Chapter 5

The Integral

In this chapter we define the Riemann integral and develop its most important properties. We also prove the Fundamental Theorem of Calculus and discuss improper integrals.

5.1 Definition of the Integral

If $[a, b]$ is a closed, bounded interval, then a *partition* P of $[a, b]$ is a finite, ordered set of points

$$P = \{a = x_0 < x_1 < \cdots < x_n = b\}$$

of $[a, b]$, beginning with a and ending with b. Such a set of points has the effect of dividing $[a, b]$ into a collection of n subintervals

$$[x_0, x_1], [x_1, x_2], \cdots, [x_{n-1}, x_n].$$

Given a partition P, as above, of $[a, b]$ and a bounded function f, defined on $[a, b]$, a *Riemann Sum* for f and P on $[a, b]$ is a sum of the form

$$\sum_{k=1}^{n} f(\bar{x}_k)(x_k - x_{k-1}) \tag{5.1.1}$$

where, for each k, $\bar{x}_k$ is some point in the interval $[x_{k-1}, x_k]$. For each k, the term $f(\bar{x}_k)(x_k - x_{k-1})$ represents the area (or minus the area, if $f(\bar{x}_k) < 0$) of a rectangle with width $x_k - x_{k-1}$ and with height $|f(\bar{x}_k)|$ (see Figure 5.1).

In calculus, the Riemann Integral of f is defined as a limit of Riemann sums, although how this limit is defined and how one shows that it actually exists for a reasonable class of functions are things that are usually left for a more advanced course. This is that course.

Here we will give a precise definition of the integral and prove that it exists for a large class of functions on $[a, b]$. In particular, we will prove that the integral of every continuous function on $[a, b]$ exists.

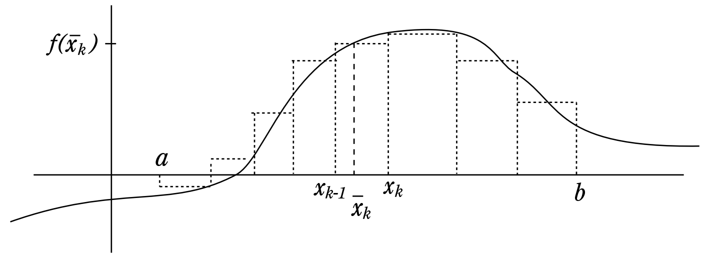

Figure 5.1: A Riemann Sum.

Upper and Lower Sums

Given a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a, b]$ and a bounded function f on $[a, b]$, we can write down two sums which have every Riemann sum for this partition and this function trapped in between them. These are the upper and lower sums for P and f:

Definition 5.1.1. Given a partition P and function f, as above, for $k = 1, \cdots, n$, we set

$$M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\} \quad \text{and} \quad m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\}.$$

Then the *upper sum* for f and P is

$$U(f, P) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}), \tag{5.1.2}$$

while the *lower sum* for f and P is

$$L(f, P) = \sum_{k=1}^{n} m_k(x_k - x_{k-1}). \tag{5.1.3}$$

Now, for any choice of $\bar{x}_k \in [x_{k-1}, x_k]$, we have

$$m_k \le f(\bar{x}_k) \le M_k.$$

This inequality remains true if we multiply through by the positive number $(x_k - x_{k-1})$. On summing the resulting inequalities, we conclude that

$$L(f, P) \le \sum_{k=1}^{n} f(\bar{x}_k)(x_k - x_{k-1}) \le U(f, P). \tag{5.1.4}$$

Thus, the upper sum $U(f, P)$ is an upper bound for all Riemann sums for f and P and the lower sum is a lower bound for all these sums. In fact, it is not hard to prove that $U(f, P)$ is the least upper bound for all Riemann sums for f and P, while $L(f, P)$ is the greatest lower bound of this set (Exercise 5.1.6).

Example 5.1.2. Find the upper sum and lower sum for the function $f(x) = x^2$ and the partition $P = \{0 < 1/4 < 1/2 < 3/4 < 1\}$ of the interval $[0, 1]$.

Solution: The function f is increasing on $[0, 1]$ and so its sup on each subinterval is achieved at the right endpoint of the interval and its inf is achieved at the left endpoint. Thus,

$$\begin{aligned} &L(f, P) \\ &= 0(1/4 - 0) + 1/16(1/2 - 1/4) + 1/4(3/4 - 1/2) + 9/16(1 - 3/4) = \frac{7}{32} \end{aligned}$$

while

$$\begin{aligned} &U(f, P) \\ &= 1/16(1/4 - 0) + 1/4(1/2 - 1/4) + 9/16(3/4 - 1/2) + 1(1 - 3/4) = \frac{15}{32}. \end{aligned}$$

Refinement of Partitions

It is useful to think of a partition of $[a, b]$ as simply a finite subset of $[a, b]$ that contains a and b. The elements of this finite set are then given labels $x_0, x_1, \cdots, x_n$ which are consistent with the order in which these elements occur in $[a, b]$. Thus, $a = x_0 < x_1 < \cdots < x_n = b$. To think of partitions as subsets of $[a, b]$ allows us to use set theoretic relations and operations such as"$\subset$" and "$\cup$" on them.

Definition 5.1.3. Let P and Q be partitions of a closed bounded interval $[a, b]$. Then we say that Q is a *refinement* of P if $P \subset Q$.

For example, the partition $0 < 1/4 < 1/3 < 1/2 < 2/3 < 3/4 < 1$ is a refinement of the partition $0 < 1/4 < 1/2 < 3/4 < 1$.

Theorem 5.1.4. *Let f be a bounded function on a closed bounded interval $[a, b]$. If Q and P are partitions of $[a, b]$ and Q is a refinement of P, then*

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P). \tag{5.1.5}$$

Proof. We will prove this in the case where Q is obtained from P by adding just one additional point to P. The general case then follows from this using an induction argument on the number of additional points needed to get from P to Q (Exercise 5.1.7).

Suppose $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ and Q is obtained by adding one point y to P. Suppose this new point lies between x_{j-1} and x_j. Then, in passing from P to Q, the subinterval $[x_{j-1}, x_j]$ is cut into the two subintervals $[x_{j-1}, y]$ and $[y, x_j]$, while all other subintervals $[x_{k-1}, x_k]$ $(k \neq j)$ remain the same. Thus, In the formulas (5.1.2) and (5.1.1) for the upper and lower sums, the terms for which $k \neq j$ are unchanged when we pass from P to Q. To prove the theorem, we just need to analyze what happens to the jth terms in (5.1.2) and (5.1.1) when P is replaced by Q.

With m_j and M_j as in Definition 5.1.1 for the partition P, we set

$$m'_j = \inf\{f(x) : x \in [x_{j-1}, y]\}, \quad M'_j = \sup\{f(x) : x \in [x_{j-1}, y]\},$$
$$m''_j = \inf\{f(x) : x \in [y, x_j]\}, \quad M''_j = \sup\{f(x) : x \in [y, x_j]\}.$$

Then $m_j = \min\{m'_j, m''_j\}$ and $M_j = \max\{M'_j, M''_j\}$, and so

$$\begin{aligned} m_j(x_j - x_{j-1}) &= m_j(y - x_{j-1}) + m_j(x_j - y) \\ &\le m'_j(y - x_{j-1}) + m''_j(x_j - y), \end{aligned}$$

while

$$\begin{aligned} &M'_j(y - x_{j-1}) + M''_j(x_j - y) \\ &\le M_j(y - x_{j-1}) + M_j(x_j - y) = M_j(x_j - x_{j-1}). \end{aligned}$$

Now (5.1.5) follows from this and the fact that the other terms in the sums defining $U(f, P)$ and $L(f, P)$ are unchanged when P is replaced by Q. □

Note that any two partitions P and Q of an interval $[a, b]$ have a common refinement. In fact, the set theoretic union $P \cup Q$ is a common refinement of P and Q. This, together with the preceding result leads to the following theorem, which says that every lower sum is less than or equal to every upper sum.

Theorem 5.1.5. *If P and Q are any two partitions of a closed bounded interval $[a, b]$ and f is a bounded function on $[a, b]$, then*

$$L(f, P) \le U(f, Q).$$

Proof. We simply apply the previous theorem to P and its refinement $P \cup Q$ and to Q and its refinement $P \cup Q$. This yields

$$L(f, P) \le L(f, P \cup Q) \le U(f, P \cup Q) \le U(f, Q).$$

□

The Integral

Given a closed bounded interval $[a, b]$ and a bounded function f on $[a, b]$, we set

$$U_a^b(f) = \inf\{U(f, Q) : Q \text{ a partition of } [a, b]\},$$
$$L_a^b(f) = \sup\{L(f, Q) : Q \text{ a partition of } [a, b]\}.$$

Theorem 5.1.5 says that every lower sum for f is less than or equal to every upper sum for f. Thus, each upper sum $U(f, P)$ is an upper bound for the set of all lower sums. Hence, it is at least as large as the least upper bound of this set; that is

$$L_a^b(f) \le U(f, P) \quad \text{for all partitions} \quad P \text{ of } [a, b].$$

This, in turn, means that $L_a^b(f)$ is a lower bound for the set of all upper sums and, hence, is less than or equal to the greatest lower bound of this set. That is,

$$L_a^b(f) \le U_a^b(f).$$

Definition 5.1.6. With f, $[a,b]$, $U_a^b(f)$, and $L_a^b(f)$ as in the above discussion, we call $L_a^b(f)$ the *lower integral* and $U_a^b(f))$ the *upper integral* of f on $[a,b]$. If the two are equal, we say that f is *integrable* on $[a,b]$ or that the Riemann Integral of f on $[a,b]$ exists. Its value is then the common value of $U_a^b(f)$ and $L_a^b(f)$ and is denoted by

$$\int_a^b f(x)\,dx.$$

Theorem 5.1.7. *The Riemann Integral of f on $[a,b]$ exists if and only if, for each $\epsilon > 0$, there is a partition P of $[a,b]$ such that*

$$U(f,P) - L(f,P) < \epsilon. \tag{5.1.6}$$

Proof. Suppose the integral exists. Then

$$\sup_P L(f,P) = L_a^b(f) = U_a^b(f) = \inf_P U(f,P),$$

where P ranges over all partitions of $[a,b]$. Thus, given $\epsilon > 0$, the number $L_a^b(f) - \epsilon/2$ is not an upper bound for the set of all $L(f,P)$ and the number $U_a^b(f) + \epsilon/2$ is not a lower bound for the set of all $U(f,P)$. This means there are partitions Q_1 and Q_2 of $[a,b]$ such that

$$L_a^b(f) - \epsilon/2 < L(f,Q_1) \le U(f,Q_2) < U_a^b(f) + \epsilon/2.$$

If P is a common refinement of Q_1 and Q_2, then Theorem 5.1.4 implies that

$$L_a^b(f) - \epsilon/2 < L(f,Q_1) \le L(f,P) \le U(f,P) \le U(f,Q_2) < U_a^b(f) + \epsilon/2.$$

Since $L_a^b(f) = U_a^b(f)$, this implies that (5.1.6) holds.

Conversely, suppose that for each $\epsilon > 0$ there is a partition P such that (5.1.6) holds. Then

$$L(f,P) \le L_a^b(f) \le U_a^b(f) \le U(f,P)$$

implies that

$$U_a^b(f) - L_a^b(f) \le U(f,P) - L(f,P) < \epsilon.$$

This means that $0 \le U_a^b(f) - L_a^b(f) < \epsilon$ for every positive ϵ, which is possible only if $U_a^b(L) - L_a^b(f) = 0$. Thus, $U_a^b(f) = L_a^b(f)$. □

The above theorem leads to a sequential characterization of the Riemann Integral which will be highly useful in proving theorems about the integral.

Theorem 5.1.8. *The Riemann Integral exists if and only if there is a sequence* $\{P_n\}$ *of partitions of* $[a, b]$ *such that*

$$\lim(U(f, P_n) - L(f, P_n)) = 0. \tag{5.1.7}$$

In this case,

$$\int_a^b f(x)\,dx = \lim S_n(f)$$

where, for each n, $S_n(f)$ *may be chosen to be* $U(f, P_n)$, $L(f, P_n)$ *or any Riemann sum (5.1.1) for* f *and the partition* P_n.

Proof. If, for every $\epsilon > 0$, we can find a partition P of $[a, b]$ such that (5.1.6) holds, then, in particular, for each $n \in \mathbb{N}$ we can find a partition P_n such that

$$U(f, P_n) - L(P_n) < 1/n.$$

Then $\lim(U(f, P_n) - L(f, P_n)) = 0$.

Conversely, if there is a sequence of partitions $\{P_n\}$ with

$$\lim(U(f, P_n) - L(f, P_n)) = 0,$$

then, given $\epsilon > 0$, there is an N such that

$$U(f, P_n) - L(f, P_n) < \epsilon \quad \text{whenever} \quad n > N.$$

By the previous theorem, this implies that the Riemann integral exists.

Now given a sequence $\{P_n\}$ satisfying (5.1.7), we know that

$$L(f, P_n) \le \int_a^b f(x)\,dx \le U(f, P_n)$$

for each n. It follows that the sequences $\{L(f, P_n)\}$ and $\{U(f, P_n)\}$ both converge to $\int_a^b f(x)\,dx$. However, by (5.1.4), we also have

$$L(f, P_n) \le S_n(f) \le U(f, P_n)$$

if $S_n(f)$ is any Riemann sum for f and the partition P_n or is $U(f, P_n)$ or $L(f, P_n)$. By the squeeze principle (Theorem 2.3.3) , we conclude

$$\int_a^b f(x)\,dx = \lim S_n(f).$$

□

Example 5.1.9. Prove that the Riemann Integral of $f(x) = x^2$ on $[0, 1]$ exists and is equal to 1/3.

Solution: The function is increasing and so its sup on any interval is achieved at the right endpoint and its inf is achieved at the left endpoint. For each $n \in \mathbb{N}$ we define a partition P_n of $[0,1]$ by

$$P_n = \{0 < 1/n < 2/n < \cdots < n/n = 1\}.$$

This divides $[0,1]$ into n subintervals, each of which has length $1/n$. The corresponding upper sum is then

$$U(f,P_n) = \sum_{k=1}^{n} \left(\frac{k}{n}\right)^2 \frac{1}{n} = \frac{1}{n^3}\sum_{k=1}^{n} k^2,$$

while the lower sum is

$$L(f,P_n) = \sum_{k=1}^{n} \left(\frac{k-1}{n}\right)^2 \frac{1}{n} = \frac{1}{n^3}\sum_{k=0}^{n-1} k^2.$$

The difference is

$$U(f,P_n) - L(f,P_n) = \frac{n^2}{n^3} = \frac{1}{n}.$$

This sequence certainly has limit 0 and so, by Theorem 5.1.8, the Riemann Integral exists. To find what it is, we need a formula for the sum $\sum_{k=1}^{n} k^2$. Such a formula exists. In fact, it can be proved by induction (Exercise 5.1.3) that

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Thus,

$$U(f,P_n) = \frac{n(n+1)(2n+1)}{6n^3} = \frac{(1+1/n)(2+1/n)}{6}.$$

This expression has limit $1/3$ as $n \to \infty$ and so $\int_0^1 x^3\,dx = 1/3$.

Exercise Set 5.1

1. Find the upper sum $U(f,P)$ and lower sum $L(f,P)$ if $f(x) = 1/x$ on $[1,2]$ and P is the partition of $[1,2]$ into four subintervals of equal length.

2. Prove that $\int_0^1 x\,dx$ exists by computing $U(f,P_n)$ and $L(f,P_n)$ for the function $f(x) = x$ and a partition P_n of $[0,1]$ into n equal subintervals. Then show that condition (5.1.7) of Theorem 5.1.8 is satisfied. Calculate the integral by taking the limit of the upper sums. Hint: use Exercise 1.2.3.

3. Prove by induction that

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

4. Prove that $\int_0^a x^2\,dx = \frac{a^3}{3}$ by expressing this integral as a limit of Riemann sums and finding the limit.

5. Let f be the function on $[0,1]$ which is 0 at every rational number and is 1 at every irrational number. Is this function integrable on $[0,1]$. Prove that your answer is correct by using the definition of the integral.

6. Prove that the upper sum $U(f,P)$ for a partition of $[a,b]$ and a bounded function f on $[a,b]$ is the least upper bound of the set of all Riemann sums for f and P.

7. Finish the proof of Theorem 5.1.4 by showing that if the theorem is true when only one element is added to P to obtain Q, then it is also true no matter how many elements need to be added to P to obtain Q.

8. Suppose m and M are lower and upper bounds for f on $[a,b]$; that is $m \le f(x) \le M$ for all $x \in [a,b]$. Prove that

$$m(b-a) \le L_a^b(f) \le U_a^b(f) \le M(b-a).$$

What conclusion about $\int_a^b f(x)\,dx$ do you draw from this if the integral exists?

9. If k is a constant and $[a,b]$ a bounded interval, prove that k is integrable on $[a,b]$ and

$$\int_a^b k\,dx = k(b-a).$$

10. Suppose f is any non-decreasing function on a bounded interval $[a,b]$. If P_n is the partition of $[a,b]$ into n equal subintervals, show that

$$U(f,P_n) - L(f,P_n) = (f(b)-f(a))\frac{b-a}{n}.$$

What do you conclude about the integrability of f?

5.2 Existence and Properties of the Integral

We first show that the integral exists for a large class of functions, a class which includes all the functions of interest to us in this course. We then show that the integral has the properties claimed for it in calculus courses.

Existence Theorems

Theorem 5.2.1. *If f is a monotone function on a closed bounded interval $[a,b]$, then f is integrable on $[a,b]$.*

Proof. This was essentially stated as an exercise (Exercise 5.1.10) in the previous section. In this exercise, it is claimed that, if f is a non-decreasing function on $[a,b]$ and P_n is the partition of $[a,b]$ into n equal subintervals, then

$$U(f,P_n) - L(f,P_n) = (f(b) - f(a))\frac{b-a}{n}. \tag{5.2.1}$$

This implies that

$$\lim(U(f,P_n) - L(f,P_n)) = 0$$

and, by Theorem 5.1.8, this implies that the Riemann Integral of f on $[a,b]$ exists.

In the case where f is non-increasing, the same proof works. The only difference is that $f(b) - f(a)$ is replaced by $f(a) - f(b)$ in (5.2.1). □

Theorem 5.2.2. *If f is a continuous function on a closed, bounded interval $[a,b]$, then f is integrable on $[a,b]$.*

Proof. Since f is continuous on the closed, bounded interval $[a,b]$, it is uniformly continuous on $[a,b]$ by Theorem 3.3.4. Thus, given $\epsilon > 0$ there is a $\delta > 0$ such that

$$|f(x) - f(y)| < \frac{\epsilon}{b-a} \quad \text{whenever} \quad |x-y| < \delta.$$

Then, if $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ is any partition of $[a,b]$ with the property that the interval $[x_{k-1}, x_k]$ has length less than δ for each k, then the maximum value M_k of f on this interval and the minimum value m_k of f on this interval differ by less than $\epsilon/(b-a)$. This implies that

$$U(f,P) - L(f,P) = \sum_{k=1}^{n}(M_k - m_k)(x_k - x_{k-1}) < \frac{\epsilon}{b-a}\sum_{k=1}^{n}(x_k - x_{k-1}) = \epsilon,$$

since $\sum_{k=1}^{n}(x_k - x_{k-1}) = b - a$. It follows from Theorem 5.1.7 that f is integrable on $[a,b]$. □

Linearity of the Integral

In the remainder of this section we adopt the following notation, introduced in Section 1.5 for the sup and inf of a function f on an interval I:

$$\sup_I f = \sup\{f(x) : x \in I\} \quad \text{and} \quad \inf_I f = \inf\{f(x) : x \in I\}.$$

The integral is a *linear transformation* from the space of integrable functions on $[a,b]$ to the real numbers. This just means that the following familiar theorem is true.

Theorem 5.2.3. *If f and g are integrable functions on a closed, bounded interval $[a,b]$ and c is a constant, then* *(a) cf is integrable and* $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$;

(b) $f+g$ is integrable and $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.

Proof. We begin by investigating the upper and lower sums for a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ and the functions cf and $f+g$. We let $I_k = [x_{k-1}, x_k]$ denote the kth subinterval determined by this partition.

If $c \ge 0$, then Theorem 1.5.10(a) tells us that

$$\sup_{I_k} cf = c\sup_{I_k} f \quad \text{and} \quad \inf_{I_k} cf = c\inf_{I_k} f$$

for $k = 1, \cdots, n$. This implies that

$$U(cf,P) = cU(f,P) \quad \text{and} \quad L(cf,P) = cL(f,P) \quad \text{if} \quad c \ge 0. \tag{5.2.2}$$

On the other hand, by Theorem 1.5.10(b),

$$\sup_{I_k}(-f) = -\inf_{I_k} f \quad \text{and} \quad \inf_{I_k}(-f) = -\sup_{I_k} f$$

for each k. This implies that

$$U(-f,P) = -L(f,P) \quad \text{and} \quad L(-f,P) = -U(f,P). \tag{5.2.3}$$

For the sum $f+g$, we have

$$\inf_{I_k} f + \inf_{I_k} g \le \inf_{I_k}(f+g) \le \sup_{I_k}(f+g) \le \sup_{I_k} f + \sup_{I_k} g$$

for each k, by 1.5.10(c). These inequalities imply that

$$L(f,P)+L(g,P) \le L(f+g,P) \le U(f+g,P) \le U(f,P)+U(g,P). \tag{5.2.4}$$

With these results in hand, the proof of the theorem becomes a relatively simple matter. We use Theorem 5.1.8. Since f is integrable, there is a sequence $\{P_n\}$ of partitions of $[a,b]$ such that

$$\lim(U(f,P_n) - L(f,P_n)) = 0. \tag{5.2.5}$$

If $c \ge 0$, then (5.2.2) implies that

$$\lim(U(cf,P_n) - L(cf,P_n) = \lim c(U(f,P_n) - L(f,P_n)) = 0$$

which implies that cf is integrable. It also follows from (5.2.2) that

$$\int_a^b cf(x)\,dx = \lim U(cf,P_n) = c\lim U(f,P_n) = c\int_a^b f(x)\,dx.$$

Similarly, using (5.2.3) yields

$$\lim(U(-f,P_n)-L(-f,P_n))=\lim(-L(f,P_n)+U(f,P_n))=0,$$

which implies that $-f$ is integrable. It also follows from (5.2.3) that

$$\int_a^b -f(x)\,dx=\lim U(-f,P_n)=-\lim L(f,P_n)=-\int_a^b f(x)\,dx.$$

Combining these results proves part (a) of the theorem.

Since, g is also integrable, there is a sequence of partitions $\{Q_n\}$ such that (5.2.5) holds with f replaced by g and P_n by Q_n. In fact, we may replace $\{P_n\}$ and $\{Q_n\}$ by the sequence of common refinements $\{P_n \cup Q_n\}$ and get a sequence of partitions that works for both f and g. Since this is so, we may as well assume that $\{P_n\}$ was chosen in the first place to be a sequence of partitions such that (5.2.5) holds and

$$\lim(U(g,P_n)-L(g,P_n))=0. \tag{5.2.6}$$

also holds. Then 5.2.4 implies that

$$0\le U(f+g,P_n)-L(f+g,P_n)\le U(f,P_n)-L(f,P_n)+U(g,P_n)-L(g,P_n).$$

Since the expression on the right has limit 0, so does $U(f+g,P_n)-L(f+g,P_n)$. Hence, $f+g$ is integrable. Also, on passing to the limit as P ranges through the sequence of partitions P_n, inequality (5.2.4) implies that

$$\int_a^b (f(x)+g(x))\,dx=\int_a^b f(x)\,dx+\int_a^b g(x)\,dx.$$

This completes the proof of part (b) of the theorem. □

The Order Preserving Property

The integral is order preserving:

Theorem 5.2.4. *If f and g are integrable functions on $[a,b]$ and $f(x)\le g(x)$ for all $x\in[a,b]$, then*

$$\int_a^b f(x)\,dx\le\int_a^b g(x)\,dx.$$

Proof. We first prove that if h is an integrable function which is non-negative on $[a,b]$, then

$$\int_a^b h(x)\,dx\ge 0.$$

In fact, this is obvious. If h is non-negative, then its inf and sup on any subinterval in any partition are also non-negative. This implies that the upper sums $U(h,P)$ and lower sums $L(h,P)$ are non-negative for any partition P. Since the integral is greater than or equal to every lower sum, it is non-negative.

To finish the proof, we apply the result of the previous paragraph to the function $h = g - f$ which is non-negative on $[a, b]$ if $f(x) \leq g(x)$ for $x \in [a, b]$. Using linearity (Theorem 5.2.3) we conclude that

$$\int_a^b g(x)\,dx - \int_a^b f(x)\,dx = \int_a^b (g(x) - f(x))\,dx \geq 0.$$

This proves the theorem. □

This has the following useful corollary. Its proof is left to the exercises.

Corollary 5.2.5. *Let f be an integrable function on the closed bounded interval $I = [a, b]$ and set $M = \sup_I f$, and $m = \inf_I f$. Then*

$$m(b - a) \leq \int_a^b f(x)\,dx \leq M(b - a)$$

Theorem 5.2.6. *If f is integrable on $[a, b]$, then $|f|$ is also integrable on $[a, b]$ and*

$$\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx$$

.

Proof. Let f be integrable on $[a, b]$. Suppose we can show that $|f|$ is also integrable on $[a, b]$. To derive the above inequality is then quite easy. The inequalties $-|f(x)| \leq f(x) \leq |f(x)|$, together with Theorem 5.2.4, imply that

$$-\int_a^b |f(x)|\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b |f(x)|\,dx$$

and this implies that

$$\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx.$$

To complete the proof, we must show that the integrability of f on $[a, b]$ implies the integrability of $|f|$.

Let I be an arbitrary subinterval of $[a, b]$. Then, by the triangle inequality,

$$|f(x)| - |f(y)| \leq |f(x) - f(y)|$$

for all $x, y \in I$. It follows from this (Exercise 5.2.7) that

$$\sup_I |f| - \inf_I |f| \leq \sup_I f - \inf_I f.$$

If we apply this as I ranges over each subinterval in a partition P, the result for the upper and lower sums is

$$U(|f|, P) - L(|f|, P) \leq U(f, P) - L(f, P).$$

It now follows from Theorem 5.1.7 that $|f|$ is integrable on $[a, b]$ if f is integrable on $[a, b]$. □

Interval Additivity

Note that, in the following theorem, we do not assume that f is integrable.

Theorem 5.2.7. *Suppose $a \le b \le c$ and f is a bounded function defined on $[a, c]$. Then the upper and lower integrals of f satisfy*

$$L_a^c(f) = L_a^b(f) + L_b^c(f) \quad \text{and} \quad U_a^c(f) = U_a^b(f) + U_b^c(f).$$

Proof. We will prove the result for the lower integral. The proof for the upper integral is essentially the same.

Let $P = \{a = x_o \le x_1 \le \cdots \le x_n = c\}$ be a partition of $[a, c]$ which has the point b as its mth partition point. Then P determines partitions

$$P' = \{a = x_0 < x_1 < \cdots < x_m = b\} \quad \text{of} \quad [a, b] \quad \text{and}$$
$$P'' = \{b = x_m < x_{m+1} < \cdots < x_n = c\} \quad \text{of} \quad [b, c].$$

In this case,

$$L(P', f) + L(P'', f) = L(P, f). \tag{5.2.7}$$

Each pair consisting of a partition P' of $[a, b]$ and a partition P'' of $[c, d]$ fit together to form a partition P of $[a, b]$ of this type. Since L_a^c is the sup of all numbers of the form $L(P, f)$ for P a partition of $[a, c]$, It follows that

$$L(P', f) + L(P'', f) \le L_a^c(f)$$

for each partition P' of $[a, b]$ and each partition P'' of $[b, c]$. On passing to the sup of the numbers on the left of this inequality, Theorem 1.5.7(c) implies that

$$L_a^b(f) + L_b^c(f) \le L_a^c(f). \tag{5.2.8}$$

On the other hand, each partition Q of $[a, c]$ has a refinement $P = Q \cup \{b\}$ which is of the type described above. That is, P determines partitions P' of $[a, b]$ and P'' of $[b, c]$ such that (5.2.7) holds. It follows that

$$L(f, Q) \le L(f, P) = L(P', f) + L(P'', f) \le L_a^b(f) + L_b^c(f).$$

On passing to the sup of the numbers on the left, we conclude that

$$L_a^c(f) \le L_a^b(f) + L_b^c(f). \tag{5.2.9}$$

Combining (5.2.8) and (5.2.9) yields the statement of the theorem in the case of the lower integral. The proof in the case of the upper integral is essentially the same. □

This theorem has as a corollary the interval additivity property for the integral. The details of how this corollary follows from the above theorem are left to the exercises.

Corollary 5.2.8. *With f and $a \le b \le c$ as in the previous theorem, f is integrable on $[a, c]$ if and only if it is integrable on $[a, b]$ and on $[b, c]$. In this case,*

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

Theorem of the Mean for Integrals

If f is an integrable function on a bounded interval $[a, b]$, then the *mean* or *average* of f on $[a, b]$ is the number

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

The following theorem is an easy consequence of the Intermediate Value Theorem. We leave its proof to the exercises.

Theorem 5.2.9. *If f is a continuous function on a closed bounded interval $[a, b]$, then there is a point $c \in [a, b]$ such that*

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Exercise Set 5.2

1. Show that if a function f on a bounded interval can be written in the form $g - h$ for functions g and h which are non-decreasing on $[a, b]$, then f is integrable on $[a, b]$.

2. Suppose f is a bounded function on a bounded interval $[a, b]$ and there is a partition $\{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a, b]$ such that f is continuous on each subinterval (x_{k-1}, x_k). Prove that such a function is integrable on $[a, b]$.

3. Prove Corollary 5.2.5.

4. Prove Corollary 5.2.8.

5. Prove that $1 \le \displaystyle\int_{-1}^1 \frac{1}{1+x^{2n}}\,dx \le 2$ for all $n \in \mathbb{N}$.

6. Prove that $\displaystyle\int_{-1}^1 \frac{x^2}{1+x^{2n}}\,dx \le 2/3$ for all $n \in \mathbb{N}$.

7. If f is a bounded function defined on an interval I, then prove that

$$\sup_I |f| - \inf_I |f| \le \sup_I f - \inf_I f$$

   by using Theorem 1.5.10(d) and the triangle inequality $|f(x)| - |f(y)| \le |f(x) - f(y)|$.

8. rove that if f is integrable on $[a, b]$ then so is f^2. Hint: if $|f(x)| \le M$ for all $x \in [a, b]$, then show that

$$|f^2(x) - f^2(y)| \le 2M|f(x) - f(y)|.$$

   for all $x, y \in [a, b]$. Use this to estimate $U(f^2, P) - L(f^2, P)$ in terms of $U(f, P) - L(f, P)$ for a given partition P.

9. Prove that if f and g are integrable on $[a, b]$, then so is fg. Hint: write fg as the difference of two squares of functions you know are integrable and then use the previous exercise.

10. Give an example of a function f such that $|f|$ is integrable on $[0, 1]$ but f is not integrable on $[0, 1]$.

11. Prove Theorem 5.2.9.

12. Let $\{f_n\}$ be a sequence of integrable functions defined on a closed bounded interval $[a, b]$. If $\{f_n\}$ converges uniformly on $[a, b]$ to a function f, prove that f is integrable and

$$\int_a^b f(x)\,dx = \lim \int_a^b f_n(x)\,dx.$$

13. If f is a bounded function defined on a closed bounded interval $[a, b]$ and if f is integrable on each interval $[a, r]$ with $a < r < b$, then prove that f is integrable on $[a, b]$ and

$$\int_a^b f(x)\ dx = \lim_{r \to b} \int_s^t f(x)\,dx.$$

Observe that the analogous result holds if $[a, r]$is replaced by $[r, b]$ in the hypothesis and in the integral on the right, and the limit is taken as $r \to a$. Hint: use Theorem 5.2.7 and Exercise 5.1.8.

14. Is the function which is $\sin 1/x$ for $x \neq 0$ and 0 for $x = 0$ integrable on $[0, 1]$? Justify your answer.

5.3 The Fundamental Theorems of Calculus

There are two fundamental theorems of calculus. Both relate differentiation to integration. In most calculus courses, the Second Fundamental Theorem is usually proved first and then used to prove the First Fundamental Theorem. Unfortunately, this results in a First Fundamental Theorem that is weaker than it could be. To prove the best possible theorems, one should give independent proofs of the two theorems. This is what we shall do.

First Fundamental Theorem

The following theorem concerns the integral of f' on $[a, b]$ where f is a function which we assume is differentiable on (a, b) but not necessarily at a or b. The reason the integral still makes sense is that, for a function that is integrable on $[a, b]$, changing its value at one point (or at finitely many points) does not affect its integrability or its integral (Exercise 5.3.9). Thus, a function which is missing values at a and/or b can be assigned values there arbitrarily and the integrability and value of the integral will not depend on how this is done.

Theorem 5.3.1. *Let $[a, b]$ be a closed bounded interval and f a function which is continuous on $[a, b]$ and differentiable on (a, b) with f' integrable on $[a, b]$. Then*

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

Proof. Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of $[a, b]$. We apply the Mean Value Theorem to f on each of the intervals $[x_{k-1}, x_k]$. This tells us there is a point $c_k \in (x_{k-1}, x_k)$ such that

$$f'(c_k)(x_k - x_{k-1}) = f(x_k) - f(x_{k-1}).$$

If we sum this over $k = 1, \cdots, n$, the result is

$$\sum_{k=1}^{n} f'(c_k)(x_k - x_{k-1}) = f(b) - f(a).$$

The sum on the left is a Riemann sum for f' and the partition P and so, by (5.1.4), it lies between the lower and upper sums for f' and P. Thus,

$$L(f', P) \le f(b) - f(a) \le U(f', P). \tag{5.3.1}$$

Since f' is integrable on $[a, b]$, there is a sequence of partitions $\{P_n\}$ for which the corresponding sequences of upper and lower sums for f' both converge to $\int_a^b f'(x)\,dx$. However, in view of (5.3.1) the only number both sequences can converge to is $f(b) - f(a)$. □

The above theorem is somewhat stronger than the one usually stated in calculus, because we only assume that the derivative f' is integrable on $[a, b]$, not that it is continuous. Are there functions which are differentiable with an integrable derivative which is not continuous?

Example 5.3.2. Find a function f which is differentiable on an interval, with an integrable derivative which is not continuous.

Solution: Let $f(x) = x^2 \sin 1/x$ if $x \neq 0$ and set $f(0) = 0$. Then, f is differentiable on all of $\mathbb{R}$ and its derivative is

$$f'(x) = 2x \sin 1/x - \cos 1/x \quad \text{if} \quad x \neq 0$$

and is 0 at $x = 0$. This follows from the Chain Rule and the Product Rule for derivatives everywhere except at $x = 0$. At $x = 0$ we calculate the derivative using the definition of derivative:

$$f'(0) = \lim_{x \to 0} \frac{x^2 \sin 1/x}{x} = \lim_{x \to 0} x \sin 1/x = 0.$$

Now the function $f'(x)$ is integrable on any closed bounded interval (see Exercise 5.2.13, but it is not continuous at 0. Thus, f is a function to which the previous theorem applies, but it does not have a continuous derivative.

Second Fundamental Theorem

So far we have defined the integral $\int_a^b f(x)\,dx$ only in the case where $a < b$. We remedy this by defining the integral to be 0 if $a = b$ and declaring

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx \quad \text{if} \quad b < a.$$

This ensures that the integral in the following theorem makes sense whether x is to the left or the right of a.

Theorem 5.3.3. *Let f be a function which is integrable on a closed bounded interval $[b, c]$. For $a, x \in [b, c]$ define a function $F(x)$ by*

$$F(x) = \int_a^x f(t)\,dt.$$

Then F is continuous on $[b, c]$. At each point x of (b, c) where f is continuous the function F is differentiable and

$$F'(x) = f(x).$$

Proof. The definition of F makes sense, because it follows from Theorem 5.2.7 that a function integrable on an interval is also integrable on every subinterval.

Since f is integrable on $[b, c]$ it is bounded on $[b, c]$. Thus, there is an M such that

$$|f(t)| \leq M \quad \text{for all} \quad t \in [b, c].$$

If $x, y \in [b, c]$ then

$$F(y) - F(x) = \int_a^y f(t)\,dt - \int_a^x f(t)\,dt = \int_x^y f(t)\,dt. \tag{5.3.2}$$

(see Exercise 5.3.11). Then by Exercise 5.3.12 ,

$$|F(y) - F(x)| = \left|\int_x^y f(t)\,dt\right| \leq M|y - x|.$$

Thus, given $\epsilon > 0$, if we choose $\delta = \epsilon/M$, then

$$|F(y) - F(x)| < \epsilon \quad \text{whenever} \quad |y - x| < \delta.$$

This shows that F is uniformly continuous on $[b, c]$.

Now suppose $x \in (b, c)$ is a point at which f is continuous. If y is also in (b, c), then

$$\int_x^y f(x)\,dt = f(x)(y - x)$$

since $f(x)$ is a constant as far as the variable of integration t is concerned. This and (5.3.2) imply that

$$\begin{aligned}\frac{F(y)-F(x)}{y-x}-f(x) &= \frac{1}{y-x}\left(\int_x^y f(t)\,dt-\int_x^y f(x)\,dt\right)\\ &= \frac{1}{y-x}\int_x^y (f(t)-f(x))\,dt.\end{aligned} \tag{5.3.3}$$

Since f is continuous at x, given $\epsilon > 0$, we may choose $\delta > 0$ such that

$$|f(t)-f(x)| < \epsilon \quad \text{whenever} \quad |x-t| < \delta.$$

Then, for y with $|y-x| < \delta$, it will be true that $|x-t| < \delta$ for every t between x and y. Thus, for such a choice of y, we have

$$\left|\frac{1}{y-x}\int_x^y (f(t)-f(x))\,dt\right| \le \frac{1}{|y-x|}\epsilon|y-x| = \epsilon$$

In view of (5.3.3), this implies that

$$\lim_{y\to x}\frac{F(y)-F(x)}{y-x} = f(x).$$

Thus, F is differentiable at x and $F'(x) = f(x)$. □

Example 5.3.4. Find $\dfrac{d}{dx}\displaystyle\int_0^{\sin x} e^{-t^2}\,dt$.

Solution: This is a composite function. If $F(u) = \displaystyle\int_0^u e^{-t^2}\,dt$, then the function we are asked to differentiate is $F(\sin x)$. By the Chain Rule, the derivative of this composite function is

$$F'(\sin x)\cos x.$$

By the previous theorem, $F'(u) = e^{-u^2}$. Thus,

$$\frac{d}{dx}\int_0^{\sin x} e^{-t^2}\,dt = F'(\sin x)\cos x = e^{-\sin^2 x}\cos x.$$

Example 5.3.5. Find $\dfrac{d}{dx}\displaystyle\int_x^{2x}\sin t^2\,dt$.

Solution: We begin by writing

$$G(x) = \int_x^{2x}\sin t^2\,dt = \int_0^{2x}\sin t^2\,dt - \int_0^x \sin t^2\,dt.$$

Then using the previous theorem and the Chain Rule yields

$$G'(x) = 2\sin 4x^2 - \sin x^2.$$

Substitution

We will not rehash all the integration techniques that are taught in the typical calculus class. However, two of these techniques are of such great theoretical importance, that it is worth discussing them again. The techniques in question are substitution and integration by parts. Each of these follows from the Fundamental Theorems and an important theorem from differential calculus – the chain rule in the case of substitution and the product rule in the case of integration by parts. We begin with substitution.

Theorem 5.3.6. *Let g be a differentiable function on an open interval I with g' integrable on I and let $J = g(I)$. Let f be continuous on J. Then for any pair $a, b \in I$,*

$$\int_a^b f(g(t))g'(t)\,dt = \int_{g(a)}^{g(b)} f(u)\,du. \tag{5.3.4}$$

Proof. The composite function $f \circ g$ is continuous on I since g is continuous on I and f is continuous on J. By Exercise 5.2.9, this implies that $f(g(t))g'(t)$ is an integrable function of t on I. We set

$$F(v) = \int_{g(a)}^{v} f(u)\,du.$$

Then $F'(v) = f(v)$ by the Second Fundamental Theorem, and so, by the Chain Rule,

$$(F(g(x)))' = f(g(x))g'(x).$$

Thus, $F \circ g$ is a differentiable function on I with an integrable derivative $f(g(x))g'(x)$. By the First Fundamental Theorem,

$$F(g(b)) - F(g(a)) = \int_a^b f(g(x))g'(x)\,dx.$$

By the definition of F, $F(g(a)) = 0$ and $F(g(b)) = \displaystyle\int_{g(a)}^{g(b)} f(u)\,du$. Thus,

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du,$$

as claimed. □

Note that the above theorem states formally what happens when we make the substitution $u = g(t)$ in the integral on the left in (5.3.4).

Integration by Parts

The integration by parts formula is a direct consequence of the Fundamental Theorems and the product rule for differentiation.

Theorem 5.3.7. *Suppose f and g are continuous functions on a closed bounded interval $[a, b]$ and suppose that f and g are differentiable on (a, b) with derivatives that are integrable on $[a, b]$. Then fg' and $f'g$ are integrable on $[a, b]$ and*

$$\int_a^b f(x)g'(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b g(x)f'(x)\,dx. \tag{5.3.5}$$

Proof. We have f and g integrable because they are continuous on $[a, b]$, while f' and g' are integrable by hypothesis. By Exercise 5.2.9, fg' and gf' are both integrable.

The product fg is differentiable on (a, b) and

$$(fg)' = fg' + gf'.$$

Thus, $(fg)'$ is also integrable and, by the First Fundamental Theorem,

$$f(b)g(b) - f(a)g(a) = \int_a^b (f(x)g(x))'\,dx = \int_a^b f(x)g'(x)\,dx + \int_a^b g(x)f'(x)\,dx.$$

Formula (5.3.5) follows immediately from this. □

Example 5.3.8. Suppose f is a continuous function on $[-\pi, \pi]$ which is differentiable on $(-\pi, \pi)$ with an integrable derivative. Also suppose $f(-\pi) = f(\pi)$. Prove that, for each $n \in \mathbb{N}$,

$$\begin{aligned} \int_{-\pi}^{\pi} f'(x)\sin nx\,dx &= -n\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\ \int_{-\pi}^{\pi} f'(x)\cos nx\,dx &= n\int_{-\pi}^{\pi} f(x)\sin nx\,dx. \end{aligned} \tag{5.3.6}$$

Solution: These are the equations relating the Fourier coefficients of the derivative of a function f to the Fourier coefficients of f itself.

The first equation is proved using the integration by parts formula (5.3.5) for $f(x)$ and $g(x) = \sin x$. Since $\sin(-n\pi) = \sin(n\pi) = 0$, the terms $f(b)g(b) - f(a)g(a)$ are 0. The first equation then follows directly from (5.3.5).

The second equation follows from (5.3.5) for $f(x)$ and $g(x) = \cos x$. However, this time the terms $f(b)g(b) - f(a)g(a)$ contribute 0 because cos is an even function and $f(-\pi) = f(\pi)$.

Exercise Set 5.3

1. Find $\displaystyle\int_{4/\pi}^{2/\pi} (2x\sin 1/x - \cos 1/x)\,dx$. Hint: see Example 5.3.2.
2. Find $\displaystyle\frac{d}{dx}\int_1^x \cos 1/t\,dt$ for $x > 0$.
3. Find $\displaystyle\frac{d}{dx}\int_0^{2x} \sin t^2\,dt$.

4. Find $\dfrac{d}{dx}\displaystyle\int_{1/x}^{x} \mathrm{e}^{-t^2}\, dt$.

5. If $f(x) = -1/x$ then $f'(x) = 1/x^2$. Thus, Theorem 5.3.1 seems to imply that
$$\int_{-1}^{1} 1/x^2\, dx = f(1) - f(-1) = -1 - 1 = -2.$$
However, $1/x^2$ is a positive function, and so its integral over $[-1, 1]$ should be positive. What is wrong?

6. If f is a differentiable function on $[a, b]$ and f' is integrable on $[a, b]$, then find
$$\int_a^b f(x)f'(x)\, dx.$$

7. Let f be a continuous function on the interval $[0, 1]$. Express
$$\int_0^{\pi/2} f(\sin\theta)\cos\theta\, d\theta$$
as an integral involving only the function f.

8. Find $\displaystyle\int_0^x t^n \ln t\, dt$ where n is an arbitrary integer.

9. Prove that if f is integrable on $[a, b]$ and $c \in [a, b]$, then changing the value of f at c does not change the fact that f is integrable or the value of its integral on $[a, b]$.

10. The function $f(x) = x/|x|$ has derivative 0 everywhere but at $x = 0$. Its derivative $f'(x) = 0$ is integrable on $[-1, 1]$ and has integral 0. However $f(1) - f(-1) = 1 - (-1) = 2$. This seems to contradict Theorem 5.3.1. Explain why it does not.

11. The interval additivity property (Theorem 5.2.7) is stated for three points a, b, c satisfying $a < b < c$. Show that it actually holds regardless of how the points a, b, and c are ordered. Hint: you will need to consider various cases.

12. Suppose f is integrable on an interval containing a and b and $|f(x)| \le M$ on I. Prove that
$$\left|\int_a^b f(x)\, dx\right| \le M|b - a|.$$
Note that we do not assume that $a < b$.

5.4 Logs, Exponentials, Improper Integrals

The following development of the log and exponential functions is the one presented in most calculus classes these days. It is such a beautiful application of the Second Fundamental Theorem that we felt obligated to include it here.

The Natural Logarithm

One consequence of the Second Fundamental Theorem is that every function f which is continuous on an open interval I has an anti-derivative on I. In fact, if a is any point of I, then

$$F(x) = \int_a^x f(t)\,dt$$

is an anti-derivative for f on I (that is, $F'(x) = f(x)$ on I).

Now $\dfrac{x^{n+1}}{n+1}$ is an antiderivative for x^n for all integers n with the exception of $n = -1$. However, since x^{-1} is continuous on $(0, +\infty)$ and on $(-\infty, 0)$, it has an antiderivative on each of these intervals. There is no mystery about what the antiderivatives are. On $(0, +\infty)$ the function

$$\int_1^x \frac{1}{t}\,dt$$

is an antiderivative for $1/x$. Obviously, this function is important enough to deserve a name.

Definition 5.4.1. We define the natural logarithm to be the function ln, defined for $x \in (0, +\infty)$ by

$$\ln x = \int_1^x \frac{1}{t}\,dt.$$

This is the unique antiderivative for $1/x$ on $(0, +\infty)$ which has the value 0 when $x = 1$.

On $(-\infty, 0)$ an antiderivative for $1/x$ is given by

$$\int_{-1}^x \frac{1}{t}\,dt.$$

Note that the x that appears in this integral is negative, and so $-x = |x|$. If we make the substitution $s = -t$, then Theorem 5.3.6 implies that

$$\int_{-1}^x \frac{1}{t}\,dt = \int_1^{-x} \frac{1}{s}\,ds = \ln(-x) = \ln|x|.$$

Thus, $\ln|x|$ is an antiderivative for $1/x$ on both $(0, +\infty)$ and $(-\infty, 0)$.

The next two theorems show that ln has the key properties that we expect of a logarithm.

Theorem 5.4.2. *For all* $a, b \in (0, +\infty)$, $\ln ab = \ln a + \ln b$.

Proof. By the Chain Rule, the derivative of $\ln ax$ is $\dfrac{1}{ax}a = \dfrac{1}{x}$. Thus, $\ln ax$ and $\ln x$ have the same derivative on the interval $(0, +\infty)$. By Corollary 4.3.4

$$\ln ax = \ln x + c$$

for some constant c. The constant may be evaluated by setting $x = 1$. Since $\ln 1 = 0$, this tells us that $c = \ln a$. Thus,

$$\ln ax = \ln x + \ln a.$$

This gives $\ln ab = \ln a + \ln b$ when we set $x = b$. □

Theorem 5.4.3. *If* $a > 0$ *and* r *is any rational number, then* $\ln a^r = r \ln a$.

Proof. The proof of this is similar to the proof of the previous theorem. The key is to compute the derivative of the function $\ln x^r$. We leave the details to Exercise 5.4.1. □

Theorem 5.4.4. *The natural logarithm is strictly increasing on* $(0, +\infty)$. *Also,*

$$\lim_{x \to \infty} \ln x = +\infty \quad \textit{and} \quad \lim_{x \to 0} \ln x = -\infty.$$

Proof. The function $\ln x$ is strictly increasing on $(0, +\infty)$ because its derivative is positive on this interval.

Since $\ln 1 = 0$ and ln is increasing, $\ln 2$ is positive. Given any number M, choose an integer m such that $m \ln 2 > M$ and set $N = 2^m$. Then

$$\ln x > \ln 2^m = m \ln 2 > M \quad \text{whenever} \quad x > N.$$

This implies that $\lim_{x \to \infty} \ln x = +\infty$. The fact that $\lim_{x \to 0} \ln x = -\infty$ follows easily from $\lim_{x \to \infty} \ln x = +\infty$ and properties of ln. The details are left to the exercises. □

The Exponential Function

The function ln is strictly increasing on $(0, +\infty)$ and, therefore, it has an inverse function. The image of $(0, +\infty)$ under ln is an open interval by Exercise 4.2.5. By Theorem 5.4.4 this open interval must be the interval $(-\infty, \infty)$. Therefore, the inverse function for ln has domain $(-\infty, \infty)$ and image $(0, \infty)$.

Definition 5.4.5. We define the exponential function to be the function with domain $(-\infty, \infty)$ which is the inverse function of ln. We will denote it by $\exp x$.

The theorems we proved about ln immediately translate into theorems about exp.

Theorem 5.4.6. *The function* exp *is its own derivative – that is,* $\exp'(x) = \exp(x)$.

Proof. By Theorem 4.2.9 we have

$$\exp'(x) = \frac{1}{\ln'(\exp(x))} = \frac{1}{1/\exp(x)} = \exp(x).$$

□

Theorem 5.4.7. *The exponential function satisfies*

(a) $\exp(a+b) = \exp a \exp b$ *for all* $a, b \in \mathbb{R}$*;*

(b) $\exp(ra) = (\exp(a))^r$ *for all* $a \in R$ *and* $r \in \mathbb{Q}$*.*

Proof. Let $x = \exp a$ and $y = \exp b$, so that $a = \ln x$ and $b = \ln y$. Then

$$\exp(a+b) = \exp(\ln x + \ln y) = \exp(\ln xy) = xy = \exp a \exp b$$

by Theorem 5.4.2. This proves (a). The proof of (b) is similar and is left to the exercises. □

We define the number e to be $\exp 1$, so that $\ln \mathrm{e} = 1$. It follows from (b) of the above theorem that, if r is a rational number, then

$$\mathrm{e}^r = (\exp 1)^r = \exp r. \tag{5.4.1}$$

Now at this point, a^r is defined for every positive a and rational r. We have not yet defined a^x if x is a real number which is not rational. However, $\exp x$ is defined for every real x. Since (5.4.1) tells us that $\mathrm{e}^r = \exp r$ if r is rational, it makes sense to *define* e^x for any real x to be $\exp x$.

More generally, if a is any positive real number, then

$$a^r = (\exp \ln a)^r = \exp(r \ln a),$$

and so it makes sense to define a^x for any real x to be $\exp(x \ln a)$. The following definition formalizes this discussion.

Definition 5.4.8. If x is any real number and a is a positive real number, we define a^x by

$$a^x = \exp(x \ln a).$$

In particular,

$$\mathrm{e}^x = \exp x.$$

With this definition of a^x, the laws of exponents

$$a^{x+y} = a^x a^y \quad \text{and} \quad a^{xy} = (a^x)^y$$

are satisfied. The proofs are left to the exercises.

The General Logarithm

We define the logarithm to the base a, $\log_a$, to be the inverse function of the function a^x. The following theorem gives a simple description of it in terms of the natural logarithm $\ln x$. The proof is left to the exercises.

Theorem 5.4.9. *For each $a > 0$, we have* $\log_a x = \dfrac{\ln x}{\ln a}$.

Improper Integrals

So far, we have defined the integral $\displaystyle\int_a^b f(x)\,dx$ only for bounded intervals $[a,b]$ and bounded functions f on $[a,b]$. Thus, our definition does not allow for integrals such as

$$\int_0^\infty \frac{1}{1+x^2}\,dx \quad \text{or} \quad \int_0^1 \frac{1}{\sqrt{x}}\,dx.$$

It turns out that a perfectly good meaning can be attached to each of these integrals. To do so requires extending our definition of the integral.

We first consider an integral of the form $\displaystyle\int_a^\infty f(x)\,dx$ where a is finite. We assume that f is integrable on each interval of the form $[a,s]$ for $a \le s < \infty$. Then we set

$$\int_a^\infty f(x)\,dx = \lim_{s\to\infty} \int_a^s f(x)\,dx,$$

provided this limit exists and is finite. In this case, we say that the *improper integral* $\displaystyle\int_a^\infty f(x)\,dx$ *converges.*

Integrals of the form $\displaystyle\int_{-\infty}^b f(x)\,dx$ are treated similarly. Assuming f is integrable on each interval of the form $[r,b]$ with $-\infty < r \le b$, we set

$$\int_{-\infty}^b f(x)\,dx = \lim_{r\to-\infty} \int_r^b f(x)\,dx,$$

provided this limit exists and is finite. In this case, we say that the *improper integral* $\displaystyle\int_{-\infty}^b f(x)\,dx$ *converges.*

For an integral of the form $\displaystyle\int_{-\infty}^\infty f(x)\,dx$, we simply break the integral up into a sum of improper integrals involving only one infinite limit of integration. That is, we write

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^0 f(x)\,dx + \int_0^\infty f(x)\,dx$$

If the two improper integrals on the right converge, we then say the improper integral on the left converges – it converges to the sum on the right.

Example 5.4.10. Find $\int_{-\infty}^{\infty} \frac{1}{1+x^2}$ or show that it fails to converge.

Solution: We write

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx.$$

Then, since $\arctan'(x) = \frac{1}{1+x^2}$, the First Fundamental Theorem implies that

$$\begin{aligned}\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx &= \lim_{r\to-\infty} \int_{r}^{0} \frac{1}{1+x^2}\,dx \\ &= \lim_{r\to-\infty} (\arctan 0 - \arctan r) = \pi/2,\end{aligned}$$

and

$$\begin{aligned}\int_{0}^{\infty} \frac{1}{1+x^2}\,dx &= \lim_{s\to\infty} \int_{0}^{s} \frac{1}{1+x^2}\,dx \\ &= \lim_{s\to\infty} (\arctan s - \arctan 0) = \pi/2,\end{aligned}$$

Thus, $\int_{-\infty}^{\infty} \frac{1}{1+x^2}$ converges to π.

Functions With Singularities

If a function f is integrable on $[r, b]$ for every r with $a < r \le b$, but unbounded on the interval $(a, b]$, then it is not integrable on $[a, b]$. It is said to have a *singularity* at a. Still, its improper integral over $[a, b]$ may exist in the sense that

$$\lim_{r\to a^+} \int_{r}^{b} f(x)\,dx$$

may exist and be finite. In this case we say that the improper integral $\int_{a}^{b} f(x)\,dx$ *converges*. Its value, of course, is the indicated limit.

Similarly, a function f may be integrable on $[a, s]$ for every s with $a \le s < b$, but not bounded on $[a, b)$. In this case, its improper integral over $[a, b]$ is

$$\lim_{s\to b^-} \int_{a}^{s} f(x)\,dx$$

provided this limit converges.

It may be that the singular point for f is an interior point c of the interval over which we wish to integrate f. That is, it may be that $a < c < b$ and f is integrable on closed subintervals of $[a, b]$ that don't contain c, but f blows up at c. In this case, we write

$$\int_{a}^{b} f(x)\,dx = \int_{a}^{c} f(x)\,dx + \int_{c}^{b} f(x)\,dx.$$

If the two improper integrals on the right converge, then we say the improper integral on the left converges and it converges to the sum on the right.

Example 5.4.11. Find $\int_{-1}^{1} x^{-1/3}\,dx$.

Solution: Here the integrand blows up at 0. An antiderivative for $x^{-1/3}$ is $\frac{3}{2}x^{2/3}$. Thus,

$$\int_{-1}^{0} x^{-1/3}\,dx = \lim_{s\to 0^-} \frac{3}{2}(s^{2/3} - (-1)^{2/3}) = -\frac{3}{2},$$

while

$$\int_{0}^{1} x^{-1/3}\,dx = \lim_{r\to 0^+} \frac{3}{2}((1)^{2/3} - (r)^{2/3}) = \frac{3}{2}.$$

Thus,

$$\int_{-1}^{1} x^{-1/3}\,dx = \int_{-1}^{0} x^{-1/3}\,dx + \int_{0}^{1} x^{-1/3}\,dx$$

converges to $-\frac{3}{2} + \frac{3}{2} = 0$.

The following is a theorem which can be used to conclude that an improper integral converges without actually carrying out the integration.

Theorem 5.4.12. *Let $\int_a^b f(x)\,dx$ be an improper integral – improper due to the fact that $a = -\infty$ or $b = \infty$ or f has a singularity at a or f has a singularity at b. If g is a non-negative function such that $|f(x)| \le g(x)$ for all $x \in (a,b)$ and if*

$$\int_a^b g(x)\,dx$$

converges, then

$$\int_a^b f(x)\,dx$$

also converges.

Proof. We will prove this in the case where the bad point is b – either $b = \infty$ or f blows up at b. The case where a is the bad point is entirely analogous.

Let $h(x) = f(x) + |f(x)|$. Then $0 \le h(x) \le 2g(x)$ for all $x \in (a,b)$. So

$$H(s) = \int_a^s h(x)\,dx \quad \text{and} \quad \int_a^s g(x)\,dx$$

are non-decreasing functions of s (Exercise 5.4.13) and

$$H(s) \le 2\int_a^s g(x)\,dx \le 2\int_a^b g(x)\,dx.$$

The integral on the right is finite by hypothesis. It follows that the non-decreasing function $H(s)$ is bounded above. By Exercise 4.1.13, $\lim_{s\to b^-} H(s)$ converges, Hence, the improper integral $\int_a^b h(x)\,dx$ converges.

The same argument, with h replaced by $|f(x)|$ shows that $\int_a^b |f(x)|\,dx$ converges. Since $f = h - |f|$, it follows that $\int_a^b f(x)\,dx$ also converges. □

Example 5.4.13. Determine whether $\int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\,dx$ converges.

Solution: Since $\mathrm{e}^{-x^2} \le \dfrac{1}{1+x^2}$ (by Exercise 4.4.3) and each of

$$\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx \quad \text{and} \quad \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$$

converges by Example 5.4.10, the same is true of the corresponding integrals for e^{-x^2}. It follows that $\int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\,dx$ converges.

Exercise Set 5.4

1. Supply the details for the proof of Theorem 5.4.3.
2. Prove that $\ln\left(\dfrac{a}{b}\right) = \ln a - \ln b$ for all $a, b \in (0, +\infty)$.
3. Finish the proof of Theorem 5.4.4 by showing that $\lim_{x\to 0} \ln x = -\infty$. Hint: this follows easily from $\lim_{x\to\infty} \ln x = +\infty$ and properties of ln.
4. Prove Part (b) of Theorem 5.4.7.
5. Using Definition 5.4.8 and the properties of exp prove the laws of exponents:
$$a^{x+y} = a^x a^y \quad \text{and} \quad a^{xy} = (a^x)^y.$$
6. Compute the derivative of a^x for each $a > 0$.
7. Find an antiderivative for a^x for each $a > 0$.
8. Prove Theorem 5.4.9.
9. For which values of $p > 0$ does the improper integral $\int_1^{\infty} \dfrac{1}{x^p}\,dx$ converge. Justify your answer.
10. For which values of $p > 0$ does the improper integral $\int_0^{1} \dfrac{1}{x^p}\,dx$ converge. Justify your answer.

11. Show that $\int_{-\infty}^{\infty} \frac{\sin x}{1+x^2}$ converges. Can you tell what it converges to?

12. Does the improper integral $\int_0^1 \ln x \, dx$ converge? If so, what does it converge to?

13. Prove that if f is an integrable function on every interval $[a, s)$ with $s < b$ and if $f(x) \geq 0$ on $[a, b]$, then the function

$$F(s) = \int_a^s f(x) \, dx$$

is a non-decreasing function on $[a, b)$.