)}(a) = n! c_n \quad \text{or} \quad c_n = \frac{f^{(n)}(a)}{n!}.$$

□

Taylor's Formula

The previous theorem tells us that the only power series, centered at a , that could possibly converge to $f(x)$ in an interval centered at a is the power series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k. \quad (6.5.2)$$

This is called the *Taylor Series* for f at a . The n th partial sum of this series,

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

is called the n th *Taylor polynomial* for f at a . The function f is analytic at a if and only if the sequence of Taylor polynomials for f converges to f in some open interval centered at a . Taylor's Formula helps decide whether this is true by providing a formula for the remainder when f is approximated by its n th Taylor polynomial.

Theorem 6.5.3. (Taylor's Formula) *Let f be a function which has continuous derivatives up through order $n+1$ in an open interval I centered at a . Then, for each $x \in I$,*

$$f(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x), \quad (6.5.3)$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{(n+1)}, \quad (6.5.4)$$

for some c between a and x .

Proof. This theorem is reminiscent of the Mean Value Theorem. In fact, in the case $n = 0$, it is the Mean Value Theorem. It is not surprising that its proof mimics the proof of the Mean Value Theorem.

We set

$$R_n(x) = f(x) - f(a) - f'(a)(x-a) - \cdots - \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

so that (6.5.3) holds. We then define a function $s(t)$ on I by

$$s(t) = f(x) - f(t) - f'(t)(x-t) - \cdots - \frac{f^{(n)}(t)}{n!}(x-t)^n - R_n(x) \left(\frac{x-t}{x-a} \right)^{n+1}.$$

Then $s(a) = s(x) = 0$, and so there must be a critical point c for s somewhere strictly between a and x . Since s is differentiable on I , this critical point must be a point where s' is 0 – that is, $s'(c) = 0$. In the calculation of s' , all the terms cancel except two at the very end, leaving

$$0 = s'(c) = -\frac{f^{(n+1)}(c)}{n!}(x-c)^n + (n+1)R_n(x) \frac{(x-c)^n}{(x-a)^{n+1}}.$$

Equation (6.5.4) follows from this when we solve for $R_n(x)$. □

Example 6.5.4. Find the Taylor series expansion of e^x at 0 and tell for which values of x this expansion converges to e^x .

Solution: The function e^x is infinitely differentiable on \mathbb{R} with k th derivative equal to e^x for all x . Thus, its k th derivative evaluated at 0 is 1 for all k . Taylor's Formula then tells us that

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + R_n(x),$$

where

$$R_n(x) = e^c \frac{x^{n+1}}{(n+1)!},$$

for some c between 0 and x .

For all values of x and c , $\lim_{n \rightarrow \infty} e^c \frac{x^{n+1}}{(n+1)!} = 0$ (Exercise 6.5.1). This implies that the Taylor polynomials for e^x converge to e^x for all $x \in \mathbb{R}$ – that is, the Taylor series expansion

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \quad (6.5.5)$$

is valid for all $x \in \mathbb{R}$.

Example 6.5.5. Find the Taylor series expansion of $\sin x$ at 0 and tell for which values of x this expansion converges to $\sin x$.

Solution: The function $f(x) = \sin x$ is infinitely differentiable on \mathbb{R} and its first 4 derivatives are

$$f'(x) = \cos x, \quad f''(x) = -\sin x, \quad f'''(x) = -\cos x, \quad f^{(4)}(x) = \sin x.$$

Since $f^{(4)} = f$, taking n th derivatives leads to $f^{(n+4)} = f^{(n)}$ for every non-negative integer n . Thus, at 0, the sin and its derivatives form the following repeating sequence with period 4:

$$0, 1, 0, -1, 0, 1, 0, -1, 0, \dots$$

Hence, Taylor's formula for $\sin x$ at $a = 0$ is

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + R_{2n+2}(x),$$

where

$$R_{2n+2}(x) = \sin^{(2n+3)}(c) \frac{x^{2n+3}}{(2n+3)!} \quad \text{for some } c.$$

The reason we use $R_{2n+2}(x)$ rather than $R_{2n+1}(x)$ for the remainder (they are actually equal, since the term of degree $2n+2$ is 0 in Taylor's Formula for $\sin x$) is that we get better estimates on the size of the remainder if we use $R_{2n+2}(x)$.

Since $|\sin^{(2n+3)}(c)| \leq 1$, we have

$$|R_{2n+2}(x)| \leq \frac{|x|^{2n+3}}{(2n+3)!}$$

which implies that the remainder has limit 0 for all x (see Exercise 6.5.1). Thus, the Taylor series expansion

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots$$

is valid for all $x \in \mathbb{R}$.

Example 6.5.6. Find an estimate for the error if $\sin x$ is approximated by $x - x^3/3!$ for x in the interval $[-\pi/4, \pi/4]$. By an estimate for the error, we mean an upper bound for the error which is as close to the actual error as possible without going to extraordinary effort.

Solution: By the previous example, the difference between $\sin x$ and its third degree Taylor polynomial has absolute value less than or equal to

$$\frac{|x|^5}{5!} \leq \frac{(\pi/4)^5}{5!} < .003 \quad \text{for} \quad -\pi/4 \leq x \leq \pi/4.$$

Lagrange's Form for the Remainder

The following integral formula for the remainder in Taylor's Formula sometimes leads to better estimates on the size of the remainder than does the usual form.

Theorem 6.5.7. *If f is a function with continuous derivatives up through order $n + 1$ on an open interval I containing a and x , then the remainder $R_n(x)$ in Taylor's formula for f at a can be written as*

$$R_n(x) = \frac{1}{n!} \int_a^x (x-t)^n f^{(n+1)}(t) dt \quad (6.5.6)$$

Proof. We prove (6.5.6) by induction on n with the base case being $n = 0$. In the case where $n = 0$, Taylor's formula is

$$f(x) = f(a) + R_0(x) \quad \text{so that} \quad R_0(x) = f(x) - f(a).$$

Equation (6.5.6) in this case is

$$f(x) - f(a) = \int_a^x f'(t) dt,$$

which is just the Fundamental Theorem of Calculus. Thus, (6.5.6) holds when $n = 0$.

For the induction step, we assume (6.5.6) holds for a given n and proceed to prove that it then holds for $n + 1$. If we apply integration by parts to the integral on the right side of (6.5.6), the result is

$$R_n(x) = \frac{f^{(n+1)}(a)}{(n+1)!} (x-a)^{n+1} + \frac{1}{(n+1)!} \int_a^x (x-t)^{n+1} f^{(n+2)}(t) dt.$$

Since, $R_{n+1}(x) = R_n(x) - \frac{f^{(n+1)}(a)}{(n+1)!} (x-a)^{n+1}$, this proves (6.5.6) holds with n replaced by $n + 1$, thus completing the induction step. \square

Example 6.5.8. Find a power series expansion for $f(x) = (1+x)^p$ which is valid on $(-1, 1)$, where p is any real number.

Solution: The derivatives of f are

$$p(1+x)^{p-1}, p(p-1)(1+x)^{p-2}, \dots, p(p-1)\cdots(p-n+1)(1+x)^{p-n} \dots$$

The n th derivative evaluated at 0 is $p(p-1)\cdots(p-n+1)$. Thus, Taylor's formula for f is

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2}x^2 + \cdots + \frac{p(p-1)\cdots(p-n+1)}{n!}x^n + R_n(x),$$

where

$$R_n(x) = \frac{p(p-1)\cdots(p-n)}{n!} \int_0^x \frac{(x-t)^n}{(1+t)^{n+1-p}} dt,$$

if we use Lagrange's form of the remainder. However, since t is between 0 and x , t and x have the same sign, and this implies that

$$\left| \frac{x-t}{t+1} \right| \leq |x|. \quad (6.5.7)$$

(Exercise 6.5.9). From this, we conclude that

$$|R_n(x)| \leq \frac{p(p-1)\cdots(p-n)}{n!} |x|^n \int_0^x (1+t)^{p-1} dt.$$

This is just the constant $\int_0^x (1+t)^{p-1} dt$ times the absolute value of the n th term in the power series

$$1 + px + \frac{p(p-1)}{2}x^2 + \cdots + \frac{p(p-1)\cdots(p-n+1)}{n!}x^n + \cdots, \quad (6.5.8)$$

which happens to be the Taylor series for $(1+x)^p$ at 0. If we can show that this series converges when $|x| < 1$, then the Term Test implies its sequence of terms converges to 0 and, by the above, this shows that the remainder $R_n(x)$ converges to 0 and, hence, that this series converges to $(1+x)^p$ when $|x| < 1$.

We prove that (6.5.8) converges on $(-1, 1)$ by using the Ratio Test. For the absolute value of the ratio of term $n+1$ to term n , we get

$$\frac{|p-n|}{n+1} |x|$$

which has limit $|x|$ as $n \rightarrow \infty$. Hence, the series (6.5.8) converges for $|x| < 1$ and it converges to $(1+x)^p$.

Note that when p is a positive integer, the series (6.5.8) terminates at $n = p$, that is, all terms with $n > p$ are zero and Taylor's Formula for $(1+x)^p$ at 0,

with $n \geq p$ is just

$$\begin{aligned} (1+x)^p &= 1 + px + \frac{p(p-1)}{2}x^2 + \cdots + \frac{p(p-1)\cdots(p-p+1)}{p!}x^p \\ &= \sum_{k=0}^p \frac{p!}{k!(p-k)!}x^k, \end{aligned} \tag{6.5.9}$$

which is the Binomial Theorem (Theorem 1.2.12) with $a = 1$ and $b = x$. The Binomial Theorem for general a and b can be deduced from this (Exercise 6.5.14).

Exercise Set 6.5

1. Prove that $\lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0$ for all x .
2. Find the Taylor Series expansion for $\cos x$ at 0 and show that it converges for all x .
3. Use Taylor's Formula to estimate the error if $\cos x$ is approximated by $1 - \frac{x^2}{2}$ on the interval $[-.1, .1]$.
4. What is the smallest n for which we can be sure that

$$1 + 1 + \frac{1}{2} + \frac{1}{3!} + \cdots + \frac{1}{n!}$$

is within .001 of e ?

5. What is Taylor's Formula for the function $f(x) = \sqrt{1+x}$ with $a = 0$?
6. What is Taylor's Formula for the function $f(x) = x^3 - x^2 - 4x + 4$ with $a = 1$?
7. What is Taylor's Formula for $\ln(1+x)$ with $a = 0$. Compare with Example 6.4.11.
8. Use the binomial series with $p = -1/2$ to get a power series expansion for $\frac{1}{\sqrt{1-x}}$ valid on $(-1, 1)$. Use this to get power series expansions for first $\frac{1}{\sqrt{1-x^2}}$, and then $\arcsin x$ on this same interval.
9. Prove that if $x \in (-1, 1)$ and t is between 0 and x (so that t and x have the same sign and $|t| \leq |x| < 1$), then

$$\left| \frac{x-t}{t+1} \right| \leq |x|.$$

10. Verify the computation of s' given in the proof of Theorem 6.5.3.

11. Prove that if f is an infinitely differentiable function on $(a - r, a + r)$ and there is a constant K such that

$$|f^{(n)}(x)| \leq K \frac{n!}{r^n}$$

for all $n \in \mathbb{N}$ and all $x \in (a - r, a + r)$, then the Taylor Series for f at a converges to f on $(a - r, a + r)$.

12. Use l'Hôpital's Rule to show that $\lim_{x \rightarrow 0} \frac{e^{-1/x^2}}{x^n} = 0$ for every n .
13. If $g(x) = e^{-1/x^2}$ for $x \neq 0$ and $g(0) = 0$, show that g is infinitely differentiable on the entire real line, but all of its derivatives at 0 are equal to 0. Argue that this means that g cannot be analytic at 0. Hint: use the previous exercise to help compute the derivatives of g at 0.
14. Prove that the Binomial Formula (Theorem 1.2.12) for a general a and b follows from the Taylor Series expansion (6.5.9) of $(1+x)^p$ for p a positive integer.
15. Give a new proof that $e^x e^y = e^{x+y}$ by using the Taylor series expansion for e^x (6.5.5) and the product formula of Theorem 6.3.6. You will also need to use the binomial formula.