

Chapter 4

The Derivative

In this chapter we will prove the standard theorems from calculus concerning differentiation – theorems such as the Chain Rule, the Mean Value Theorem, and L'Hôpital's Rule.

We begin with the concept of the limit of a function.

4.1 Limits of Functions

Definition 4.1.1. Let I be an open interval, a a point of I , and f a function defined on I except possibly at a itself. Then we will say the limit of $f(x)$ as x approaches a is L and write

$$\lim_{x \rightarrow a} f(x) = L$$

if, for each $\epsilon > 0$, there is a $\delta > 0$ such that

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad x \in I \text{ and } 0 < |x - a| < \delta.$$

Note that the condition $0 < |x - a|$ in the above definition means that, in defining the limit of f as x approaches a , we only care about values of f at points of I other than a itself.

Note also, that the domain of f may be larger than I and may not be an interval at all, but, in order to define the limit of f at a we want f to be defined at least at all points, except a itself, in some open interval containing a .

Remark 4.1.2. On comparing the above definition with the definition of continuity (Definition 3.1.1), we conclude that, if f is defined on an open interval containing a , then f is continuous at a if and only if $\lim_{x \rightarrow a} f(x) = f(a)$.

This means that if f is not continuous at a (or not defined at a), but it has a limit L as x approaches a , then we can make f continuous at a by redefining (or defining) it at a by setting $f(a) = L$.

Example 4.1.3. Find $\lim_{x \rightarrow 1} f(x)$ if $f(x)$ is the function $\frac{x^3 - 1}{x - 1}$ on $\mathbb{R} \setminus \{1\}$.

Solution: For $x \in \mathbb{R} \setminus \{1\}$, we have

$$f(x) = \frac{x^3 - 1}{x - 1} = x^2 + x + 1.$$

The function on the right is continuous at 1 (since it is a polynomial) and has the value 3 there. Thus, if we extend f to all of \mathbb{R} by giving it the value 3 at $x = 1$, then it becomes the continuous function $x^2 + x + 1$. By the above remark, $\lim_{x \rightarrow 1} f(x) = 3$.

Example 4.1.4. Can the function $\frac{\sin x}{x}$ on $\mathbb{R} \setminus \{0\}$ be defined at 0 in such a way that it becomes continuous at 0?

Solution: We learned in calculus that $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$. Thus, if $\frac{\sin x}{x}$ is given the value 1 at $x = 0$, it will be continuous there.

One Sided Limits, Limits at $\pm\infty$

Example 4.1.5. Give an intuitive discussion of the behavior of the function $f(x) = x/|x|$ as x approaches 0.

Solution: We have $f(x) = 1$ if $x > 0$ and $f(x) = -1$ if $x < 0$. Thus, as x approaches 0, $f(x)$ approaches 1 if we keep x to the right of 0, while $f(x)$ approaches -1 if we keep x to the left of 0. However, $\lim_{x \rightarrow 0} f(x)$ does not exist, since in the definition of limit, we allow x to be on either side of a .

The above example suggests that it may be useful to define one-sided limits that depend only on the behavior of the function on one side of the point a . If a function is defined on an unbounded interval, then it may also be useful to discuss its limit at $+\infty$ or $-\infty$. Correctly formulated, the same definition can be used to cover the cases of one sided limits and of limits at $\pm\infty$.

Definition 4.1.6. Let f be a function defined on an open interval (a, b) , where a could be $-\infty$ and b could be $+\infty$. We say that the limit from the right of $f(x)$ as x approaches a is L and write

$$\lim_{x \rightarrow a^+} f(x) = L$$

if for every $\epsilon > 0$ there is an $m \in (a, b)$ such that

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad a < x < m.$$

Similarly, we say the limit of $f(x)$ as x approaches b from the left is L , and write

$$\lim_{x \rightarrow b^-} f(x) = L$$

if for every $\epsilon > 0$ there is a $m \in (a, b)$ such that

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad m < x < b.$$

Note that, if a is finite, then to say that there is a $m \in (a, b)$ such that $|f(x) - L| < \epsilon$ whenever $a < x < m$ is the same thing as saying there is a $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $|x - a| < \delta$ and $x \in (a, b)$ (this is clear if we let m and δ determine each other by the formula $\delta = m - a$). This is just like the ordinary definition of limit of f at a except x is restricted to lie to the right of a . A similar analysis holds for the limit from the left at b in the case where b is finite.

In the case where $b = \infty$, the condition $m < x < b$ just means that $m < x$, while in the case where $a = -\infty$, the condition $a < x < m$ just means that $x < m$. Stated this way, the above definition is the traditional definition of limit at ∞ or at $-\infty$.

For limits at ∞ or $-\infty$, we will simply write “ $\lim_{x \rightarrow \infty} f(x)$ ” or “ $\lim_{x \rightarrow -\infty} f(x)$ ” rather than “ $\lim_{x \rightarrow \infty^-} f(x)$ ” or “ $\lim_{x \rightarrow -\infty^+} f(x)$ ”.

In view of the above discussion, the following theorem is almost obvious. Its proof is left to the exercises.

Theorem 4.1.7. *Let I be an open interval and a a point of I . If f is defined on I except possibly at a then*

$$\lim_{x \rightarrow a} f(x) = L \quad \text{if and only if} \quad \lim_{x \rightarrow a^+} f(x) = L = \lim_{x \rightarrow a^-} f(x).$$

In other words the limit of $f(x)$ as x approaches a exists if and only if the limits from the left and the right both exist and are equal. Of course, the limit is then this common value of the limits from the left and right.

Example 4.1.8. For the function

$$f(x) = \begin{cases} 1 - x & \text{if } x < 0 \\ \sin x & \text{if } x > 0, \end{cases}$$

Find $\lim_{x \rightarrow 0^-} f(x)$, $\lim_{x \rightarrow 0^+} f(x)$, and $\lim_{x \rightarrow 0} f(x)$ if they exist.

Solution: Since, to the left of 0, f agrees with the continuous function $1 - x$, its limit from the left is $\lim_{x \rightarrow 0^-} (1 - x) = 1$. On the other hand, to the right of 0, f agrees with the continuous function $\sin x$, and so its limit from the right is $\lim_{x \rightarrow 0^+} \sin x = \sin 0 = 0$. Because the limits from the left and the right are not the same, $\lim_{x \rightarrow 0} f(x)$ does not exist.

Example 4.1.9. Find $\lim_{x \rightarrow \infty} \frac{x^2 + 3x + 1}{2x^2 - 4}$.

Solution: We do this just as we would if we were finding the limit of a sequence as $n \rightarrow \infty$. We divide both numerator and denominator by the highest power of x that occurs. This yields

$$\frac{x^2 + 3x + 1}{2x^2 - 4} = \frac{1 + 3/x + 1/x^2}{2 - 4/x^2}.$$

From this, we guess that the limit is $1/2$. If we want to prove this is true, using only the above definition, we proceed as follows:

$$\left| \frac{x^2 + 3x + 1}{2x^2 - 4} - \frac{1}{2} \right| = \left| \frac{3x + 3}{2x^2 - 4} \right|.$$

Now if $x \geq 3$, then $2x^2 - 4 \geq x^2$ and $3x + 3 < 4x$. In this case, it follows from the above that

$$\left| \frac{x^2 + 3x + 1}{2x^2 - 4} - \frac{1}{2} \right| \leq \frac{4x}{x^2} = \frac{4}{x}.$$

Thus, given $\epsilon > 0$, if we choose $m = \max(3, 4/\epsilon)$, then

$$\left| \frac{x^2 + 3x + 1}{2x^2 - 4} - \frac{1}{2} \right| \leq \frac{4}{x} < \epsilon \quad \text{whenever} \quad m < x.$$

This proves that the limit is $1/2$, as we expected.

Of course, once we prove some theorems about limits, it becomes much easier to do limit problems like the one above. It turns out that all the theorems about limits of sequences, proved in the last chapter, have analogues for limits of functions.

Limit Theorems

As was the case with continuity, the limit of a function can be characterized in terms of limits of sequences. The following theorem is just like Theorem 3.1.5 and is proved the same way. The only difference is that L replaces $f(a)$. We will not repeat the proof

Theorem 4.1.10. *Let (a, b) be a (possibly infinite) interval and let u be a^+ or b^- or a point in the interval (a, b) . If f is a function, defined on (a, b) , then*

$$\lim_{x \rightarrow u} f(x) = L$$

if and only if $f(a_n) \rightarrow L$ whenever $\{a_n\}$ is a sequence of points in (a, b) , distinct from u , with $a_n \rightarrow u$.

As was the case with continuity in section 3.1, this theorem means that each theorem about convergence of sequences yields a theorem about limits of functions. For example, the Main Limit Theorem for sequences, together with the previous theorem implies the Main Limit Theorem for functions:

Theorem 4.1.11. (Main Limit Theorem) *Let (a, b) be a (possibly infinite) interval, let $u = a^+$ or b^- or a point in the interval (a, b) , and let c be a constant. Let f and g be functions defined on (a, b) . If $\lim_{x \rightarrow u} f(x) = K$ and $\lim_{x \rightarrow u} g(x) = L$, then*

$$(a) \lim_{x \rightarrow u} c = c;$$

- (b) $\lim_{x \rightarrow u} cf(x) = cK$;
- (c) $\lim_{x \rightarrow u} (f(x) + g(x)) = K + L$;
- (d) $\lim_{x \rightarrow u} f(x)g(x) = KL$;
- (e) $\lim_{x \rightarrow u} f(x)/g(x) = K/L$, provided $L \neq 0$.

There is also a theorem about the limit of a composite function which is similar to Theorem 3.1.10 and has the same proof.

Theorem 4.1.12. *Let (a, b) be a (possibly infinite) interval and let $u = a^+$ or b^- . If g is defined on (a, b) and $\lim_{x \rightarrow u} g(x) = L$, f is defined on an interval containing L and the image of g , and f is continuous at L , then*

$$\lim_{x \rightarrow u} f(g(x)) = f(L).$$

Proof. Let $\{a_n\}$ be a sequence in I converging to u . Then, by Theorem 4.1.10, $\lim_{x \rightarrow u} g(x) = L$ implies $g(a_n) \rightarrow L$. Then, by Theorem 3.1.5, the continuity of f at L implies that $f(g(a_n)) \rightarrow f(L)$. Again using Theorem 4.1.10, we conclude that $\lim_{x \rightarrow u} f(g(x)) = f(L)$. \square

Example 4.1.13. Prove that if g is a non-negative function, defined on an interval I except possibly at one point $a \in I$, and if $\lim_{x \rightarrow a} g(x) = L$, then

$$\lim_{x \rightarrow a} g^r(x) = L^r \quad \text{for all rational } r > 0.$$

Solution: If $r > 0$ is rational and we set $f(x) = x^r$, then f is continuous on $[0, \infty)$ by Theorem 3.1.6. Since $g^r(x) = f(g(x))$, it follows immediately from the previous theorem that $\lim_{x \rightarrow a} g^r(x) = L^r$.

Infinite Limits

Just as with sequences, for a function f it is sometimes useful to know that, even though f may not have a finite limit as $x \rightarrow u$, it does approach either $+\infty$ or $-\infty$. In analogy with Definition 2.4.4, we define infinite limits as follows.

Definition 4.1.14. If f is a function defined on an interval (a, b) , then we say $\lim_{x \rightarrow a^+} f(x) = \infty$ if, for each M , there is an $m \in (a, b)$ such that

$$f(x) > M \quad \text{whenever } a < x < m.$$

Infinite limits at b^- and what it means for the limit to be $-\infty$ are defined analogously (see the exercises).

If $c \in (a, b)$ and $\lim_{x \rightarrow c^-} f(x)$ and $\lim_{x \rightarrow c^+} f(x)$ are both ∞ , then we write $\lim_{x \rightarrow c} f(x) = \infty$. The analogous statement holds if the limits are both $-\infty$.

The following theorem reduces statements about infinite limits to statements about finite limits. Its proof is left to the exercises.

Theorem 4.1.15. Let f be defined on (a, b) and let $u = a^+$ or b^- or a point in the interval (a, b) . If f is positive on (a, b) , then

$$\lim_{x \rightarrow u} f(x) = \infty \quad \text{if and only if} \quad \lim_{x \rightarrow u} \frac{1}{f(x)} = 0.$$

Similarly, if f is negative on (a, b) , then

$$\lim_{x \rightarrow u} f(x) = -\infty \quad \text{if and only if} \quad \lim_{x \rightarrow u} \frac{1}{f(x)} = 0.$$

Example 4.1.16. Analyze the behavior of $f(x) = \frac{x}{1-x}$ as x approaches 1.

Solution: We have $\lim_{x \rightarrow 1} \frac{1}{f(x)} = \lim_{x \rightarrow 1} \frac{1-x}{x} = 0$, and so the limits of this function from the left and the right at 1 are both 0. On $(0, 1)$ the function f is positive and so $\lim_{x \rightarrow 1^-} f(x) = \infty$ by the previous theorem. On $(1, \infty)$ the function f is negative and so $\lim_{x \rightarrow 1^+} f(x) = -\infty$, also by the previous theorem.

Exercise Set 4.1

In each of the next 6 exercises find the indicated limit and prove that your answer is correct.

1. $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$.

2. $\lim_{x \rightarrow 2} \frac{x^2 + x - 2}{x - 1}$.

3. $\lim_{x \rightarrow 2} \left(\frac{x^2 - 4}{x - 2} \right)^{3/2}$.

4. $\lim_{x \rightarrow 0} \cos(x^2 - x)$.

5. $\lim_{x \rightarrow 2} \frac{x^2 - 3x + 1}{2x^2 + 1}$.

6. $\lim_{x \rightarrow \infty} \frac{x^2 - 3x + 1}{2x^2 + 1}$.

7. If $f(x) = \frac{\sin x}{|x|}$, find $\lim_{x \rightarrow 0^+} f(x)$ and $\lim_{x \rightarrow 0^-} f(x)$. Does $\lim_{x \rightarrow 0} f(x)$ exist?

8. If $f(x) = \sin 1/x$, do $\lim_{x \rightarrow 0^+} f(x)$ and $\lim_{x \rightarrow 0^-} f(x)$ exist?

9. If, in Example 4.1.8, f is defined to be $-x$ for $x < 0$ instead of $1 - x$, does $\lim_{x \rightarrow 0} f(x)$ exist? Why?

10. Prove Theorem 4.1.7.

11. Let f be defined on a bounded interval (a, b) and let u be a^+ , b^- or a point of (a, b) . Prove that if $\lim_{x \rightarrow u} f(x)$ exists and is positive, then there is a $\delta > 0$ such that $f(x) > 0$ whenever $|x - u| < \delta$ and $x \in (a, b)$. Hint: recall the proof of Theorem 2.2.3.
12. Let f be a non-negative function on an interval (a, b) and let $u = a^+$ or b^- . If $\lim_{x \rightarrow u} f(x)$ exists, prove that it is a non-negative number.
13. Prove that if f is a bounded, non-decreasing function on the interval (a, b) , then $\lim_{x \rightarrow a^+} f(x)$ and $\lim_{x \rightarrow b^-} f(x)$ both exist and are finite.
14. State an appropriate definition for the statement $\lim_{x \rightarrow b^-} f(x) = -\infty$.
15. Prove Theorem 4.1.15

4.2 The Derivative

The definition of the derivative is familiar from calculus.

Definition 4.2.1. Let f be a function defined on an open interval containing $a \in \mathbb{R}$. If

$$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

exists and is finite, then we denote it by $f'(a)$, and we say f is differentiable at a with derivative $f'(a)$. If f is defined and differentiable at every point of an open interval I , then we say that f is differentiable on I .

The derivative f' of f is a new function with domain consisting of those points in the domain of f at which f is differentiable.

Remark 4.2.2. When convenient, we will make the change of variables $h = x - a$ and write the derivative in the form

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}. \quad (4.2.1)$$

Equivalently, when it is convenient to use x for the independent variable in the function f' , we will write the derivative in the form

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}.$$

We don't intend to repeat the computation of the derivatives of all the elementary functions. This is done in calculus. We will assume the student knows how to differentiate polynomials, rational functions, trigonometric functions, inverse trigonometric functions, and exponentials and logarithms. We will, however, compute a couple of derivatives directly from the above definition, just to remind the student of how this is done, and we will occasionally compute a derivative, as an example, to illustrate the use of some theorem.

Example 4.2.3. If $f(x) = x^3$, find the derivative of f using just Definition 4.2.1.

Solution: We have

$$\begin{aligned} f'(a) &= \lim_{x \rightarrow a} \frac{x^3 - a^3}{x - a} = \lim_{x \rightarrow a} \frac{(x - a)(x^2 + xa + a^2)}{x - a} \\ &= \lim_{x \rightarrow a} (x^2 + xa + a^2) = 3a^2. \end{aligned}$$

Thus, $f'(a) = 3a^2$.

Example 4.2.4. If $f(x) = \sqrt{x}$, find $f'(x)$ for $x > 0$ using just Definition 4.2.1.

Solution: We have

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h \rightarrow 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}. \end{aligned}$$

Thus, $f'(x) = \frac{1}{2\sqrt{x}}$.

Differentiation Theorems

We will use what we know about limits to prove the main theorems concerning differentiation. Some of these are proved in the typical calculus course and some are not.

Theorem 4.2.5. *If f is differentiable at a , then f is continuous at a .*

Proof. If f is defined in an open interval containing a and x , and if $x \neq a$, then

$$f(x) = f(a) + \frac{f(x) - f(a)}{x - a}(x - a).$$

We take the limit of both sides as $x \rightarrow a$. If f is differentiable at a , then $\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$ exists and is finite. Since $\lim_{x \rightarrow a} (x - a) = 0$, this implies that $\lim_{x \rightarrow a} f(x) = f(a)$. Thus, f is continuous at a . \square

Theorem 4.2.6. *Let f and g be functions defined on an open interval I containing a and suppose f and g are both differentiable at a and c is a constant. Then cf , $f + g$, fg are differentiable at a , as is f/g provided $g(a) \neq 0$, and*

- (a) $(cf)'(a) = cf'(a)$;
- (b) $(f + g)'(a) = f'(a) + g'(a)$;
- (c) $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$;
- (d) $\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g^2(a)}$.

Proof. We will prove (c) and (d) and leave (a) and (b) to the exercises.

To prove (c), we write

$$\frac{f(x)g(x) - f(a)g(a)}{x - a} = \frac{f(x) - f(a)}{x - a}g(x) + f(a)\frac{g(x) - g(a)}{x - a} \quad (4.2.2)$$

By the previous theorem, $\lim_{x \rightarrow a} g(x) = g(a)$, and so the Main Limit Theorem implies that the limit of the right side of (4.2.2) as $x \rightarrow a$ exists and is equal to $f'(a)g(a) + f(a)g'(a)$. Thus, the limit of the left side of this equality as $x \rightarrow a$ exists as well. Hence, $(fg)'(a)$ exists and is equal to $f'(a)g(a) + f(a)g'(a)$.

To prove part (d), we first prove that $1/g$ is differentiable at a and

$$\left(\frac{1}{g}\right)'(a) = -\frac{g'(a)}{g^2(a)}.$$

In fact

$$\frac{1/g(x) - 1/g(a)}{x - a} = \frac{g(a) - g(x)}{g(a)g(x)(x - a)} = \frac{g(a) - g(x)}{x - a} \frac{1}{g(a)g(x)}.$$

If we take the limit of both sides and use the Main Limit Theorem, the conclusion is that $(1/g)'(a)$ exists and is equal to $-\frac{g'(a)}{g^2(a)}$, as claimed.

Now part (d) of the theorem follows from the computation

$$\begin{aligned} \left(\frac{f}{g}\right)'(a) &= \left(f\frac{1}{g}\right)'(a) = f'(a)\frac{1}{g(a)} - f(a)\frac{g'(a)}{g^2(a)} \\ &= \frac{f'(a)g(a) - f(a)g'(a)}{g^2(a)}. \end{aligned}$$

□

The Chain Rule

Theorem 4.2.7. *Suppose g is defined in an open interval I containing a and f is defined in an open interval containing $g(I)$. If g is differentiable at a and f is differentiable at $g(a)$, then $f \circ g$ is differentiable at a and*

$$(f \circ g)'(a) = f'(g(a))g'(a).$$

Proof. We let $b = g(a)$ and we define a function h by

$$h(y) = \begin{cases} \frac{f(y) - f(b)}{y - b} & \text{if } y \neq b \\ f'(y) & \text{if } y = b. \end{cases}$$

Then, since

$$\lim_{y \rightarrow b} \frac{f(y) - f(b)}{y - b} = f'(b),$$

the function h is continuous at $b = g(a)$. Furthermore,

$$\frac{f(g(x)) - f(g(a))}{x - a} = h(g(x)) \frac{g(x) - g(a)}{x - a}.$$

Since h is continuous at $b = g(a)$ and g is continuous at a , we conclude that $h(g(x))$ is continuous at $x = a$. Thus, if we take the limit of both sides of the above identity, we conclude that

$$(f \circ g)'(a) = \lim_{x \rightarrow a} \frac{f(g(x)) - f(g(a))}{x - a} = h(g(a)) \lim_{x \rightarrow a} \frac{g(x) - g(a)}{x - a} = f'(g(a))g'(a).$$

□

Example 4.2.8. Find $(\sin \sqrt{x})'$ using the Chain Rule.

Solution: The derivative of \sin is \cos and the derivative of \sqrt{x} is $\frac{1}{2\sqrt{x}}$. Thus, by the Chain Rule,

$$(\sin \sqrt{x})' = (\cos \sqrt{x}) \frac{1}{2\sqrt{x}} = \frac{\cos \sqrt{x}}{2\sqrt{x}}.$$

Derivative of an Inverse Function

If f is continuous and strictly monotone on an interval I , then it has a continuous inverse function g , defined on $J = f(I)$, such that $g(J) = I$ (Theorem 3.2.6). If I is an open interval and a is a point of I , then J is also an open interval and $b = f(a) \in J$ (Exercise 4.2.5).

Theorem 4.2.9. *If f is strictly monotone on an open interval I containing a , f is differentiable at a , and $f'(a) \neq 0$, then the inverse function g of f is differentiable at $b = f(a)$ and*

$$g'(b) = \frac{1}{f'(a)} = \frac{1}{f'(g(b))}.$$

Proof. For $y \in J$, we set $x = g(y) \in I$. Then $f(x) = y$. We also have $b = f(a)$ and $a = g(b)$. Then

$$\frac{g(y) - g(b)}{y - b} = \frac{x - a}{f(x) - f(a)}.$$

If we denote by h the function of x on the right, then, since f is strictly monotone on I , h is defined everywhere on I except at $x = a$. Since $\lim_{x \rightarrow a} h(x) = \frac{1}{f'(a)}$, the function h will be defined and continuous at a if we give it the value $\frac{1}{f'(a)}$ at $x = a$. Then

$$\frac{g(y) - g(b)}{y - b} = h(g(y)).$$

If we pass to the limit as $y \rightarrow b$, then, by Theorem 4.1.12, the expression on the right has limit $h(g(b)) = \frac{1}{f'(g(b))}$, since g is continuous at b . This implies the expression on the left has the same limit, which means that $g'(b)$ exists and equals $\frac{1}{f'(g(b))}$. \square

Example 4.2.10. Find the derivative of $\sin^{-1}(x)$.

Solution: The function $\sin x$, when restricted to the domain $[-\pi/2, \pi/2]$ is strictly increasing. Its inverse function $\sin^{-1}(x)$ is also increasing and has domain $[-1, 1]$ – the image of $[-\pi/2, \pi/2]$ under \sin . Thus, \sin^{-1} has a non-negative derivative on $(-1, 1)$ and by Theorem 4.2.9, it is given by

$$(\sin^{-1} x)' = \frac{1}{\cos(\sin^{-1} x)} = \frac{1}{\sqrt{1 - \sin^2(\sin^{-1} x)}} = \frac{1}{\sqrt{1 - x^2}},$$

since $\sin(\sin^{-1} x) = x$.

Exercise Set 4.2

- Using just the definition of the derivative, show that the derivative of $1/x$ is $-1/x^2$.
- Using just the definition of the derivative, find $(x^2 + 3x)'$.
- Show how to derive the expression for the derivative of $\tan x$ if you know the derivatives of $\sin x$ and $\cos x$.
- Using theorems from this section, find the derivative of $\tan\left(\frac{x}{x^2 + 1}\right)$.
- We know that the image of a closed interval under a continuous function is a closed interval or a point (Theorem 3.2.4). Show that the image of an open interval under a continuous, strictly monotone function is an open interval.
- If $f \circ g \circ h(x) = f(g(h(x)))$ is the composition of three functions, find an expression for its derivative. You may use the Chain Rule.
- Using Theorem 4.2.9, derive the expression for the derivative of \sqrt{x} .
- Using Theorem 4.2.9, derive the expression for the derivative of $\tan^{-1} x$.
- Prove that if f is defined on an open interval I and has a positive derivative at a point $a \in I$, then there is an open interval J , containing a and contained in I , such that $f(x) < f(a) < f(y)$ whenever $x, y \in J$ and $x < a < y$. Hint: see Exercise 4.1.11.

10. If f is a monotone function on an interval and g is its inverse function, then

$$f \circ g(y) = y$$

for every y in the domain J of g . Use the Chain Rule on this identity to derive the expression for the derivative of the inverse function g . This argument is not a substitute for the proof in Theorem 4.2.9. Why?

11. Is the function defined by

$$f(x) = \begin{cases} x \sin 1/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

differentiable at 0? How about the function

$$f(x) = \begin{cases} x^2 \sin 1/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}?$$

12. Is the function defined by

$$f(x) = \begin{cases} x^2 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

differentiable at 0?

4.3 The Mean Value Theorem

Critical Points

The proof of the Mean Value Theorem rests on the fact that a continuous function on a closed bounded interval $[a, b]$ takes on its maximum and minimum values only at critical points. A *critical point* for f on $[a, b]$ is a point $c \in [a, b]$ which satisfies one of the following:

1. c is an endpoint (a or b);
2. c is a stationary point, meaning $c \in (a, b)$ and $f'(c) = 0$; or
3. c is a singular point, meaning $c \in (a, b)$ and $f'(c)$ does not exist.

Theorem 4.3.1. *If f is a continuous function on a closed bounded interval $[a, b]$ and $c \in [a, b]$ is a point at which f assumes a maximum or a minimum value on $[a, b]$, then c is a critical point for f on $[a, b]$.*

Proof. Assume f has a maximum at c . The proof in the case where it has a minimum is the same, except that the inequalities reverse.

We will prove that if c is not an endpoint or a singular point, then it must be a stationary point. This implies that it has to be one of the three.

If c is not an endpoint and not a singular point, then $a < c < b$ and f has a derivative at c . Since $f(x) \leq f(c)$ for all $x \in [a, b]$, we have

$$\frac{f(x) - f(c)}{x - c} \begin{cases} \leq 0 & \text{for } x > c, \\ \geq 0 & \text{for } x < c. \end{cases}$$

It follows from Exercise 4.1.12 that

$$\lim_{x \rightarrow c^+} \frac{f(x) - f(c)}{x - c} \leq 0 \quad \text{and} \quad \lim_{x \rightarrow c^-} \frac{f(x) - f(c)}{x - c} \geq 0.$$

Since these two one-sided limits must be equal if the limit itself exists, we conclude that the limit must be 0. That is, $f'(c) = 0$. Hence c is a stationary point. \square

The Mean Value Theorem

The Mean Value Theorem is one of the most heavily used tools of calculus. It says that if f is continuous on $[a, b]$ and differentiable on (a, b) , then for at least one point between a and b the graph of f has tangent line parallel to the line joining $(a, f(a))$ to $(b, f(b))$; this may happen at several points (see Figure 4.1). More precisely,

Theorem 4.3.2. *If a function f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) , then there is at least one point $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}. \quad (4.3.1)$$

Proof. The function whose graph is the line joining $(a, f(a))$ to $(b, f(b))$ is

$$g(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a).$$

If we subtract this from f the result is the function s , where

$$s(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

The function s is also continuous on $[a, b]$ and differentiable on (a, b) . By Theorem 3.2.1, s assumes both a maximum value and a minimum value on $[a, b]$. However,

$$s(a) = s(b) = 0,$$

and so s is either identically zero or it assumes a non-zero maximum or a non-zero minimum on (a, b) . In each of these cases, s has a critical point in (a, b) . Let c be such a critical point. Since s is differentiable on (a, b) , c must be a point at which s' is 0. Thus,

$$s'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} = 0,$$

which implies that c satisfies (4.3.1). \square

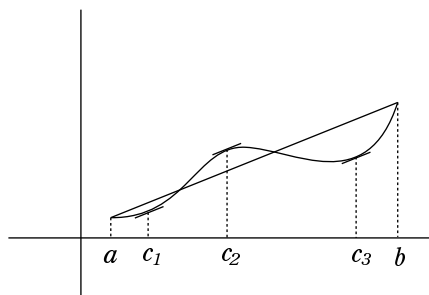


Figure 4.1: Three choices for the c in the Mean Value Theorem.

The Mean Value Theorem has a wide variety of applications. Many of the frequently used facts that we take for granted in calculus are direct consequences of this theorem. It is also used to prove many new facts that go beyond standard calculus material.

Functions with Vanishing Derivative

Theorem 4.3.3. *If f is a differentiable function on an open interval (a, b) and f' is identically 0 on (a, b) , then f is a constant.*

Proof. let x, y be any two points of (a, b) with $x < y$. Then the Mean Value Theorem implies that there is a number c between x and y such that

$$f'(c) = \frac{f(y) - f(x)}{y - x}.$$

Since $f'(c) = 0$, this implies that $f(x) - f(y) = 0$, or $f(x) = f(y)$. Thus, f has the same value at any two points of (a, b) and this means that it is constant. \square

Corollary 4.3.4. *If f and g are differentiable functions on (a, b) and $f'(x) = g'(x)$ for all $x \in (a, b)$, then there is a constant c such that $f(x) = g(x) + c$ on (a, b) .*

Proof. We apply the previous theorem to $f - g$. \square

Another way to say this corollary is: If a function h has an antiderivative on (a, b) , then any two of its antiderivatives differ by a constant. We use this fact all the time in integration theory.

Monotone Functions

Theorem 4.3.5. *If f is a function which is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b) , then f is strictly increasing on $[a, b]$ if $f'(x) > 0$ for all $x \in (a, b)$, while f is strictly decreasing on $[a, b]$ if $f'(x) < 0$ for all $x \in (a, b)$.*

Proof. If x and y are any two points of $[a, b]$ with $x < y$, then the Mean Value Theorem tells us there is a $c \in (x, y) \subset (a, b)$ at which

$$f'(c) = \frac{f(y) - f(x)}{y - x}.$$

Since the denominator is positive, this means that $f'(c)$ and $f(y) - f(x)$ have the same sign. This implies that f is strictly increasing (resp. decreasing) on $[a, b]$ if $f'(c)$ is positive (resp. negative) for all $c \in (a, b)$. \square

This is the basis for the familiar graphing technique which uses the sign of the derivative of f to determine intervals on which f is increasing or decreasing.

The converse of Theorem 4.3.5 is not true, since a function which is strictly increasing on an interval (a, b) can have a derivative that is 0 at some points of (a, b) (for example, $f(x) = x^3$ is strictly increasing on $(-\infty, +\infty)$, but its derivative is 0 at 0). However, the following related result is an “if and only if” Theorem. Its proof is left to the exercises.

Theorem 4.3.6. *Let f be a continuous function on $[a, b]$ which is differentiable on (a, b) . Then f is non-decreasing on $[a, b]$ if and only if $f'(x) \geq 0$ for all $x \in (a, b)$, while if f is non-increasing on $[a, b]$ if and only if $f'(x) \leq 0$ for all $x \in (a, b)$.*

Example 4.3.7. Find the intervals on which the function $f(x) = x^3 - 3x + 5$ is increasing, decreasing.

Solution: The derivative of f is $f'(x) = 3x^2 - 3 = 3(x - 1)(x + 1)$. This function is positive for $x > 1$ and $x < -1$ and is negative for $-1 < x < 1$. Thus, by Theorem 4.3.5, f is increasing on $(-\infty, -1]$ and $[1, +\infty)$ and it is decreasing on $[-1, 1]$.

Example 4.3.8. Prove that $\sin x < x$ for all $x > 0$.

Solution: Let $f(x) = x - \sin x$. Then $f(0) = 0$ and $f'(x) = 1 - \cos x \geq 0$ for all x . In fact, $f'(x) > 0$ except at multiples of 2π . By Theorem 4.3.5, f is increasing on $[0, 2\pi]$. Since it is 0 at $x = 0$, it must be positive on $(0, 2\pi]$. Thus, $\sin x < x$ for $x \in (0, 2\pi]$. It is obvious that $\sin x < x$ for $x > 2\pi$ (since $\sin x \leq 1$ for all x).

Uniform Continuity

We know that a continuous function on a closed, bounded interval I is uniformly continuous. If the interval I is not closed or not bounded, then continuous functions on I need not be uniformly continuous. However, we have the following application of the Mean Value Theorem:

Theorem 4.3.9. *If f is a differentiable function on a (possibly infinite) open interval (a, b) , and if f' is bounded on (a, b) , then f is uniformly continuous on (a, b) .*

Proof. Let M be an upper bound for $|f'|$ on (a, b) . Then $|f'(x)| \leq M$ for all $x \in (a, b)$. By the Mean Value Theorem, if $x, y \in (a, b)$, then there is a c between x and y such that

$$\frac{f(x) - f(y)}{x - y} = f'(c).$$

If we take the absolute value of both sides and multiply by $|x - y|$ and , this yields

$$|f(x) - f(y)| = |f'(c)||x - y| \leq M|x - y|.$$

Thus, given $\epsilon > 0$, if we choose $\delta = \epsilon/M$, then

$$|f(x) - f(y)| \leq \epsilon \quad \text{whenever} \quad |x - y| < \delta.$$

This proves that f is uniformly continuous on (a, b) . □

Exercise Set 4.3

1. If f is a continuous function on $[-1, 1]$ which is differentiable on $(-1, 1)$ and satisfies $f(-1) = 0$, $f(0) = 0$, and $f(1) = 1$, then show that f' takes on the values $0, 1/2$, and 1 on $[-1, 1]$.
2. Prove that $|\sin x - \sin y| \leq |x - y|$ for all $x, y \in \mathbb{R}$.
3. If $r > 0$ prove that $\ln y - \ln x \leq \frac{y - x}{r}$ if $r \leq x \leq y$.
4. Suppose f is a continuous function on $[0, \infty)$ which is differentiable on $(0, \infty)$. If $f(0) = 0$ and $|f'(x)| \leq M$ for all $x \in (0, \infty)$, then prove that $|f(x)| \leq Mx$ on $[0, \infty)$.
5. Prove that if f is a differentiable function on $(0, \infty)$ and f and f' both have finite limits at ∞ , then $\lim_{x \rightarrow \infty} f'(x) = 0$. Hint: apply the Mean Value Theorem to f for large values of a and b .
6. If $f(x) = 2x^3 + 3x^2 - 12x + 5$, find the intervals on which f is increasing and those on which it is decreasing.
7. Prove that $\ln x \leq x - 1$ for all $x > 0$. Hint: analyze where $x - 1 - \ln x$ is increasing and where it is decreasing.
8. Find where $e^{-x} x^e$ is increasing and where it is decreasing. Which is bigger e^π or π^e ?
9. Prove Theorem 4.3.6.
10. Suppose f is a differentiable function on an interval (a, b) and that f' takes on both positive and negative values on (a, b) . Prove that f' must take on the value 0 as well. Hint: show that if $f'(x) > 0$ and $f'(y) < 0$ for points x, y with $a < x < y < b$, then the maximum of f on $[x, y]$ occurs at some point strictly between x and y ; the same argument will show that if $f'(x) < 0$ and $f'(y) > 0$, then the minimum of f on $[x, y]$ occurs at a point strictly between x and y .

11. Use the result of the previous exercise to show that, if f is differentiable on (a, b) and f' takes on two values c and d on (a, b) , then it takes on every value between c and d . This is the Intermediate Value Theorem for Derivatives. Note that we do not assume f' is continuous on $[a, b]$.
12. Let f be differentiable on \mathbb{R} . Prove that, if there is an $r < 1$ such that $|f'(x)| \leq r$ for all $x \in \mathbb{R}$, then $|f(x) - f(y)| \leq r|x - y|$ for all $x, y \in \mathbb{R}$. A function with this property is called a *contraction mapping*.
13. Let f satisfy the conditions of the previous exercise. Show there is a fixed point for f – that is, an $x \in \mathbb{R}$ such that $f(x) = x$. Hint: construct a sequence $\{x_n\}$ inductively by setting $x_1 = 0$ and $x_{n+1} = f(x_n)$. Show that this sequence is Cauchy and that it converges to a fixed point for f .
14. Prove that if f is increasing on $[a, b]$ and on $[b, c]$, then f is also increasing on $[a, c]$.
15. The following is a partial converse to Theorem 4.3.9: Prove that if f is differentiable on a, possibly infinite, interval (a, b) and if $\lim_{x \rightarrow b} f'(x) = \infty$, then f is not uniformly continuous on (a, b) . The same conclusion holds if $\lim_{x \rightarrow a} f'(x) = \infty$.
16. Show that $\ln x$ is uniformly continuous on $[1, \infty)$, but not on $(0, 1]$.

4.4 L'Hôpital's Rule

In this section we prove the familiar L'Hôpital's Rule – a tool from calculus, useful in calculating limits of indeterminate forms. It has two forms, depending on whether the indeterminate form is of type $0/0$ or of type ∞/∞ . The proof uses the following generalization of the Mean Value Theorem.

Cauchy's Mean Value Theorem

Theorem 4.4.1. *Let f and g be functions which are continuous on a closed, bounded interval $[a, b]$ and differentiable on (a, b) . Assume that $g'(x) \neq 0$ for all $x \in (a, b)$. Then there exists $c \in (a, b)$ such that*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}. \quad (4.4.1)$$

Proof. We begin by observing that g is strictly monotone on $[a, b]$. This follows from the fact that g' is never 0 on (a, b) . If it is never 0, then it cannot take on both positive and negative values on (a, b) (Exercise 4.3.10). Thus, it is always positive or always negative, and this implies that it is strictly monotone on $[a, b]$. In particular, $g(b) \neq g(a)$.

The proof now follows the same strategy as the proof of the ordinary Mean Value Theorem (Theorem 4.3.2). The only difference is that $x - a$ and $b - a$

are replaced by $g(x) - g(a)$ and $g(b) - g(a)$ in the definition of the function s . Thus, we set

$$s(x) = f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}(g(x) - g(a)).$$

Note that s is continuous on $[a, b]$ and differentiable on (a, b) . By Theorem 3.2.1, s assumes both a maximum value and a minimum value on $[a, b]$. However,

$$s(a) = s(b) = 0,$$

and so s is either identically zero or it assumes a non-zero maximum or a non-zero minimum on (a, b) . In any of these cases, s has a critical point in (a, b) . Let c be such a critical point. Since s is differentiable on (a, b) , c must be a point at which s' is 0. Thus,

$$s'(c) = f'(c) - \frac{f(b) - f(a)}{g(b) - g(a)}g'(c) = 0,$$

which implies that c satisfies (4.4.1). □

Example 4.4.2. Prove that $|\cos x - 1| \leq \frac{x^2}{2}$ for all x .

Solution: We use Cauchy's Mean Value Theorem with $f(x) = \cos x$ and $g(x) = x^2$. It implies that there is c between 0 and x such that

$$\frac{\cos x - 1}{x^2} = \frac{\cos x - \cos 0}{x^2 - 0^2} = \frac{-\sin c}{2c}.$$

Since $|\sin c| \leq |c|$ by Exercise 4.3.2, this implies that

$$\left| \frac{\cos x - 1}{x^2} \right| \leq \frac{1}{2},$$

which implies that $|\cos x - 1| \leq \frac{x^2}{2}$.

L'Hôpital's Rule

The problem of finding

$$\lim_{x \rightarrow 1} \frac{\ln x}{x^2 - 1}$$

cannot be attacked by using the part of the Main Limit Theorem which deals with limits of quotients, because the limit of the denominator is 0. In fact, both numerator and denominator have limit 0. A limit problem of this type is called a 0/0 form.

Similarly, the problem of finding

$$\lim_{x \rightarrow \infty} \frac{e^x}{x^2}$$

cannot be attacked using the limit of quotients part of the Main Limit Theorem. This time the problem is that both numerator and denominator have limit $+\infty$. A limit problem of this type is called an ∞/∞ form.

Problems of this type can often be solved by using the following theorem.

Theorem 4.4.3. (L'Hôpital's Rule) *Let f and g be differentiable functions on a (possibly infinite) interval (a, b) and let u stand for a^+ or b^- . Suppose, $g(x)$ and $g'(x)$ are non-zero on all of (a, b) and*

1. $\lim_{x \rightarrow u} f(x) = 0 = \lim_{x \rightarrow u} g(x)$, or
2. $\lim_{x \rightarrow u} f(x) = \infty = \lim_{x \rightarrow u} g(x)$.

Then

$$\lim_{x \rightarrow u} \frac{f(x)}{g(x)} = \lim_{x \rightarrow u} \frac{f'(x)}{g'(x)}, \quad (4.4.2)$$

provided the limit on the right exists.

Proof. We will present the proof in the case where $u = a^+$ and the limit on the right in (4.4.2) is a finite number L . The case where this limit is infinite can be reduced to the finite case (Exercise 4.4.16). The proof in the case $u = b^-$ is entirely analogous.

If $x, y \in (a, b)$, then Cauchy's Mean Value Theorem tells us that there is a c between x and y such that

$$f(x) - f(y) = (g(x) - g(y)) \frac{f'(c)}{g'(c)},$$

or

$$\frac{f(x)}{g(x)} = \frac{f(y)}{g(x)} + \left(1 - \frac{g(y)}{g(x)}\right) \frac{f'(c)}{g'(c)}$$

On subtracting L and performing some algebra, this becomes

$$\frac{f(x)}{g(x)} - L = \frac{f(y)}{g(x)} + \left(1 - \frac{g(y)}{g(x)}\right) \left(\frac{f'(c)}{g'(c)} - L\right) - L \frac{g(y)}{g(x)}.$$

On applying the triangle inequality, this yields

$$\left| \frac{f(x)}{g(x)} - L \right| \leq \left| \frac{f(y)}{g(x)} \right| + \left(1 + \left| \frac{g(y)}{g(x)} \right| \right) \left| \frac{f'(c)}{g'(c)} - L \right| + \left| L \frac{g(y)}{g(x)} \right|. \quad (4.4.3)$$

Given $\epsilon > 0$, we will show how to make each of the terms on the right in this inequality be less than $\epsilon/3$ by choosing x sufficiently close to a .

At this point the proof splits into two cases, depending on whether hypothesis (1) or (2) holds. If (1) holds, then since $\lim_{x \rightarrow a^+} f'(x)/g'(x) = L$, Definition 4.1.6 tells us there is an $m \in (a, b)$ so that

$$\left| \frac{f'(c)}{g'(c)} - L \right| < \epsilon/6 \quad (4.4.4)$$

whenever $a < c < m$. This condition will be satisfied if x is any number with $a < x < m$ and y any number with $a < y < x$ (since c is between x and y). Now, given any x , we can choose a y (depending on x) so that $a < y < x$ and

$$\left| \frac{f(y)}{g(x)} \right| < \frac{\epsilon}{3}, \quad \text{and} \quad (4.4.5)$$

$$\left| \frac{g(y)}{g(x)} \right| < \min \left(1, \frac{\epsilon}{3|L|} \right). \quad (4.4.6)$$

This is possible because $\lim_{y \rightarrow a^+} f(y) = 0 = \lim_{y \rightarrow a^+} g(y)$ holds by hypothesis (1). Taken together, inequalities (4.4.3) through (4.4.6) imply that

$$\left| \frac{f(x)}{g(x)} - L \right| < \epsilon \quad \text{whenever} \quad a < x < m.$$

This implies that $\lim_{x \rightarrow a^+} \frac{f(x)}{g(x)} = L$ and completes the proof in the case where (1) holds.

In the case where hypothesis (2) holds, the proof is almost the same. We still use (4.4.3), but the order in which x , y , and m are chosen changes and x and y reverse positions in the interval (a, b) . We first choose y such that (4.4.4) holds whenever $a < c < y$. This is possible because $\lim_{c \rightarrow a^+} f'(c)/g'(c) = L$.

We then choose $m \in (a, y)$ in such a way that (4.4.5) and (4.4.6) hold whenever $a < x < m$. This is possible because $\lim_{x \rightarrow a^+} g(x) = \infty$ holds by hypothesis (2). Because $m < y$, such a choice of x will force $x < y$ and, hence, $c < y$ (again, since c is between x and y).

As before, inequalities (4.4.3) through (4.4.6) imply that

$$\left| \frac{f(x)}{g(x)} - L \right| < \epsilon \quad \text{whenever} \quad a < x < m.$$

This implies that $\lim_{x \rightarrow a^+} \frac{f(x)}{g(x)} = L$ and completes the proof in the case where (2) holds. \square

Example 4.4.4. Find $\lim_{x \rightarrow 1} \frac{\ln x}{x^2 - 1}$.

Solution: This is a $0/0$ form since $\lim_{x \rightarrow 1} \ln x = 0 = \lim_{x \rightarrow 1} (x^2 - 1)$. If we differentiate numerator and denominator, and take the limit of the resulting fraction, we get

$$\lim_{x \rightarrow 1} \frac{1/x}{2x} = \frac{1}{2}.$$

We conclude that

$$\lim_{x \rightarrow 1} \frac{\ln x}{x^2 - 1} = \frac{1}{2}$$

as well.

Example 4.4.5. Find $\lim_{x \rightarrow \infty} \frac{x^2}{e^x}$.

Solution: This is an ∞/∞ form since $\lim_{x \rightarrow \infty} e^x = \infty = \lim_{x \rightarrow \infty} x^2$. If we differentiate numerator and denominator, and take the limit of the resulting fraction, we get

$$\lim_{x \rightarrow \infty} \frac{2x}{e^x}.$$

This is still an ∞/∞ form. If we again differentiate numerator and denominator and pass to the limit, we get

$$\lim_{x \rightarrow \infty} \frac{2}{e^x} = 0.$$

We conclude from L'Hôpital's Rule that

$$\lim_{x \rightarrow \infty} \frac{2x}{e^x} = 0,$$

and, hence, that

$$\lim_{x \rightarrow \infty} \frac{x^2}{e^x} = 0.$$

Example 4.4.6. Find $\lim_{n \rightarrow \infty} (1 + r/n)^n$.

Solution: This is the limit of a sequence. However, we may compute this limit by replacing the integer valued variable n with the real valued variable x . If we find that $\lim_{x \rightarrow \infty} (1 + r/x)^x$ has a limit, then $\lim_{n \rightarrow \infty} (1 + r/n)^n$ must have the same limit.

We set $f(x) = (1 + r/x)^x$, $g(x) = \ln(f(x)) = x \ln(1 + r/x)$, and $y = 1/x$. Then

$$\lim_{x \rightarrow \infty} g(x) = \lim_{y \rightarrow 0} g(1/y) = \lim_{y \rightarrow 0} \frac{\ln(1 + ry)}{y}.$$

This is a $0/0$ form and L'Hôpital's Rule implies that this limit is

$$\lim_{y \rightarrow 0} \frac{r}{1 + ry} = r.$$

Then

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} e^{g(x)} = e^r.$$

by Theorem 4.1.12.

Exercise Set 4.4

1. Prove that if $r > 0$ and $x > 1$, then $\ln x \leq \frac{x^r - 1}{r}$. Hint: use Cauchy's form of the Mean Value Theorem with $f(x) = \ln x$ and $g(x) = x^r$.
2. Prove that $|\sin x - x| \leq \frac{1}{6}|x|^3$.
3. Prove that $1 + x^2 \leq e^{x^2}$ for all $x \in \mathbb{R}$.

4. If f is a function which is differentiable on an open interval I containing 0 and if $f(0) = 0$, then prove that there is a c between 0 and x at which

$$f(x) = \frac{f'(c)}{c^{n-1}} \frac{x^n}{n}.$$

Hint: apply the Cauchy Mean Value Theorem to $f(x)$ and $g(x) = x^n$.

5. Use the previous exercise and induction to prove that if f is n -times differentiable on an open interval I containing 0 and if the k th derivative, $f^{(k)}$ of f is 0 at 0 for $k = 0, 1, \dots, n-1$, then there is a c between 0 and x at which

$$f(x) = f^{(n)}(c) \frac{x^n}{n!}.$$

Find each of the following limits if they exist:

6. $\lim_{x \rightarrow \infty} \frac{\ln x}{x^r}$ where $r > 0$.
7. $\lim_{x \rightarrow 0} x \ln x$.
8. $\lim_{x \rightarrow 0} \frac{\sin x - x}{x^3}$.
9. $\lim_{x \rightarrow 0} \frac{1 + \cos x}{x^2}$.
10. $\lim_{x \rightarrow 0} x^x$.
11. $\lim_{x \rightarrow \infty} x^{1/x}$.
12. $\lim_{x \rightarrow \infty} (\sqrt{x+1} - \sqrt{x})$.
13. $\lim_{n \rightarrow \infty} \frac{\ln n}{\sqrt{n}}$
14. Let f be a differentiable function on $(0, \infty)$. Prove that if $\lim_{x \rightarrow \infty} f(x) = \infty$ and $\lim_{x \rightarrow \infty} f'(x) = L$, then

$$\lim_{x \rightarrow \infty} \frac{e^{f(x)}}{\int_0^x e^{f(t)} dt} = L.$$

15. let f be a differentiable function on an interval of the form $(a, +\infty)$. Prove that if there is a number $r \neq 0$ such that $\lim_{x \rightarrow \infty} (rf'(x) + f(x)) = L$ is finite, then $\lim_{x \rightarrow \infty} f'(x) = 0$ and $\lim_{x \rightarrow \infty} f(x) = L$. Hint: apply L'Hôpital's Rule to $\frac{e^{x/r} f(x)}{e^{x/r}}$.
16. Finish the proof of Theorem 4.4.3 by showing that if the theorem is true whenever $\lim_{x \rightarrow u} f'(x)/g'(x)$ is finite, then it is also true whenever this limit is infinite.