Chapter 5

Residue Theory

The Residue Theorem (Theorem 4.4.3) has a wide range of applications. This chapter is devoted to exploring some of them. We begin with a section on techniques for computing residues.

5.1 Computing Residues

Recall the discussion of residues in Section 4.4. If f is analytic in a neighborhood of z_0, except at z_0 itself, then f has an isolated singularity at z_0. In this case, it has a Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

in $D \setminus \{z_0\}$ for some disc D, centered at z_0. The *residue* of f at z_0. denoted $\mathrm{Res}(f, z_0)$ is then the coefficient a_{-1} of $(z - z_0)^{-1}$ in this expansion. Computing the residue can be easy. It can also be hard.

If the singularity of f at z_0 is a pole of order k, then there is a formula for $\mathrm{Res}(f, z_0)$ which makes it, in principal, computable. If f has a pole of order k at z_0, then it has the form

$$f(z) = \frac{g(z)}{(z - z_0)^k},$$

where g is analytic and non-vanishing in a neighborhood of z_0.

Theorem 5.1.1. *If* $f(z) = \dfrac{g(z)}{(z - z_0)^k}$ *where* g *is analytic in a neighborhood of* z_0*, then* $Res(f, z_0)$ *is the coefficient of degree* $k-1$ *in the power series expansion of* g *about* z_0*. That is,*

$$Res(f, z_0) = \frac{g^{(k-1)}(z_0)}{(k-1)!}.$$

Proof. Since g is analytic in a nieghborhood of z_0, Theorem 3.2.5 implies that it has a power series expansion

$$g(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n,$$

where, by Theorem 3.2.1,

$$b_n = \frac{g^{(n)}(z_0)}{n!}.$$

The Laurent expansion of f is then given by

$$f(z) = b_0(z - z_0)^{-k} + \cdots + b_{k-1}(z - z_0)^{-1} + \sum_{n=k}^{\infty} b_n (z - z_0)^{n-k}.$$

Thus, the residue of f at z_0 is

$$\text{Res}(f, z_0) = b_{k-1} = \frac{g^{k-1}(z_0)}{(k-1)!},$$

as claimed. □

The first couple of cases of the preceding theorem are worth highlighting. We do this in the following corollary, which follows immediately from the theorem.

Corollary 5.1.2. *Given a function g, analytic in a neighborhood of z_0,*

(a) if $f(z) = \dfrac{g(z)}{z - z_0}$, then Res$(f, z_0) = g(z_0)$;

(b) if $f(z) = \dfrac{g(z)}{(z - z_0)^2}$, then Res$(f, z_0) = g'(z_0)$.

Example 5.1.3. Find Res$(f, 1)$ if $f(z) = \dfrac{z^2 + 1}{(z - 1)(z^2 - 2z + 5)}$.

Solution: The function f is of the form

$$f(z) = \frac{g(z)}{z - 1},$$

where

$$g(z) = \frac{z^2 + 1}{z^2 - 2z + 5}.$$

Since g is analytic in a neighborhood of 1, Corollary 5.1.2(a) implies that

$$\text{Res}(f, 1) = g(1) = \frac{1}{2}.$$

Example 5.1.4. Find $\mathrm{Res}(f,0)$ if $f(z)=\dfrac{\sin z}{z^2}$.

Solution: This is clearly a situation where Corollary 5.1.2(b) applies, with $g(z)=\sin z$. Thus,

$$\mathrm{Res}(f,0)=g'(0)=\cos 0=1.$$

We could also have done this one directly, by writing out the power series expansion for $\sin z$ about 0, and dividing by z^2 to obtain the Laurent series expansion of f.

Long Division

Often the function f, whose residue we wish to compute at some z_0, is of the form

$$f(z)=\frac{p(z)}{h(z)},$$

where p and h are analytic functions in a neighborhood of z_0 with known power series expansions about z_0, and h has a zero of order k at z_0. In this situation, we can always derive a formula for $\mathrm{Res}(f,z_0)$ using Theorem 5.1.1. We simply write h as $h(z)=(z-z_0)^k q(z)$ where q is analytic in a neighborhood of z_0 and $q(z_0)\neq 0$. Then

$$f(z)=\frac{g(z)}{(z-z_0)^k} \quad \text{where} \quad g(z)=\frac{p(z)}{q(z)}. \tag{5.1.1}$$

We then apply Theorem 5.1.1. Of course, to apply this theorem, one has to compute $g^{(k-1)}(z_0)$ (or, equivalently, the coefficient of $(z-z_0)^{k-1}$ in the power series expansion of g about z_0), and this may be quite difficult. Rather than doing this directly, through repeated differentiation, it is possible to compute the power series coefficients of g using power series methods – specifically, the method of long division of power series.

Note that p and q are both analytic functions in a neighborhood of z_0 and, since the power series expansions of p and h about z_0 are known, the same thing is true of the power series expansion of q about z_0. Since $q(z_0)\neq 0$, $g=p/q$ is also analytic in a neighborhood of z_0 and, hence, has a power series expansion in some disc centered at z_0. The method of long division is an algorithm for finding the power series coefficents of p/q in terms of those of p and q.

For simplicity, in our discussion of long division, we will work with power series in z – that is, power series centered at 0. Power series centered at some point z_0 other than 0 can always be put in this form with a change of variables. The coefficient formulas we derive below are not affected by such a change.

Suppose that p has a zero of order at least m at 0 (m might be 0 and it might be less than the actual order of the zero of p at 0). Also suppose that $q(0)\neq 0$. Then p and q have power series expansions

$$\begin{aligned} p(z)&=a_m z^m+a_{m+1}z^{m+1}+a_{m+2}z^{m+2}+\cdots+a_n z^n+\cdots,\\ q(z)&=b_0+b_1 z+b_2 z^2+\cdots+b_n z^n+\cdots, \end{aligned} \tag{5.1.2}$$

which converge in some disc $D_r(0)$, with $r > 0$. Since $q(0) = b_0 \neq 0$, $q(z)$ is non-vanishing in some disc $D_\delta(0)$, with $0 < \delta < r$. Then the quotient p/q is analytic in $D_\delta(0)$, with a zero of order at least m at 0, and has a convergent power series expansion

$$\frac{p(z)}{q(z)} = c_m z^m + c_{m+1} z^{m+1} + c_{m+2} z^{m_2} + \cdots + c_n z^n + \cdots . \tag{5.1.3}$$

How can we determine the coefficients c_n if we know the coefficients a_n and b_n? A method often taught in calculus for doing this is the method of long division of power series. This is a recursive method that determines the coefficients a_n one after another. It is described as follows: We have

$$\frac{p(z)}{q(z)} = \frac{a_m}{b_0} z^m + \frac{p(z) - (a_0/b_0) z^m q(z)}{q(z)}, \tag{5.1.4}$$

where the fraction on the right has the same denominator as the original fraction, but the numerator has changed and is now a power series with lowest order term of degree at least $m+1$ whereas the original numerator had lowest order term of degree at least m. If we write out explicitely the polynomials involved, then 5.1.4 becomes

$$\begin{aligned} \frac{p(z)}{q(z)} &= \frac{a_m z^m + a_{m+1} z^{m+1} + a_{m+2} z^{m+2} + \cdots}{b_0 + b_1 z + b_2 z^2 + \cdots} \\ &= \frac{a_m}{b_0} z^m + \frac{(a_{m+1} - b_1 a_m/b_0) z^{m+1} + (a_{m+2} - b_2 a_m/b_0) z^{m+2} + \cdots}{b_0 + b_1 z + b_2 z^2 + \cdots}. \end{aligned} \tag{5.1.5}$$

This tells us that the first coefficient in (5.1.3) is

$$c_m = a_m/b_0. \tag{5.1.6}$$

The next coefficient c_{m+1} is obtained by repeating the procedure on the new fraction that appears on the right in the above equation. This procedure may be repeated as often as is needed to obtain a given coefficient c_n. For example, in Exercises 5.1.14 and 5.1.15 you are asked to use this procedure to show that

$$c_{m+1} = \frac{a_{m+1}}{b_0} - \frac{a_m b_1}{b_0^2} \tag{5.1.7}$$

and

$$c_{m+2} = \frac{a_{m+2}}{b_0} - \frac{a_{m+1} b_1}{b_0^2} + \frac{a_m b_1^2}{b_0^3} - \frac{a_m b_2}{b_0^2}. \tag{5.1.8}$$

Residue of a Quotient

We now return to the problem of finding the residue at z_0 of a quotient, $f = p/h$, where h has a zero of order k at z_0. We define q and $g = p/q$ as in (5.1.1). It follows from Theorem 5.1.1 that:

Theorem 5.1.5. *Suppose* $f(z) = \dfrac{p(z)}{h(z)}$, *where* $p(z)$ *and* $h(z)$ *are analytic in a neighborhood of* z_0 *and* h *has a zero of order* k *at* z_0. *If we write* $h(z) = (z - z_0)^k q(z)$ *where* q *is analytic in a neighborhood of* z_0 *and* $q(0) \neq 0$, *then*

$$\mathit{Res}(f, z_0) = c_{k-1}$$

where c_{k-1} *is the coefficient of the* $k-1$ *degree term in the power series expansion of* $g = p/q$ *about* z_0. *The number* c_{k-1}, *may be computed using long division on the quotient* p/q; *that is, they may be computed through repeated use of (5.1.5).*

For small k, the expression for c_{k-1}, given by carrying out the method described in the above theorem, is reasonably simple. However, the complexity increases rapidly with increasing k.

When $k = 1$, the above theorem and (5.1.6), with $m = 0$, say that if

$$f(z) = \frac{p(z)}{h(z)} = \frac{p(z)}{(z - z_0)q(z)},$$

with $q(z_0) \neq 0$, then

$$\mathrm{Res}(f, z_0) = c_0 = \frac{a_0}{b_0} = \frac{p(z_0)}{q(z_0)}.$$

This is just Corollary 5.1.2(a). We can restate it in terms of the original functions p and h as follows: since $h(z) = (z - z_0)q(z)$, we have

$$q(z_0) = h'(z_0)$$

and so

Corollary 5.1.6. *Let* p *and* h *be analytic in a neighborhood of* z_0, *where* h *has a zero of order* 1 *at* z_0. *Then*

$$\mathit{Res}(p/h, z_0) = \frac{p(z_0)}{h'(z_0)}.$$

Example 5.1.7. Find $\mathrm{Res}(f, 0)$ if

$$f(z) = \frac{\mathrm{e}^z}{\sin z}.$$

Solution: Since $\sin z$ has a zero of order 1 at $z = 0$, we may apply Corollary 5.1.6 with $p(z) = \mathrm{e}^z$ and $h(z) = \sin z$. Then

$$\mathrm{Res}(f, 0) = \frac{\mathrm{e}^0}{\cos 0} = 1.$$

Example 5.1.8. Find $\mathrm{Res}(f, 0)$ if

$$f(z) = \frac{1}{\mathrm{e}^z - 1 - z}.$$

Solution: Here the denominator has a zero of order 2. We set $p(z) = 1$ and

$$h(z) = e^z - 1 - z = \frac{z^2}{2} + \frac{z^3}{3!} + \cdots .$$

Then $h(z) = z^2 q(z)$ where

$$q(z) = \frac{1}{2} + \frac{z}{3!} + \cdots .$$

By Theorem 5.1.5, the residue we seek is the coefficient of the degree 1 term in the power series expansion of $1/q$ and this may be obtained by long division. In fact, we have

$$\frac{1}{1/2 + z/3! + \cdots} = 2 - \frac{z/3 + \cdots}{1/2 + z/3! + \cdots} = 2 - (2/3)z + \cdots ,$$

and so the residue is $-2/3$.

Example 5.1.9. Find the residue at 0 of $f(z) = \dfrac{\cot z}{z^2}$.

Solution: We proceed as in Theorem 5.1.5. The function f has a pole of order 3 at 0. We set $p(z) = \cos z$, $h(z) = z^2 \sin z$, and

$$q(z) = \frac{h(z)}{z^3} = \frac{\sin z}{z} = 1 - \frac{z^2}{6} + \frac{z^4}{120} - \cdots .$$

We then use long division to find the first three power series coefficients, c_0, c_1 and c_2 of p/q. By Theorem 5.1.5 The residue we seek will then be c_2.

$$\begin{aligned}\frac{p(z)}{q(z)} &= \frac{\cos z}{z^{-1}\sin z} = \frac{1 - z^2/2 + \cdots}{1 - z^2/6 + \cdots} \\ &= 1 + \frac{(-1/2 + 1/6)z^2 + \cdots}{1 - z^2/6 + \cdots} \\ &= 1 + \frac{-(1/3)z^2 + \cdots}{1 - z^2/6 + \cdots} \\ &= 1 - (1/3)z^2 + \cdots\end{aligned}$$

Thus, $c_0 = 1$, $c_1 = 0$, and $c_2 = -1/3$. We conclude that $\text{Res}(f, 0) = -1/3$.

Exercise Set 5.1

1. Find $\text{Res}(f, 0)$ and $\text{Res}(f, 1)$ if $f(z) = \dfrac{z + 2}{z(z - 1)(z + 4)}$.
2. Find $\text{Res}(f, 0)$ and $\text{Res}(f, 2)$ if $f(z) = \dfrac{\cos z}{2z - z^2}$.
3. Find $\text{Res}(f, 0)$ if $f(z) = \mathrm{e}^{1/z}$.

4. Find $\mathrm{Res}(f,0)$ if $f(z)=\cot z$.

5. Find $\mathrm{Res}(f,\pi)$ if $f(z)=\cot z$.

6. Find $\mathrm{Res}(f,0)$ if $f(z)=\cot^2 z$.

7. Find $\mathrm{Res}(f,0)$ if $f(z)=\dfrac{\mathrm{e}^z}{\cos z-1}$.

8. Find $\mathrm{Res}(f,0)$ if $f(z)=\dfrac{1}{z^2\sin z}$.

9. Find $\mathrm{Res}(f,0)$ if $f(z)=\dfrac{1}{z-\log(1+z)}$.

10. Use long division of power series to find the power series expansion about 0 of $\tan z=\dfrac{\sin z}{\cos z}$ through terms of degree five.

11. Prove that if f has a simple pole at z_0 and g is analytic in a neighborhood of z_0, then $Res(gf,z_0)=g(z_0)\mathrm{Res}(f,z_0)$.

12. Show by example that it is not true that $Res(gf,z_0)=g(z_0)\mathrm{Res}(f,z_0)$ if g is analytic in a neighborhood of z_0 and f has a pole of degree greater than 1 at z_0.

13. Prove that a meromorphic function which is even and has an isolated singularity at 0 has residue 0 at 0.

14. Derive formula (5.1.7).

15. Derive formula (5.1.8).

5.2 Evaluating Integrals Using Residues

The Residue Theorem (Theorem 4.4.3) tells us that if f is a function which is analytic in a set U, except at a discrete set of isolated singularities, then we can evaluate its integral around a closed path γ in U, which is homologous to 0 in U and doesn't hit any of these singularities, provided we can find the residues of f at singularities where γ has non-zero index. As we shall see, this turns out to be a very practical method for evaluating a wide variety of integrals.

Sines and Cosines

Because of the identities

$$\cos\theta=\frac{\mathrm{e}^{i\theta}+\mathrm{e}^{-i\theta}}{2},\quad \sin\theta=\frac{\mathrm{e}^{i\theta}-\mathrm{e}^{-i\theta}}{2i},$$

the cosine and sine functions may be regarded as the restrictions to the unit circle (parameterized by $z = \mathrm{e}^{i\theta}, 0 \le \theta \le 2\pi$) of the meromorphic functions

$$\frac{z+z^{-1}}{2}, \quad \text{and} \quad \frac{z-z^{-1}}{2i}.$$

This means that an integral of the form

$$\int_0^{2\pi} f(\theta)\,d\theta, \tag{5.2.1}$$

where f is a rational expression in $\cos\theta$ and $\sin\theta$, may be reformulated as a path integral around the unit circle of a certain meromorphic function of z. We simply let $g(z)$ be the function of z obtained by replacing the functions $\cos\theta$ and $\sin\theta$, that make up f, with the functions $(z+z^{-1})/2$ and $(z-z^{-1})/2i$. It will then be true that

$$g(\mathrm{e}^{i\theta}) = f(\theta).$$

With $\gamma(\theta) = \mathrm{e}^{i\theta}$, we then have

$$\int_\gamma \frac{g(z)}{iz}\,dz = \int_0^{2\pi} f(\theta)\,d\theta \tag{5.2.2}$$

The integral on the left is $2\pi i$ times the sum of the residues of $g(z)/iz$ inside the unit circle, provided $g(z)/iz$ has no poles on the unit circle itself.

Example 5.2.1. Find $\displaystyle\int_0^{2\pi} \frac{d\theta}{2+\sin\theta}$.

Solution: If we replace the $\sin\theta$ in the integrand by $(z-z^{-1})/2i$, the resulting function is

$$\frac{1}{2+(z-z^{-1})/2i} = \frac{2iz}{4iz+z^2-1}.$$

This is the g of equation (5.2.2). We divide this by iz and integrate around the unit circle to get our answer. Thus, we need to evaluate the integral

$$\int_{|z|=1} \frac{2}{z^2+4iz-1}\,dz = \int_{|z|=1} \frac{2}{(z+(2-\sqrt{3})i)(z+(2+\sqrt{3})i)}\,dz.$$

Here we have factored $z^2+4iz-1$ by finding its roots using the Quadratic Formula. The only pole of the integrand that is inside the unit circle is the one at the point $(-2+\sqrt{3})i$. The residue at this pole is found using Corollary 5.1.2(a). That is, we simply evaluate the function

$$\frac{2}{z+(2+\sqrt{3})i}$$

at the pole $z = (-2+\sqrt{3})i$. The resulting residue is

$$\frac{2}{2\sqrt{3}\,i} = -\frac{\sqrt{3}}{3}i.$$

By the Residue Theorem, the integral we seek is this number times $2\pi i$. Thus,

$$\int_0^{2\pi} \frac{d\theta}{2+\sin\theta} = \frac{2\sqrt{3}}{3}\,\pi.$$

This technique may be used to evaluate any integral of the form (5.2.1) where $f(\theta)$ is a rational function of $\sin\theta$ and $\cos\theta$ with a denominator which does not vanish for $\theta \in [0, 2\pi]$.

Improper Integrals

Improper Integrals of the form

$$\int_{-\infty}^{\infty} f(t)\,dt \tag{5.2.3}$$

can often be evaluated using residues.

There are two senses in which such an integral may converge. If the expression

$$\int_{-y}^{x} f(t)\,dt$$

has a limit as x and y approach $+\infty$ independently, then the improper interval is said to converge. This means, given $\epsilon > 0$, there is an M such that the integral is within ϵ of the limit whenever $x \geq M$ and $y \geq M$. This is equivalent to the convergence of the two one-sided improper integrals

$$\int_0^{\infty} f(t)\,dt \quad \text{and} \quad \int_{-\infty}^{0} f(t)\,dt.$$

On the other hand, if the limit of the symmetric integral,

$$\lim_{x\to\infty} \int_{-x}^{x} f(t)\,dt,$$

exists, then the integral (5.2.3) is said to converge in the *principal value* sense. This is weaker than ordinary convergence of the integral – that is, (5.2.3) may converge in the principal value sense and not converge in the ordinary sense.

An improper integral (5.2.3) is said to converge absolutely if the integral of $|f|$ converges. This implies the convergence of (5.2.3) and, in fact, we have the following integral analogue of the comparison test for convergence of series.

Theorem 5.2.2. *If f and g are continuous functions on the line, with $g(t) \geq 0$ and $|f(t)| \leq g(t)$ for all $t \in \mathbb{R}$, then the convergence of the integral of g on $\mathbb{R}$ implies the convergence of the integral of f on $\mathbb{R}$ and*

$$\left|\int_{-\infty}^{\infty} f(t)\,dt\right| \leq \int_{-\infty}^{\infty} g(t)\,dt.$$

Proof. We will prove the existence of the improper integral $\int_0^\infty f(t)\,dt$. The existence of $\int_{-\infty}^0 f(t)\,dt$ follows from the same argument. We will use the fact that, if a function h on $(0,\infty)$ has the property that $\lim_{n\to\infty} h(x_n)$ exists for every increasing sequence $\{x_n\}$ converging to infinity, then the limits of all such sequences are the same number L and $\lim_{x\to\infty} h(x) = L$ (Exercise 5.2.11).

Let $\{x_k\}_{k=1}^\infty$ be an increasing sequence of positive numbers converging to infinity. We set $x_0 = 0$ and, for $k \geq 1$, set

$$a_k = \int_{x_{k-1}}^{x_k} f(t)\,dt, \quad b_k = \int_{x_{k-1}}^{x_k} g(t)\,dt.$$

Then the hypothesis that $|f(t)| \leq g(t)$ implies that $|a_k| \leq b_k$ for all n. Furthermore, for each positive integer n,

$$\int_0^{x_n} f(t)\,dt = \sum_{k=1}^n a_k \quad \text{and} \quad \int_0^{x_n} g(t)\,dt = \sum_{k=1}^n b_k.$$

Thus, the convergence of the improper integral of $g(t)$ implies the convergence of the series of positive terms $\sum_{k=1}^\infty b_k$. This, in turn implies the abolute convergence of the series $\sum_{k=1}^\infty a_k$ and the inequality

$$\left|\sum_{k=1}^\infty a_k\right| \leq \sum_{k=1}^\infty b_k. \tag{5.2.4}$$

This means that, if $h(x) = \int_0^x f(t)\,dt$, then the sequence $\{h(x_n)\}$ has a limit for every increasing sequence $\{x_n\}$ converging to infinity. By the remark in the first paragraph, $\lim_{x\to\infty} h(x)$ exists and is the common limit of all the sequences $\{h(x_n)\}$. Hence the improper integral of f exists and is, by definition, this number L.

It follows from 5.2.4 that

$$\left|\int_0^\infty f(t)\,dt\right| = |L| \leq \sum_{k=1}^\infty b_k = \int_0^\infty g(t)\,dt.$$

The same argument can be used to show that $\int_{-\infty}^0 f(t)\,dt$ exists and satisfies the analogous inequality. The theorem follows on combining these results. ☐

Evaluating Integrals on the Line

Often the integrand f of an integral of the form (5.2.3) is the restriction to the real line of a function which is analytic, except for a discrete set of singularities, on an open set containing either the closed upper half plane $\{z : \operatorname{Im}(z) \geq 0\}$ or the closed lower half plane $\{z : \operatorname{Im}(z) \leq 0\}$. If this function has no singularities on the real line and it decreases rapidly enough as $z \to \infty$, then the integral can be approximated arbitrarily closely by the integral of f around a semicirclular

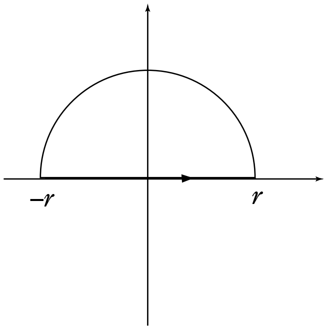

Figure 5.1: The path γ_r.

closed path in the appropriate half plane. Such an integral can be evaluated using residues.

Specifically, suppose f is analytic in U, except on a discrete subset of U, where U is an open set containing the closed upper half plane. Suppose also that p, R and C are positive numbers, with $p > 1$, such that, for $\operatorname{Im}(z) \geq 0$,

$$|f(z)| \leq C|z|^{-p} \quad \text{when} \quad |z| > R. \tag{5.2.5}$$

This condition ensures that the improper integral (5.2.3) converges (Exercise 5.2.12).

Let γ_r be the path which begins at $-r$ on the real line, traverses the interval $[-r, r]$ and then returns to $-r$ along the semicircle which is the upper half of the circle $|z| = r$ (see Figure 5.1).

We have

$$\int_{\gamma_r} f(z)\,dz = \int_{-r}^{r} f(x)\,dx + \int_0^{\pi} f(re^{it})ire^{it}\,dt. \tag{5.2.6}$$

If $r > R$ then (5.2.5) implies that $|f(re^{it})ire^{it}| \leq Cr^{1-p}$ and so

$$\left|\int_0^{\pi} f(re^{it})ire^{it}\,dt\right| \leq \pi C r^{1-p}.$$

Since $p > 1$, the right side of this inequality has limit 0 as $r \to \infty$. We conclude from this and (5.2.6) that

$$\lim_{r\to\infty} \int_{\gamma_r} f(z)\,dz = \lim_{r\to\infty} \int_{-r}^{r} f(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx.$$

Now, by the Residue Theorem, $\displaystyle\int_{\gamma_r} f(z)\,dz$ is $2\pi i$ times the sum of the residues of f at singularities inside γ_r. The condition (5.2.5) ensures that there are no singularities of f in the upper half plane outside the circle $|z| = R$. This implies there are only finitely many singularieties in the upper half plane. It also implies that the integral is independent of r, for $r > R$, and is equal to $2\pi i$ times the sum of the residues of f at this finite set of singularities.

If f is analytic in an open set containing the lower half plane, then all of the above still holds with the following changes: the lower half plane replaces the upper half plane, the path γ_r is replaced by its reflection through the x-axis, and, since the new path has negative orientation, the resulting integral is $-2\pi i$ times the sum of the residues of f in the lower half plane.

This proves the following theorem:

Theorem 5.2.3. *Let H be either the closed upper or lower half plane. If f is analytic, except at a discrete set of singularities, in an open set containing H, has no singularities on $\mathbb{R}$, and there are positive numbers R, C, and $p > 1$ such that (5.2.5) is satisfied for $z \in H$, then*

$$\int_{-\infty}^{\infty} f(x)\,dx = \sigma(H)2\pi i \sum_{j=1}^{m} Res(f, z_j),$$

where $z_1, z_2, \cdots, z_m$ are the singularities of f in the half plane H, and $\sigma(H) = 1$ if H is the upper half plane and -1 if H is the lower half plane.

Example 5.2.4. Find $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\,dx$.

Solution: We use the previous theorem with

$$f(z) = \frac{z^2}{1+z^4} = \frac{1}{z^{-4}+1} z^{-2}$$

If we fix an $R > 1$, we have

$$|f(z)| \le \frac{1}{1-R^{-4}} |z|^{-2} \quad \text{if} \quad |z| \ge R$$

and so condition (5.2.5) is satisfied with $p = 2$ and $C = (1 - R^{-4})^{-1}$.

The poles of f occur at the 4th roots of -1 and these are

$$z_1 = \frac{\sqrt{2}}{2}(1+i), \quad z_2 = \frac{\sqrt{2}}{2}(-1+i),$$
$$z_3 = \frac{\sqrt{2}}{2}(-1-i), \quad z_4 = \frac{\sqrt{2}}{2}(1-i).$$

Thus

$$f(z) = \frac{z^2}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}.$$

Only z_1 and z_2 are in the upper half plane. We evaluate the residue of f at each of these points using Corollary 5.1.2(a). This means, for $j = 1, 2$, we evaluate at z_j what is left of f when $z - z_j$ is removed from the denominator. The results are

$$\begin{aligned} \operatorname{Res}(f, z_1) &= \sqrt{2}\frac{(1+i)^2}{(2)(2+2i)(2i)} = -\frac{\sqrt{2}}{8}(1+i) \\ \operatorname{Res}(f, z_2) &= \sqrt{2}\frac{(-1-i)^2}{(-2)(2i)(-2+2i)} = \frac{\sqrt{2}}{8}(1-i). \end{aligned}$$

The sum of these is $-\frac{\sqrt{2}}{4}i$ and, when multiplied by $2\pi i$, this yields

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\,dx = \frac{\sqrt{2}}{2}\pi.$$

Symmetries

It can be useful to exploit symmetries of the integrand in computing integrals using residues.

Example 5.2.5. Find $\int_0^\infty \frac{1}{(1+x^2)^2}\,dx$.

Solution: Since the integrand is an even function, its integral over the entire real line is twice its integral from 0 to ∞. Thus,

$$\int_0^\infty \frac{1}{(1+x^2)^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^2}\,dx.$$

We evaluate the latter integral using Theorem 5.2.3. We set

$$f(z) = \frac{1}{(1+z^2)^2} = \frac{1}{(z-i)^2(z+i)^2}.$$

The only pole in the upper half plane is the one at i, and this is a pole of order 2. Corollary 5.1.2(b) applies, and it tells us that the residue of f at this pole is obtained by differentiating

$$g(z) = \frac{1}{(z+i)^2},$$

and evaluating at $z = i$. The result is

$$g'(i) = \frac{-2}{(2i)^3} = -\frac{i}{4}.$$

Thus,

$$\int_0^\infty \frac{1}{(1+x^2)^2}\,dx = \frac{1}{2}\cdot 2\pi i\cdot\frac{-i}{4} = \frac{\pi}{4}.$$

In the next example, we exploit the fact that the integrand is unchanged if we rotate coordinates by an angle π/n.

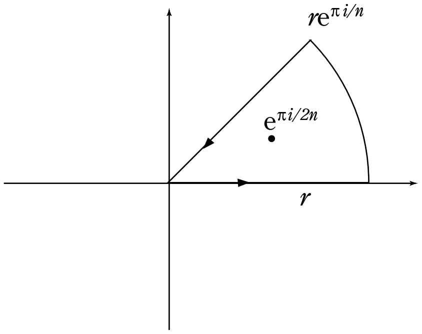

Figure 5.2: Path for example 5.2.6.

Example 5.2.6. Find $\int_0^\infty \frac{1}{1+x^{2n}}\,dx$.

Solution: The integrand is even, so we could just proceed as in the preceding example, replacing this integral by one over all of $\mathbb{R}$ and then applying Theorem 5.2.3. However, this would mean evaluating residues at each of the n poles in the upper half plane. There is a better way. We set

$$f(z) = \frac{1}{1+z^{2n}},$$

and note that

$$f(e^{\pi i/n}z) = f(z) \tag{5.2.7}$$

for each $z \in \mathbb{C}$. This means f has the same values along the ray $\{e^{\pi i/n}x : 0 \le x\}$ as it does along the ray $[0, +\infty)$. To exploit this, we choose $r > 1$ and use the path γ_r which traverses the boundary of the sector indicated in Figure 5.2. That is, the path γ_r which traverses the interval $[0, r]$, then the arc $re^{it}, 0 \le t \le \pi/n$, and then returns to 0 along the interval $[re^{\pi i/n}, 0]$.

Then

$$\begin{aligned}\int_{\gamma_r} f(z)\,dz &= \int_0^r f(x)\,dx + \int_0^{\pi/n} f(re^{it})rie^{it}\,dt - \int_0^r f(e^{\pi i/n}x)e^{\pi i/n}\,dx \\ &= (1-e^{\pi i/n})\int_0^r f(x)\,dx + \int_0^{\pi/n} f(re^{it})rie^{it}\,dt.\end{aligned}$$

The second integral on the right clearly goes to 0 as $r \to \infty$ and the integral on the left is $2\pi\mathrm{Res}(f, z_1)$, where $z_1 = e^{\pi i/2n}$ is the only pole of f inside γ_r. Thus,

$$\int_0^\infty f(x)\,dx = 2\pi i\frac{\mathrm{Res}(f, z_1)}{1-e^{\pi i/n}}.$$

All that remains is to evaluate this residue.

Since $z_1^{2n} = -1$, we have

$$z^{2n} + 1 = z^{2n} - z_1^{2n} = (z - z_1)(z^{2n-1} + z^{2n-2}z_1 + \cdots + z_1^{2n-1}).$$

Corollary 5.1.2(a) says that, to find the residue of f at z_1, we simply evaluate the inverse of the second factor on the right at $z = z_1$. The result is

$$\operatorname{Res}(f, z_1) = \frac{1}{2nz_1^{2n-1}} = -\frac{z_1}{2n} = -\frac{\mathrm{e}^{\pi i/2n}}{2n},$$

and this implies

$$\int_0^\infty \frac{1}{1+x^{2n}}\,dx = -\frac{2\pi i}{2n}\frac{\mathrm{e}^{\pi i/2n}}{1-\mathrm{e}^{\pi i/n}} = \frac{\pi/2n}{\sin(\pi/2n)}.$$

Exercise Set 5.2

1. Find $\displaystyle\int_0^{2\pi} \frac{d\theta}{5-4\cos\theta}$.
2. Find $\displaystyle\int_0^{2\pi} \frac{d\theta}{10+6\sin\theta}$.
3. Find $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2+2x+2}\,dx$.
4. Find $\displaystyle\int_0^{\infty} \frac{x^2}{(1+x^2)^2}\,dx$.
5. Find $\displaystyle\int_{-\infty}^{\infty} \frac{x}{(x^2+2x+2)^2}\,dx$
6. Find $\displaystyle\int_0^{\infty} \frac{1}{1+x^6}\,dx$ (see Example 5.2.6).
7. Find $\displaystyle\int_0^{\infty} \frac{1}{1+x^3}\,dx$ using contour integration over the boundary of the sector defined in polar coordinates by $\{re^{i\theta} : 0 \le r \le R,\ 0 \le \theta \le 2\pi/3\}$ with $R > 1$ (see Example 5.2.6).
8. Find $\displaystyle\int_{-\infty}^{\infty} \frac{\mathrm{e}^{ix}}{1+x^2}\,dx$.
9. Find $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(x)}{(1+x^2)^2}\,dx$ by first finding $\displaystyle\int_{-\infty}^{\infty} \frac{\mathrm{e}^{ix}}{(1+x^2)^2}\,dx$ and and then taking the real part of the result.
10. Find $\displaystyle\int_0^{\infty} \frac{\cos x}{1+x^2}\,dx$.
11. Prove the fact, used in the proof of Theorem 5.2.2, that if h is a function on $(0,\infty)$ and $\lim_{k\to\infty} h(x_k)$ exists for every increasing sequence $\{x_k\}$ of positive numbers converging to ∞, then these limits are all the same number L and $\lim_{x\to\infty} h(x) = L$.

12. Prove that an improper integral of the form (5.2.3) converges if f is a continuous function on $\mathbb{R}$ which satisfies an inequality $|f(x)| \leq C|x|^{-p}$ for $|x| > R$, where C and R are positive constants, and $p > 1$.

13. Suppose f is a continuous function on $(0, \infty)$ (which may have a singularity at 0). Show how to use the substitution $t = \mathrm{e}^s$ to convert the improper integral $\int_0^\infty f(t)\,dt$ into an improper integral on $(-\infty, \infty)$. Use this to prove a theorem analogous to Theorem 5.2.2 for improper integrals on $(0, \infty)$.

14. Find $\displaystyle\int_0^\infty \frac{dt}{t + t\log^2 t}$. Hint: Use the substitution of the previous exercise.

5.3 Fourier and Mellin Transforms

The technique of evaluating integrals using residue theory is particularly useful in the study of integral transforms such as the Fourier and Mellin Transforms. Often the integrals involved cannot be evaluated using standard integration techniques from calculus.

The Fourier Transform

The Fourier Transform is widely used in Differential Equations, Physics, and Engineering. Its generalizations are the subject of a major area of mathematical research, with applications that impact every branch of mathematics.

Definition 5.3.1. If f is a function defined on the real line and $t \in \mathbb{R}$, we set

$$\hat{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-itx}\,dx,$$

whenever this improper integral exists. The resulting function of t is called the *Fourier Transform* of f.

We will show how to use residue theory to evaluate Fourier Transforms of functions that are analytic on $\mathbb{C}$, except at a discrete set of singularities, and that vanish at infinity. The key to this technique is the fact that $|\mathrm{e}^{-itz}| = \mathrm{e}^{ty}$ vanishes rapidly as $y = \mathrm{Im}\,(z) \to \infty$ if $t < 0$, while it vanishes rapidly as $y \to -\infty$ if $t > 0$. In the first case, we are able to express the integral defining $\hat{f}$ as a sum of residues in the upper half plane, and in the second case we express it as a sum of residues in the lower half plane. The final result is the following theorem.

Theorem 5.3.2. *Let f be a function analytic on $\mathbb{C}$, except at a discrete set of singularities, with no singularities on the real line and with* $\lim_{z\to\infty} f(z) = 0$*. Then $\hat{f}(t)$ exists if $t \neq 0$. It is equal to $\sqrt{2\pi}\,i$ times the sum of the residues of $f(z)e^{-izt}$ in the upper half plane if $t < 0$, and is equal to $-\sqrt{2\pi}\,i$ times the sum of the residues of $f(z)e^{-izt}$ in the lower half plane if $t > 0$.*

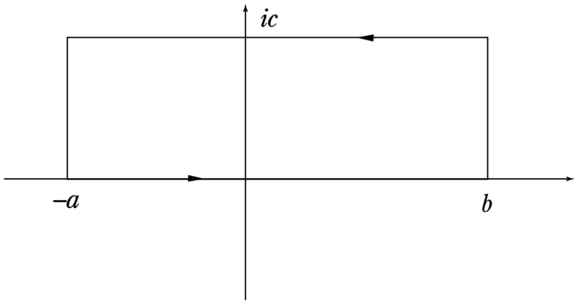

Figure 5.3: Path γ of Theorem 5.3.2.

Proof. We give the proof in case $t < 0$. The case $t > 0$ is the same except that the lower half plane is used instead of the upper half plane.

Since $\lim_{z\to\infty} f(z) = 0$, f has only finitely many singularities. For positive numbers a, b, and c let γ be the path which traverses the boundary of the rectangle with vertices $-a, b, b+ic, -a+ic$ in the positive direction (see Figure 5.3).

If a, b, and c are large enough, all singularities of f in the upper half plane will be inside γ. In particular, there will be no singularities on $\gamma(I)$. Then,

$$\begin{aligned}\int_\gamma f(z)\mathrm{e}^{-izt}\,dz = \int_{-a}^{b} f(x)\mathrm{e}^{-itx}\,dx + i\int_0^c f(b+iy)\mathrm{e}^{-it(b+iy)}\,dy \\ - \int_{-a}^{b} f(x+ic)\mathrm{e}^{-it(x+ic)}\,dx - i\int_0^c f(-a+iy)\mathrm{e}^{-it(-a+iy)}\,dy.\end{aligned} \tag{5.3.1}$$

We pass to the limit, first as $c \to \infty$, and then as a and b approach ∞ independently. We will show that each of the last three integrals on the right in (5.3.1) vanishes when we do this.

Since $|f(z)|$ is a continuous function on the complement of its set of singularities, and it has limit 0 at ∞, it has a maximum value on each line which doesn't hit one of the singularities. Let M_1 denote its maximum on the vertical line $\mathrm{Re}\,(z) = b$, M_2 its maximum on the horizontal line $\mathrm{Im}\,(z) = c$, and M_3 its maximum on the vertical line $\mathrm{Re}\,(z) = -a$. These numbers are functions of b, c, and a, respectively, and they each have limit 0 as a, b and c approach ∞, because $\lim_{z\to\infty} f(z) = 0$.

For the third integral on the right in (5.3.1), we have

$$\left|\int_{-a}^{b} f(x+ic)\mathrm{e}^{-it(x+ic)}\,dx\right| \le M_2\int_{-a}^{b} \mathrm{e}^{tc}\,dx = M_2\mathrm{e}^{tc}(b+a).$$

Since $t < 0$, this has limit 0 as $c \to \infty$ for fixed values of a and b.

For the second integral on the right in (5.3.1) we have

$$\left|\int_0^c f(b+iy)\mathrm{e}^{-it(b+iy)}\,dy\right| \le M_1\int_0^c \mathrm{e}^{ty}\,dy = \frac{M_1}{t}(\mathrm{e}^{tc}-1).$$

This has limit $-M_1/t$ as $c \to \infty$. Then, since M_1 has limit 0 as a, b approach ∞, the same is true of this integral. The argument that the fourth integral on the right in (5.3.1) has limit 0 as first c, then a, b approach ∞ is the same.

Of course, the integral on the left in (5.3.1) doesn't change once a, b, and c are large enough that the rectangle they determine contains all the singularities of f in the upper half plane. Once this is true, its value is $2\pi i$ times the sum of the residues at these singularities. Thus, after passing to the limit as, first c, then a and b approach ∞, (5.3.1) becomes

$$\begin{aligned} 2\pi i \sum_{j=1}^{n} \operatorname{Res}(f, z_j) &= \lim_{(a,b)\to(\infty,\infty)} \int_{-a}^{b} f(x)\mathrm{e}^{-itx}\, dx \\ &= \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-itx}\, dx = \sqrt{2\pi}\hat{f}(t), \end{aligned} \tag{5.3.2}$$

where $z_1, \cdots, z_n$ are the singularities of f in the upper half plane. On dividing by $\sqrt{2\pi}$, the proof of the theorem is complete in the case $t < 0$.

Note that the existence of the limit in (5.3.2) is exactly what is meant by the convergence of the improper integral defining $\hat{f}(t)$.

The proof in the case $t < 0$ is almost identical. The difference is: the number c is negative. Then the path γ lies in the lower half plane with negative rather that positive orientation. This results in the left side of (5.3.1) being $-2\pi i$ times the sum of the residues of $f\mathrm{e}^{-itz}$ in the lower half plane for large enough a, b, and $-c$. □

Example 5.3.3. Find the Fourier Transform of $f(x) = \dfrac{1}{1+x^2}$.

Solution: We set

$$f(z) = \frac{1}{1+z^2}.$$

If $t < 0$, the previous theorem applies, and it tells us that $\hat{f}(t)$ is $\sqrt{2\pi}\, i$ times the sum of the residues of $\mathrm{e}^{-itz} f(z)$ in the upper half plane. Now f has only one pole in the upper half plane and that is at i. The residue of $\mathrm{e}^{-itz} f(z)$ at this point can be computed using Corollary 5.1.2(a), since

$$f(z) = \frac{1}{(z+i)(z-i)}.$$

The residue is obtained by evaluating $\mathrm{e}^{-itz}/(z+i)$ at $z = i$. The result is

$$\frac{\mathrm{e}^{t}}{2i},$$

and so

$$\hat{f}(t) = \sqrt{\frac{\pi}{2}}\, \mathrm{e}^{t} \quad \text{if} \quad t < 0.$$

For $t > 0$, we compute the residue of $\mathrm{e}^{-itz} f(z)$ at $-i$ and then multiply by $-\sqrt{2\pi}\, i$. The result is

$$\hat{f}(t) = \sqrt{\frac{\pi}{2}}\, \mathrm{e}^{-t} \quad \text{if} \quad t > 0.$$

At $t = 0$ we have

$$\hat{f}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \sqrt{\frac{\pi}{2}},$$

by the case $n = 1$ of Example 5.2.6 (or simply by noting that arctan is an antiderivative for the integrand).

Putting the cases $t < 0, t > 0, t = 0$ together, we conclude that

$$\hat{f}(t) = \sqrt{\frac{\pi}{2}}\,\mathrm{e}^{-|t|} \quad \text{for all} \quad t \in \mathbb{R}.$$

Example 5.3.4. Show that $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\,dx = \pi/\mathrm{e}$.

Solution: The integral we seek is the real part of the integral

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{-ix}}{1+x^2}\,dx$$

which is $\sqrt{2\pi}$ times the Fourier Transform of $\dfrac{1}{1+x^2}$ evaluated at $t = 1$. By the previous example, this is π/e.

Example 5.3.5. Find the Fourier Transform of $f(x) = \dfrac{1}{x+i}$.

Solution: We set

$$f(z) = \frac{1}{z+i}.$$

This function is meromorphic, with no poles on the real axis, and it vanishes at infinity. Thus, Theorem 5.3.2 applies.

The only pole of f occurs at $z = -i$. Since there are no poles in the upper half plane, Theorem 5.3.2 tells us that $\hat{f}(t) = 0$ if $t < 0$.

For $t > 0$, $\hat{f}(t)$ is $-\sqrt{2\pi}\,i$ times the residue of $f(z)\mathrm{e}^{-itz}$ at $z = -i$. This residue is e^{-t} and so

$$\hat{f}(t) = \left\{ \begin{array}{cc} 0 & \text{if} \quad t < 0 \\ -\sqrt{2\pi}\,i\mathrm{e}^{-t} & \text{if} \quad t > 0 \end{array} \right\}.$$

Note the jump discontinuity at $t = 0$.

What about the value of $\hat{f}$ at $t = 0$? We have

$$\hat{f}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\,dx$$

We can attempt to evaluate this using the fact that $f(z)$ has $\log(z + i)$ as an antiderivative, given any branch of the log function. If we use the principal branch of log, then the line $\mathrm{Im}\,(z) = i$ is contained in its domain of definition, and so the real line is contained in the domain on which $\log(z + i)$ is defined and has $f(z)$ as its derivative. We conclude that

$$\begin{aligned}\int_{-a}^{b} f(x)\,dx &= \log(b+i) - \log(-a+i)\\ &= \frac{1}{2}(\log(b^2+1) - \log(a^2+1)) + i(\arg(b+i) - \arg(-a+i)).\end{aligned}$$

The limit as $(a,b) \to (\infty,\infty)$ of this expression does not exist, since its real part approaches $+\infty$ as $b \to \infty$ with a fixed and approaches $-\infty$ as $a \to \infty$ with b fixed. Thus, the improper integral defining $\hat{f}(0)$ does not converge. However, if we consider the symmetric integral

$$\int_{-a}^{a} f(x)\,dx = i(\arg(a+i) - \arg(-a+i)).$$

This has limit $-\pi i$ as $a \to \infty$. In this case, we say that the integral has *principal value* $-\pi i$, even though it is not a convergent improper integral. If we define $\hat{f}(0)$ using this value for the integral, then we have

$$\hat{f}(0) = -\sqrt{\frac{\pi}{2}}\,i,$$

which is exactly half way between $\lim_{t\to 0^+} \hat{f}(t)$ and $\lim_{t\to 0^-} \hat{f}(t)$.

The next example involves a different technique – no residues – but the Cauchy Theorems are still involved.

Example 5.3.6. Find the Fourier Transform of the normal distribution function

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

Solution: We have

$$\hat{f}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-(x^2+2itx)/2}\,dx.$$

By completing the square in the exponent, we can write this as

$$\begin{aligned}\hat{f}(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-(x+it)^2/2} e^{-t^2/2}\,dx\\ &= \frac{e^{-t^2/2}}{2\pi}\int_{-\infty}^{\infty} e^{-(x+it)^2/2}\,dx.\end{aligned} \tag{5.3.3}$$

We will use Cauchy's Integral Theorem to show that the last integral above is independent of t.

Fix $a, b > 0$ and consider the path γ that traces once in the positive direction around the boundary of the rectangle with vertices at $-a, b, b+ti, -a+ti$ (see Figure 5.4).

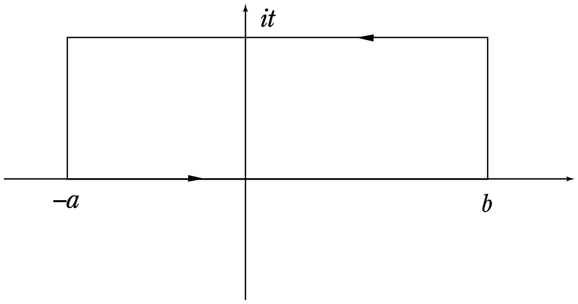

Figure 5.4: Path for the Fourier Transform in Example 5.3.6.

The integral of $\mathrm{e}^{-z^2/2}$ over this path is 0 by Cauchy's Theorem. Hence,

$$\begin{aligned} 0 = \int_{-a}^{b} \mathrm{e}^{-x^2/2}\,dx &+ \int_0^t \mathrm{e}^{-(b+iy)^2/2} i\,dy \\ &- \int_{-a}^{b} \mathrm{e}^{-(x+it)^2/2}\,dx - \int_0^t \mathrm{e}^{-(-a+iy)^2/2} i\,dy \end{aligned} \tag{5.3.4}$$

The second of these integrals satisfies the following estimate:

$$\left| \int_0^t \mathrm{e}^{-(b+iy)^2/2} i\,dy \right| \leq |t|\mathrm{e}^{y^2-b^2},$$

which implies that it has limit 0 as $b \to \infty$. A similar estimate on the fourth integral in (5.3.4) shows that it has limit 0 as $a \to 0$. It follows that

$$\int_{-\infty}^{\infty} \mathrm{e}^{-(x+it)^2/2}\,dx = \int_{-\infty}^{\infty} \mathrm{e}^{-x^2/2}\,dx$$

which means that the integral on the left is independent of t and can be evaluated by evaluating the integral on the right. The integral on the right is a standard calculus problem. Its value is $\sqrt{2\pi}$, and it is obtained by expressing the square of the integral as a double integral over the plane and then evaluating this using polar coordinates. We leave it as an exercise (Exercise 5.3.1).

In view of (5.3.3) and the above calculation, we have

$$\hat{f}(t) = \frac{\mathrm{e}^{-t^2/2}}{2\pi}\sqrt{2\pi} = \frac{\mathrm{e}^{-t^2/2}}{\sqrt{2\pi}} = f(t).$$

In other words, the normal distribution function is its own Fourier Transform.

The Mellin Transform

If f is a function defined on the positive real numbers, then the function of t defined by

$$\int_0^{\infty} f(x)x^{t-1}\,dx, \tag{5.3.5}$$

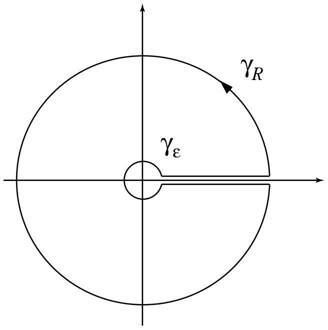

Figure 5.5: Path for Computing the Mellin Transform.

with $t > 0$, is called the Mellin Transform of f. Of course, this improper integral may converge for only certain values of t and possibly for no values of t. Using residue theory, we will derive some conditions on f and t which ensure this integral does converge, and give a formula for computing it.

We assume that f is analytic on $\mathbb{C}$ except at a finite set of singularities, none of which lie on the positive real axis. We define a function z^{t-1}, analytic on $\mathbb{C} \setminus [0, \infty)$, in the following way: We cut the plane along the positive real axis and denote by $\log(z)$ for $z \neq 0$ the branch of the log function determined by restricting $\arg(z)$ to lie in the interval $[0, 2\pi]$. Note that, since we allow both $\arg(z) = 0$ and $\arg(z) = 2\pi$, this results in there being two values of log at each point on the positive real axis. Think of the cut along $(0, \infty)$ as resulting in two copies of the positive real axis – an upper edge ($\arg(z) = 0$) and a lower edge ($\arg(z) = 2\pi$). Our log function is the ordinary real valued natural log function along the upper edge of the cut, and is this plus $2\pi i$ along the lower edge of the cut. We then define

$$z^{t-1} = \mathrm{e}^{(t-1)\log z} \quad \text{for} \quad z \neq 0.$$

We will compute the integral (5.3.5) by using path integration on the function $f(z)z^{t-1}$.

The path γ for our path integral is the path which goes along the upper edge of the cut from $\epsilon > 0$ to R, then along the circle γ_R of radius R centered at the origin, then along the lower edge of the cut from R to ϵ and finally around the origin along the circle γ_ϵ of radius ϵ (see Figure 5.5).

Let $z_1, z_2, \cdots, z_n$ be the non-zero singularities of f and assume that R is large enough and ϵ small enough that all of these singularities lie inside the circle of radius R and outside the circle of radius ϵ. Then γ has index 1 about each z_j. If we set

$$g(z) = f(z)z^{t-1} \quad \text{for} \quad z \notin [0, +\infty),$$

then, by the Residue Theorem, we have

$$\begin{aligned} 2\pi i \sum_{j=1}^{n} Res(g, z_j) &= \int_{\gamma} f(z) z^{t-1}\, dz \\ &= \int_{\epsilon}^{R} f(x) x^{t-1}\, dx + \int_{\gamma_R} f(z) z^{1-t}\, dz \\ &- \mathrm{e}^{2\pi i(t-1)} \int_{\epsilon}^{R} f(x) x^{t-1}\, dx - \int_{\gamma_\epsilon} f(z) z^{t-1}\, dz. \end{aligned} \tag{5.3.6}$$

If, as $R \to \infty$ and $\epsilon \to 0$, the integrals around γ_R and γ_ϵ have vanishing limits, then

$$2\pi i \sum_{j=1}^{n} Res(g, z_j) = (1 - \mathrm{e}^{2\pi i t}) \int_{0}^{\infty} f(x) x^{t-1}\, dx.$$

If t is not an integer, we may solve for the integral. This yields

$$\int_{0}^{\infty} f(x) x^{t-1}\, dx = -\pi \frac{\mathrm{e}^{-\pi i t}}{\sin(\pi t)} \sum_{j=1}^{n} Res(g, z_j) \tag{5.3.7}$$

It remains to derive conditions that ensure the integrals around the two circles in (5.3.6) have limits 0 as $R \to \infty$ and $\epsilon \to 0$. If $M(R)$ is the maximum value of $|f(z)|$ on the circle of radius R, then

$$\left| \int_{\gamma_R} f(z) z^{t-1}\, dz \right| \le 2\pi M(R) R^t.$$

Hence, this integral will have limit 0 as $R \to \infty$ if

$$\lim_{z \to \infty} |f(z)||z|^t = 0.$$

Similarly, if $M(\epsilon)$ is the maximum value of f on the circle of radius ϵ, then

$$\left| \int_{\gamma_\epsilon} f(z) z^{t-1}\, dz \right| \le 2\pi M(\epsilon) \epsilon^t,$$

and so this integral will have limit 0 as $\epsilon \to 0$ if

$$\lim_{z \to 0} |f(z)||z|^t = 0.$$

Thus, we have proved the following theorem

Theorem 5.3.7. *Let f be a function analytic on $\mathbb{C}$, except at finitely many singularities, none of which are on the positive real axis, and let $z_1, z_2, \cdots, z_n$ be the non-zero singularities of f. Suppose t is not an integer and the function $|f(z)||z|^t$ has limit 0 as $z \to 0$ and as $z \to \infty$. Then the Mellin Transform of f at t exists and is given by*

$$\int_{0}^{R} f(x) x^{t-1}\, dx = -\pi \frac{e^{-\pi t i}}{\sin(\pi t)} \sum_{j=1}^{n} Res(g, z_j), \tag{5.3.8}$$

where $g(z) = f(z) z^{t-1}$ on $\mathbb{C} \setminus [0, +\infty)$.

The calculation of the Mellin Transform for the function $\dfrac{1}{1+x}$ is a key step in the developement of the properties of the Gamma function in Chapter 9.

Example 5.3.8. Show that the Mellin Transform of $f(x) = \dfrac{1}{1+x}$ is $\dfrac{\pi}{\sin(\pi t)}$.

Solution: According to the preceding theorem, the Mellin Transform of this function f will exist if $0 < t < 1$, since it is for these values of t that

$$\lim_{z\to 0} \frac{|z|^t}{|1+z|} = 0 = \lim_{z\to\infty} \frac{|z|^t}{|1+z|}.$$

The only singularity of $f(z)$ occurs when $z = -1$, and the corresponding residue, $\mathrm{Res}(g, -1)$ for $g(z) = f(z)z^{t-1}$, is

$$(-1)^{t-1} = \mathrm{e}^{\pi(t-1)i} = -\mathrm{e}^{\pi t i}.$$

Thus, by Theorem 5.3.7, if $0 < t < 1$, then

$$\int_0^\infty \frac{x^{t-1}}{1+x}\,dx = \frac{\pi}{\sin(\pi t)}.$$

Exercises Set 5.3

1. Prove that
$$\int_{-\infty}^{\infty} \mathrm{e}^{-x^2/2}\,dx = \sqrt{2\pi}$$
by expressing the square of the integral as a double integral over $\mathbb{R}^2$ which can then be evaluated using polar coordinates.

2. Find the Fourier Transform of $f(x) = \dfrac{x}{1+x^2}$.

3. Find the Fourier Transform of $f(x) = \dfrac{1}{1+x^4}$.

4. Find the Fourier Transform of $f(x) = \sqrt{\dfrac{\pi}{2}}\,\mathrm{e}^{-|x|}$.

5. Find the Fourier Transform of $f(x) = \dfrac{1}{x^2+4x+5}$.

6. If f and f' are both continuous and have Fourier Transforms which exist at t, and if f has limit 0 at both ∞ and $-\infty$, prove that the Fourier Transform of f' at t is $it\hat{f}(t)$. Hint: use integration by parts.

7. Find $\displaystyle\int_{-\infty}^{\infty} \frac{x\sin x}{1+x^2}\,dx$.

8. Find the Mellin Transform of $f(x) = \dfrac{1}{1+x^2}$.

9. Find the Mellin Transform of $f(x) = \dfrac{1}{1+x^3}$.
10. Find $\displaystyle\int_0^\infty \frac{x^{2/3}}{1+x}\,dx$.
11. Show that if $h(y) = f(\mathrm{e}^y)$, then the Mellin Transform of f can be expressed as $\displaystyle\int_{-\infty}^{\infty} h(y)\mathrm{e}^{ty}\,dy$. Note the similarity to the Fourier Transform.
12. Verify the second equality in Equation 5.3.6.

5.4 Summing Infinite Series

In this section we demonstrate a method for using the Residue Theorem to calculate the sum of an infinite series of the form

$$\sum_{n=-\infty}^{\infty} f(n), \tag{5.4.1}$$

where f is an analytic function with isolated singularities in the complex plane.

The idea is this: If we can find an analytic function g, such that g has a simple pole with residue 1 at each integer n, then fg will have a simple pole with residue $f(n)$ at each integer n (see Exercise 5.1.11). If we can also choose an expanding sequence of simple closed paths $\{\gamma_N\}$ so that each singularity of fg is contained in γ_n for sufficiently large N, and so that the integral of fg around γ_N has limit 0 as $N \to \infty$, then the Residue Theorem will imply that the sum of all residues of fg constitutes a convergent series with sum 0. If f has only finitely many singularities and none of them are integers, then the sum (5.4.1) will necessarily converge to the negative of the sum of the residues of fg at the singularities of f. If some of the singularities of f occur at integers, then the result will need to be adjusted accordingly.

This program actually works under reasonable hypotheses on f, if g is chosen well. A particularly good choice is $g(z) = \pi\cot(\pi z)$.

Some Properties of the Cotangent

Since $\sin(\pi z)$ has a zero of order one at every integer and no other zeroes on $\mathbb{C}$, the meromorphic function

$$\cot(\pi z) = \frac{\cos(\pi z)}{\sin(\pi z)}$$

has a simple pole at each integer and no other poles. The residue of this function is $1/\pi$ at each n (Exercise 5.1.5). Thus,

Lemma 5.4.1. *The function* $\pi\cot(\pi z)$ *has a simple pole with residue* 1 *at each integer, and no other poles.*

The function $\pi \cot(\pi z)$ is obviously not bounded near ∞, since it has a pole at each integer. However, it turns out that there is a sequence of closed paths, converging to ∞, on which this function is bounded. The paths in question are the paths γ_N, where N is a positive integer and γ_N traces once in the positive direction around the square with vertices at the points $(N+1/2)(1+i)$, $(N+1/2)(-1+i)$,$(N+1/2)(-1-i)$, and $(N+1/2)(1-i)$ (see Figure 5.6).

Lemma 5.4.2. *There is a positive number R such that, if $N \geq R$, then $|\cot(\pi z)| \leq 2$ on $\gamma_N(I)$.*

Proof. We have

$$\begin{aligned}\cot(\pi z) &= \frac{\cos(\pi z)}{\sin(\pi z)} = i\frac{e^{\pi i z} + e^{-\pi i z}}{e^{\pi i z} - e^{-\pi i z}} \\ &= i\frac{e^{2\pi i z} + 1}{e^{2\pi i z} - 1} = i\frac{e^{2\pi i x}e^{-2\pi y} + 1}{e^{2\pi i x}e^{-2\pi y} - 1},\end{aligned}$$

where $z = x + iy$. Thus,

$$|\cot(\pi z)| = \left|\frac{e^{2\pi i x}e^{-2\pi y} + 1}{e^{2\pi i x}e^{-2\pi y} - 1}\right|.$$

If $x = N + 1/2$ for an integer N, then $e^{2\pi i x} = -1$ and $|\cot(\pi z)| \leq 1$. Thus, $|\cot(\pi z)| \leq 1$ on the vertical sides of each γ_N. On the horizontal sides, where y is fixed at $\pm(N+1/2)$, the maximum of $|\cot(\pi z)|$ occurs when $e^{2\pi i x} = 1$, that is, at integer values of x. Thus, the maximum is

$$\left|\frac{e^{\pm(2N+1)\pi} + 1}{e^{\pm(2N+1)\pi} - 1}\right|.$$

Since this expression has limit 1 as $N \to \infty$, there is an M such that it is less than 2 when $N > M$. □

Lemma 5.4.3. *With γ_N as above,*

$$\int_{\gamma_N} \frac{\pi \cot(\pi z)}{z} = 0 \tag{5.4.2}$$

for each positive integer N.

Proof. The function $h(z) = \dfrac{\pi \cot(\pi z)}{z}$ is analytic inside γ_N except at integer points, where it has poles. By the Residue Theorem

$$\int_{\gamma_N} \frac{\pi \cot(\pi z)}{z} = \sum_{n=-N}^{N} \operatorname{Res}(h, n). \tag{5.4.3}$$

Since h is an even function, its residue at 0 is 0 (Exercise 5.1.13). The poles of h at non-zero integers are simple poles and may be calculated using Corollary 5.1.6. The result is that the residue of h at n is $1/n$. Since this is an odd function of n and the range of summation in (5.4.3) is symmetric about the origin, the integral in (5.4.2) is 0. □

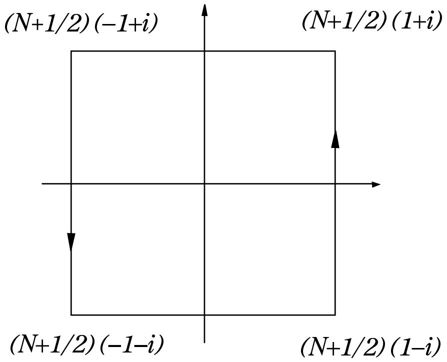

Figure 5.6: Path γ_N.

A Summation Theorem

We now have enough information about the function $g(z) = \pi \cot(\pi z)$ to prove the following summation theorem.

Theorem 5.4.4. *Suppose f is analytic on $\mathbb{C}$ except at a finite set*

$$E = \{z_1, z_2, \cdots, z_n\}$$

of isolated singularities. Also suppose that there exist positive constants R and M such that

$$|f(z)| \le \frac{M}{|z|} \quad \textit{for} \quad |z| \ge R. \tag{5.4.4}$$

Then

$$\lim_{N \to \infty} \sum_{n \in \mathbb{Z}_0^N} f(n) = -\sum_{j=1}^{m} \mathit{Res}(\pi f(z) \cot(\pi z), z_j),$$

where Z_0^N is the set of integers in $[-N, N]$ which do not belong to E .

Proof. The singularities of $\pi f(z) \cot(\pi z)$ occur at the integers and at the points $z_1, \cdots, z_n$. At an integer n not in E, this function has residue $f(n)$, since $\cot(\pi z)$ has a simple pole with residue 1 at n and f has no singularity at n. If N is large enough so that the z_j are all inside γ_N, then the Residue Theorem implies

$$\frac{1}{2\pi i} \int_{\gamma_N} \pi f(z) \cot(\pi z)\, dz = \sum_{n \in \mathbb{Z}_0^N} f(n) + \sum_{j=1}^{m} \mathrm{Res}(\pi f(z) \cot(\pi z), z_j). \tag{5.4.5}$$

Thus, to prove the theorem, we simply need to show that the integral on the left converges to 0 as $N \to \infty$.

Choose R and M large enough that (5.4.4) holds, the inequality in Lemma 5.4.2 holds, and $|z_j| \le R$ for each j. Then f is analytic in the annulus

$$A = \{z : R < |z| < \infty\}$$

and vanishes at infinity. It therefore has a Laurent expansion, in this annulus, involving only negative powers of z. Say,

$$f(z) = \frac{c_{-1}}{z} + \frac{c_{-2}}{z^2} + \cdots + \frac{c_{-n}}{z^n} + \cdots .$$

By Lemma 5.4.3,

$$\int_{\gamma_N} \frac{\pi \cot(\pi z)}{z}\, dz = 0.$$

and so

$$\frac{1}{2\pi i} \int_{\gamma_N} \pi f(z) \cot(\pi z)\, dz = \frac{1}{2\pi i} \int_{\gamma_N} \pi \left(f(z) - \frac{c_{-1}}{z} \right) \cot(\pi z)\, dz. \tag{5.4.6}$$

However, the function

$$f(z) - \frac{c_{-1}}{z} = \frac{c_{-2}}{z^2} + \cdots + \frac{c_{-n}}{z^n} + \cdots$$

has the form $\frac{q(z)}{z^2}$ in A, where q is analytic in A and has limit c_{-2} at infinity. This implies that, for some $R_1 > R$, the function $|q|$ is bounded by some positive constant M_1 on $\{z : |z| \ge R_1\}$. Then, by Lemma 5.4.2, the integrand of the integral on the right in (5.4.6) is bounded above by $2MM_1/|z|^2$. Since $|z| > N$ on γ_N and the length of the path γ_N is $8N + 4$, we have that

$$\left| \frac{1}{2\pi i} \int_{\gamma_N} \pi f(z) \cot(\pi z)\, dz \right| \le \frac{2MM_1(8N+4)}{N^2},$$

which converges to 0 as $N \to \infty$. This completes the proof. $\square$

Example 5.4.5. Prove that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Solution: We apply the preceding theorem with $f(z) = 1/z^2$. It clearly satisfies the hypothesis of the theorem. The only singularity of this function is the one at 0 and so the theorem tells us that

$$\sum_{n \in \mathbb{Z}_0} \frac{1}{n^2} = -\mathrm{Res}\left(\frac{\pi \cot(\pi z)}{z^2}, 0 \right),$$

where $\mathbb{Z}_0$ is the set of non-zero integers. The residue of $\frac{\cot z}{z^2}$ at 0 is $-1/3$ by Example 5.1.9. Since the coefficient of z in the Laurent expansion of $\pi \cot(\pi z)$ is π^2 times the coefficient of z in the Laurent expansion of $\cot z$, we conclude that $\frac{\pi \cot(\pi z)}{z^2}$ has residue $\pi^2/3$ at 0. Then,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{2} \sum_{n \in \mathbb{Z}_0} \frac{1}{n} = \frac{\pi^2}{6}.$$

Exercise Set 5.4

1. Prove that, like $\pi\cot(\pi z)$, the function $\dfrac{2\pi i}{\mathrm{e}^{2\pi i z}-1}$ has a simple pole with residue 1 at each integer and it has no other poles.

2. Find $\displaystyle\sum_{n=1}^{\infty}\frac{1}{1+n^2}$.

3. Find $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(n-i)^2}$.

4. Find $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^4}$.

5. Prove that if $w\in\mathbb{C}\setminus\mathbb{Z}$, then $\displaystyle\pi\cot\pi w=\lim_{N\to\infty}\sum_{n=-N}^{N}\frac{1}{n+w}$.

6. Assuming the result of the previous exercise, prove that
$$\pi\cot\pi w=\frac{1}{w}+\sum_{n=1}^{\infty}\frac{1}{w^2-n^2}.$$

7. Prove that the function $\dfrac{\pi}{\sin(\pi z)}$ has a pole at each integer n with residue $(-1)^n$ and no other poles.

8. Derive a method for summing a series of the form $\displaystyle\sum_{-\infty}^{\infty}(-1)^n f(n)$, where f is a meromorphic function with a finite number of poles. Hint: use the preceding exercise and the method of Theorem 5.4.4.