

Chapter 3

Continuous Functions

In this chapter we begin our study of functions of a real variable. The concepts of limit and continuity for such functions are of critical importance.

3.1 Continuity

We will be dealing with functions from a subset of \mathbb{R} to \mathbb{R} . Usually in this chapter, the domain of a function will be an interval – closed, open, or half-open, bounded or unbounded – or a finite union of intervals. However, it is certainly possible to consider functions which have much more complicated subsets of \mathbb{R} as domain.

To define a function from a subset of \mathbb{R} to \mathbb{R} , we must specify a domain for the function and the rule or formula that specifies the value of the function at each point of that domain. For example, the following are descriptions of functions:

1. $f(x) = 1/x$ on $(0, \infty)$;
2. $g(x) = 1/x$ on $\mathbb{R} \setminus \{0\}$;
3. $h(x) = \sin x$ on $[0, 2\pi]$;
4. $k(x) = \sin x$ on \mathbb{R} ;
5. $e(x) = e^x$ on $[0, 1)$.

Although a function may have a *natural domain* – that is, a largest subset of \mathbb{R} on which the formula describing it makes sense – we are at liberty to choose a smaller domain for the function if we wish.

There are a number of special types of functions that we will deal with on a regular basis

1. **Polynomials:** functions of the form $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$, where the a_k are constants for $k = 0, \cdots, n$. If $a_n \neq 0$, then the *degree* of the polynomial is n . The natural domain of a polynomial is \mathbb{R} ;

2. **Rational functions:** functions of the form p/q with p and q polynomials. The natural domain of a function of this form is the set of all real numbers where the denominator q is non-zero;
3. **Trigonometric functions:** \sin , \cos , \tan , \cot , \sec , \csc ;
4. **Inverse trigonometric functions:** \sin^{-1} , \tan^{-1} , etc;
5. **Exponential and log functions:** e^x and $\ln x$.
6. **Power functions:** x^a for $a \in \mathbb{R}$. The natural domain is $\{x \in \mathbb{R}; x \geq 0\}$ unless a is a rational number with an odd denominator – in this case x^a is defined for all real numbers x .

Elementary functions are functions that can be constructed from functions of the above types using addition, multiplication, quotients and composition. It is *not* the case that all the functions we wish to consider are elementary functions.

Continuity

Definition 3.1.1. Let f be a function with domain $D \subset \mathbb{R}$ and let a be an element of D . We will say that f is *continuous* at a if, for each $\epsilon > 0$, there is a $\delta > 0$, such that

$$|f(x) - f(a)| < \epsilon \quad \text{whenever} \quad x \in D \quad \text{and} \quad |x - a| < \delta. \quad (3.1.1)$$

There is a subtle difference between the definition of continuity given above and the one that is usually given in calculus courses. The difference is that our definition depends on the domain of the function. A given expression may not be continuous at a point a if given one domain containing a , and yet it may be continuous at a if it is given a smaller domain.

Example 3.1.2. Give an example of a function which is not continuous at a certain point of its domain, but it is continuous at this point if a smaller domain is chosen for the function.

Solution: Each $x \in \mathbb{R}$ is in exactly one of the intervals $[n, n + 1)$ for $n \in \mathbb{Z}$. Consider the function defined on \mathbb{R} by

$$f(x) = x - n \quad \text{if} \quad x \in [n, n + 1), \quad n \in \mathbb{Z}.$$

The graph of this function is shown in Figure 3.1, which shows why this function is called the *sawtooth function*. We will show that this function is not continuous at 0 (or at any other integer for that matter). However, if its domain is restricted to be the interval $[0, 1)$, then it is continuous at 0.

Now $f(x) = x$ on $[0, 1)$ and $f(x) = x + 1$ on $[-1, 0)$. Suppose ϵ is greater than 0 but less than $1/2$. Then, for any $\delta > 0$, the interval $(-\delta, \delta)$ will contain points of $(-1/2, 0)$ and for any such point x ,

$$|f(x) - f(0)| = |x + 1 - 0| > 1/2 > \epsilon.$$

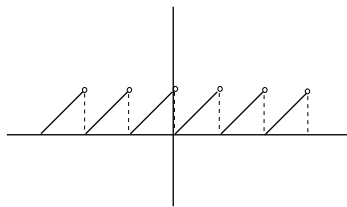


Figure 3.1: The Sawtooth Function.

Thus, there is no way to choose δ such that $|f(x) - f(0)| < \epsilon$ whenever $|x - 0| < \delta$. This means that f is not continuous at 0. The same argument works at any other integer n .

On the other hand, suppose we define a new function g which is the same as f , but with domain cut down to be just $D = [0, 1)$. Then $g(x) = x$ on D . If, for a given $\epsilon > 0$, we choose $\delta = \epsilon$, then

$$|g(x) - g(0)| = |x| < \epsilon \quad \text{whenever} \quad x \in D, \text{ and } |x - 0| = |x| < \delta.$$

Thus, g is continuous at 0.

Definition 3.1.3. We will simply say that a function with domain D is *continuous* if it is continuous at every point of D .

Example 3.1.4. Prove that $f(x) = x^2$ is continuous at $x = 2$.

Solution: We have

$$|f(x) - f(2)| = |x^2 - 4| = |x + 2||x - 2|.$$

If we insist that $|x - 2| < 1$, then $1 < x < 3$ and so $|x + 2| < 5$. Thus, given $\epsilon > 0$, if we choose $\delta = \min\{1, \epsilon/5\}$, then

$$|f(x) - f(2)| = |x + 2||x - 2| < 5|x - 2| < \epsilon \quad \text{whenever} \quad |x - 2| < \delta.$$

This proves that f is continuous at 2.

An Alternate Characterization of Continuity

There is an alternate characterization of continuity that will allow us to use the theorems of the previous chapter to easily prove the standard theorems concerning continuous functions:

Theorem 3.1.5. Let f be a function with domain D and suppose $a \in D$. Then f is continuous at a if and only if, whenever $\{x_n\}$ is a sequence in D which converges to a , then the sequence $\{f(x_n)\}$ converges to $f(a)$.

Proof. We first prove the "only if" – that is, we assume f is continuous and proceed to prove the statement about sequences. Let $\{x_n\}$ be a sequence in D with $x_n \rightarrow a$. Given $\epsilon > 0$, there is a $\delta > 0$ such that

$$|f(x) - f(a)| < \epsilon \quad \text{whenever} \quad x \in D \quad \text{and} \quad |x - a| < \delta.$$

For this δ , there is an N such that

$$|x_n - a| < \delta \quad \text{whenever} \quad n > N.$$

On combining these statements, we conclude

$$|f(x_n) - f(a)| < \epsilon \quad \text{whenever} \quad n > N.$$

Thus, $f(x_n) \rightarrow f(a)$. This completes the proof of the "only if" half of the theorem.

We will prove the "if" part, by proving the contrapositive – that is, we will prove that if f is not continuous at a , then there is a sequence $\{x_n\}$ in D such that $x_n \rightarrow a$ but $\{f(x_n)\}$ does not converge to $f(a)$.

The assumption that f is not continuous at a means that there is an $\epsilon > 0$ for which no δ can be found for which (3.1.1) is true. This means that, no matter what δ we choose, there is always an $x \in D$ such that

$$|x - a| < \delta \quad \text{but} \quad |f(x) - f(a)| \geq \epsilon.$$

In particular, for each of the numbers $1/n$ for $n \in \mathbb{N}$ we may choose an $x_n \in D$ such that

$$|x_n - a| < 1/n \quad \text{but} \quad |f(x_n) - f(a)| \geq \epsilon.$$

These numbers form a sequence $\{x_n\}$ which converges to a (since $1/n \rightarrow 0$), but the image sequence $\{f(x_n)\}$ does not converge to $f(a)$. This completes the proof of the "if" part of the theorem. \square

Combining this with the Main Limit Theorem yields the following:

Theorem 3.1.6. *If r is a positive rational number, then the function $f(x) = x^r$ is continuous on its natural domain.*

Proof. The natural domain D of $f(x) = x^r$ is \mathbb{R} if r has an odd denominator and is the set of non-negative real numbers if r has an even denominator when written in lowest terms. In either case, if $a \in D$ and $\{x_n\}$ is a sequence in D which converges to a , then $\{x_n^r\}$ converges to a^r by parts (e) and (f) of the Main Limit Theorem (Theorem 2.3.6). This implies that x^r is continuous by the previous theorem. \square

Remark 3.1.7. We will eventually prove that the functions x^a for $a \in \mathbb{R}$, e^x , $\ln x$, and the inverse trigonometric functions are all continuous. In the meantime, we will assume this is true whenever it is convenient to do so in an exercise or example. The continuity of the trigonometric functions is usually proved adequately in elementary calculus and so we will use the continuity of these functions whenever it is needed.

Combinations of Continuous Functions

If f and g are functions with domains D_f and D_g , then $f + g$ and fg have domain $D = D_f \cap D_g$, and f/g has domain $\{x \in D : g(x) \neq 0\}$.

Theorem 3.1.8. *Let f and g be functions with domains D_f and D_g . Assume f and g are both continuous at a point $a \in D = D_f \cap D_g$, and let c be a constant. Then*

- (a) cf is continuous at a ;
- (b) $f + g$ is continuous at a ;
- (c) fg is continuous at a ;
- (d) f/g is continuous at a , provided $g(a) \neq 0$;

Proof. These are all proved using the same technique used to prove the previous theorem – combine Theorem 3.1.5 with the corresponding part of the Main Limit Theorem. We will do (b) to illustrate this technique, pose part (d) as an exercise, and let it go at that.

If f and g are continuous at a and $\{x_n\}$ is any sequence in D which converges to a , then Theorem 3.1.5 tells us that $\{f(x_n)\}$ converges to $f(a)$ and $\{g(x_n)\}$ converges to $g(a)$. By part (b) of the Main Limit Theorem (Theorem 2.3.6), $\{f(x_n) + g(x_n)\}$ converges to $f(a) + g(a)$. Therefore, by Theorem 3.1.5 again, $f + g$ is continuous at a . \square

Example 3.1.9. Prove that each polynomial is continuous on all of \mathbb{R} and each rational function is continuous at all points where its denominator is not zero.

Solution: Every positive integral power of x is continuous on \mathbb{R} by Theorem 3.1.6. By (a) of the above theorem, each constant times a power of x is also continuous. Then (b) of the theorem implies that every polynomial is continuous on \mathbb{R} and (d) implies that every rational function is continuous at points where its denominator is not zero.

Composition of Continuous Functions

If f is a function with domain D_f and g is a function with domain D_g , then the composite function $f \circ g$ has domain $D_{f \circ g} = \{x \in D_g : g(x) \in D_f\}$. Suppose a is in this set, so that $a \in D_g$ and $g(a) \in D_f$. Then we can ask if $f \circ g$ is continuous at a . The following theorem answers this question. Its proof is left to the exercises.

Theorem 3.1.10. *With f and g as above, let a be in the domain of $f \circ g$. Then $f \circ g$ is continuous at a if g is continuous at a and f is continuous at $g(a)$.*

Example 3.1.11. Prove that $f(x) = \frac{1}{\sqrt{1-x^2}}$ is continuous as a function on its natural domain.

Solution: The function f has as natural domain the interval $(-1, 1)$, since it is for points in this interval and those points alone that $\sqrt{1-x^2}$ is defined and non-zero. The function $1-x^2$ is continuous on $(-1, 1)$ because it is a polynomial. The square root function is continuous on $[0, \infty)$ by Theorem 3.1.6. Thus, the composition $\sqrt{1-x^2}$ is continuous by Theorem 3.1.10. Finally, f is continuous by part (d) of Theorem 3.1.8.

Exercise Set 3.1

1. If f is a function with domain $[0, 1]$, what is the domain of $f(x^2 - 1)$?
2. What is the natural domain of the function $\frac{x^2 + 1}{x^2 - 1}$. With this as its domain, is this function continuous? Why?
3. We know \sqrt{x} is continuous at all $a \geq 0$, by Theorem 3.1.6. Give another proof of this fact using only the definition of continuity (Definition 3.1.1).
4. Prove that $\frac{1}{1+x^2}$ has natural domain \mathbb{R} and is continuous.
5. At which points is the function $f(x) = |x|$ continuous?
6. Assuming \sin is continuous, prove that $\sin(x^3 - 4x)$ is continuous.
7. Prove (d) of Theorem 3.1.8.
8. Prove Theorem 3.1.10.
9. Consider the function

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0. \end{cases}$$

Is this function continuous if its domain is \mathbb{R} ? Is it continuous if its domain is cut down to $\{x \in \mathbb{R} : x \geq 0\}$? How about if its domain is $\{x \in \mathbb{R} : x \leq 0\}$?

10. Let f be a function with domain D and suppose f is continuous at some point $a \in D$. Prove that, for each $\epsilon > 0$, there is a $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon \quad \text{whenever } x, y \in D \cap (a - \delta, a + \delta).$$

11. Prove that the function $f(x) = \begin{cases} \sin 1/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ is not continuous at 0.
12. Prove that the function $f(x) = \begin{cases} x \sin 1/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ is continuous at 0.

3.2 Properties of Continuous Functions

Continuous functions on closed bounded intervals have a number of highly useful properties. We explore some of these in this section.

Maximum and Minimum Values

A function f with domain D is said to be *bounded* above on $S \subset D$ if and only if the set $f(S) = \{f(x) : x \in S\}$ is bounded above. This is true if and only if

$$\sup_S f = \sup\{f(x) : x \in S\}$$

is finite. Similarly, f is bounded below on S if $f(S)$ is bounded below and this is true if and only if

$$\inf_S f = \inf\{f(x) : x \in S\}$$

is finite. If f is bounded above and below on S , then we say f is *bounded* on S . If f is bounded on its domain D , then it is said to be a bounded function.

Just as a bounded set may have a finite sup, but may not have a maximum element (the sup may not belong to the set), a function f may be bounded above on S without having a maximum value (this happens if $\sup_S f$ is not a value that f assumes on S). However, if f is a continuous function on a closed bounded interval, then the situation is particularly nice.

Theorem 3.2.1. *If f is a continuous function on a closed bounded interval I , then f is bounded on I and, in fact, it assumes both a minimum and a maximum value on I .*

Proof. We will prove that $M = \sup_{x \in I} f(x)$ is finite and, in fact, is a value that f takes on somewhere on I . The proof of the analogous fact for $\inf_{x \in I} f(x)$ has the same proof.

We will inductively construct a nested sequence of closed intervals $\{I_n\}$ with the following properties:

- (1) $I_1 = I$;
- (2) I_k is the closed left or right half of I_{k-1} for each $k > 1$;
- (3) $\sup_{I_k} f(x) = M$ for each k .

The first condition tells us how to pick I_1 . Suppose that I_1, \dots, I_n have been chosen satisfying (1), (2), (3) for $k \leq n$. We choose I_{n+1} as follows: If I_n is cut in half at its midpoint, yielding two closed intervals with union I_n and with intersection the midpoint of I_n , then the sup of f on at least one of these intervals must be the same as the sup of f on I_n . This is M by our induction assumption. If this is true of only one of the two halves of I_n , we choose this half to be I_{n+1} . If it is true of both halves, then we choose I_{n+1} to be the right half of I_n . This completes the induction step of the definition and proves that a sequence $\{I_n\}$ satisfying (1), (2), (3) can be constructed.

Given a nest of intervals $\{I_n\}$ as above, the Nested Interval Property (Theorem 2.5.1) implies that there is a point $a \in \bigcap_n I_n$. This is, in particular, a point of $I = I_1$. We know f is continuous at this point and so, given $\epsilon > 0$, there is a $\delta > 0$ such that

$$f(a) - \epsilon < f(x) < f(a) + \epsilon \quad \text{whenever } x \in I, |x - a| < \delta. \quad (3.2.1)$$

Now the length of I_n is $L/2^{n-1}$, where L is the length of I . Since $\lim L/2^{n-1} = 2L \lim (1/2)^n = 0$, the length of I_n will be less than δ for n sufficiently large. Suppose n is this large. Then $|x - a| < \delta$ for all $x \in I_n$, since $a \in I_n$. By (3.2.1)

$$f(a) - \epsilon < \sup_{I_n} f \leq f(a) + \epsilon.$$

That is,

$$f(a) - \epsilon < M \leq f(a) + \epsilon.$$

This implies that M is finite and that $|f(a) - M| \leq \epsilon$ for every positive ϵ . This is possible only if $f(a) = M$. Thus we have proved that $\sup_{x \in I} f(x)$ is finite and that it is a value assumed by f at some point a of I . \square

Each of the hypotheses of the above theorem is necessary in order for the conclusion to hold. This is illustrated by the following example and some of the exercises.

Example 3.2.2. Give examples of functions on $[0, 1]$ which are

- (1) unbounded;
- (2) bounded, but with no maximum value.

Solution: (1) Let

$$f(x) = \begin{cases} 1 & \text{if } x \leq 1/2 \\ \frac{1}{2x-1} & \text{if } x > 1/2. \end{cases};$$

this function is clearly unbounded on $[0, 1]$ since it blows up as x approaches $1/2$ from the right. Note that f is not continuous at $1/2$.

(2) Let

$$f(x) = \begin{cases} 2x & \text{if } x < 1/2 \\ 0 & \text{if } x \geq 1/2. \end{cases};$$

this function is bounded on $[0, 1]$ and its sup on this interval is 1, but it never takes on the value 1 on the interval. Again, this function is not continuous at $1/2$.

Exercises 3.2.4 and 3.2.5 ask the student to come up with examples showing that the conclusion of the theorem fails for a function which is continuous on an interval I , but I is not closed or is not bounded.

Intermediate Value Theorem

The next theorem says that if a continuous function on an interval takes on two values, then it takes on every value in between. Its proof is almost identical to the proof of the previous theorem.

Theorem 3.2.3. (Intermediate Value Theorem) *Let f be defined and continuous on an interval containing the points a and b and assume that $a < b$. If y is any number between $f(a)$ and $f(b)$, then there is a number c with $a \leq c \leq b$ such that $f(c) = y$.*

Proof. Let $a_1 = a$ and $b_1 = b$ and consider the closed interval $I_1 = [a_1, b_1]$. We are given that y lies between $f(a_1)$ and $f(b_1)$. We will construct a nested sequence of closed intervals with the same property. That is, we will prove by induction that there is a sequence of closed intervals $\{I_k = [a_k, b_k]\}$ such that, for all $k > 1$,

- (1) $[a_k, b_k]$ is the closed left or right half of the interval $[a_{k-1}, b_{k-1}]$;
- (2) y lies between $f(a_k)$ and $f(b_k)$.

Suppose it is possible to choose $\{I_1, \dots, I_n\}$ so that (1) and (2) hold for $k \leq n$. Then we cut I_n into two halves that have only the midpoint c_n of I_n in common. If y lies between $f(a_n)$ and $f(b_n)$ then it either lies between $f(a_n)$ and $f(c_n)$ or it lies between $f(c_n)$ and $f(b_n)$. If only one of these is true, then choose I_{n+1} to be the corresponding half of I_n . If both are true, then choose I_{n+1} to be the right half of I_n . This results in a choice for I_{n+1} that satisfies (1) and (2) for $k = n + 1$. This completes the induction step of the construction and, hence, the proof that a nested sequence of intervals satisfying (1) and (2) can be constructed.

By the Nested Interval Property, there is a point c in the intersection of all the intervals I_n . By hypothesis f is continuous at c and so, given $\epsilon > 0$, there is a $\delta > 0$ such that

$$f(c) - \epsilon < f(x) < f(c) + \epsilon \quad \text{whenever} \quad x \in I, \quad |x - c| < \delta. \quad (3.2.2)$$

Now the length of I_n is $L/2^{n-1}$, where L is the length of I . Since $\lim L/2^{n-1} = 2L \lim (1/2)^n = 0$, the length of I_n will be less than δ for n sufficiently large. Suppose n is this large. Then $|x - c| < \delta$ for all $x \in I_n$, since $c \in I_n$. By (3.2.2)

$$f(c) - \epsilon < f(a_n) < f(c) + \epsilon \quad \text{and} \quad f(c) - \epsilon < f(b_n) < f(c) + \epsilon.$$

Taken together with the fact that y lies between $f(a_n)$ and $f(b_n)$, these inequalities imply that

$$f(c) - \epsilon < y < f(c) + \epsilon \quad \text{or} \quad |f(c) - y| < \epsilon.$$

This is only possible for all positive ϵ if $f(c) = y$. This completes the proof. \square

This is another example of a theorem which is not true if the function is not required to be continuous (see Exercise 3.2.6).

Image of an Interval

Theorem 3.2.4. *Suppose f is a continuous function defined on a closed bounded interval $I = [a, b]$. Then $f(I)$ is also a closed, bounded interval or it is a single point.*

Proof. By Theorem 3.2.1, f has a maximum value M and a minimum value m on I . By Theorem 3.2.3 f takes on every value between m and M on I . Therefore the image of I is exactly $[m, M]$. This is a closed interval if $m \neq M$, and is a point otherwise. \square

Inverse Functions

We learn in calculus that a function which is monotone increasing or monotone decreasing on an interval has an inverse function. Here a function f is *monotone increasing* on I if $f(x) < f(y)$ whenever $x, y \in I$ and $x < y$. A function f is *monotone decreasing* on I if $f(x) > f(y)$ whenever $x, y \in I$ and $x < y$. A function which is monotone increasing or monotone decreasing on I is said to be *strictly monotone* on I . For strictly monotone functions, there is a converse to the previous theorem.

Theorem 3.2.5. *If f is strictly monotone on I and its range $f(I)$ is an interval, then f is continuous on I .*

Proof. Suppose f is monotone increasing. Let $f(I) = [s, t]$. Given $c \in I$, we will prove that f is continuous at c . We do this first in the case where c is not an endpoint of $I = [a, b]$.

Given $\epsilon > 0$, let $u = \max\{s, f(c) - \epsilon\}$ and $v = \min\{t, f(c) + \epsilon\}$. Then u and v are points of $[s, t]$ and

$$f(c) - \epsilon \leq u \leq f(c) \leq v \leq f(c) + \epsilon.$$

Note that the only way one of the inequalities $u \leq f(c) \leq v$ can be an equality is if $f(c)$ is one of the endpoints s or t . However, this cannot happen, since c is not an endpoint of I . Thus, $u < f(c) < v$.

Since $f(I) = [s, t]$, there are points $p, q \in I$ such that $f(p) = u$ and $f(q) = v$. Since f is monotone increasing,

$$p < c < q.$$

We choose $\delta = \min\{q - c, c - p\}$. Then $|x - c| < \delta$ implies $p < x < q$ and this implies

$$f(c) - \epsilon \leq u < f(x) < v \leq f(c) + \epsilon \quad \text{that is} \quad |f(x) - f(c)| < \epsilon.$$

This proves that f is continuous at c in the case where c is not an endpoint of I .

If c is an endpoint of I , then the argument is the same except that we only have to concern ourselves with points that lie to one side of c and of $f(c)$. The details are left to the exercises.

It remains to prove that a monotone *decreasing* function on I with a closed interval for its range is continuous. However, if g is monotone decreasing, then $f = -g$ is monotone increasing, also has a closed interval as image and, hence, is continuous by the above. But if $-g$ is continuous, then so is $g = (-1)(-g)$. \square

Theorem 3.2.6. *A continuous, strictly monotone function on a closed interval I has a continuous inverse function defined on $J = f(I)$. That is, there is a continuous function g , with domain J , such that $g(f(x)) = x$ for all $x \in I$ and $f(g(y)) = y$ for all $y \in J$.*

Proof. Since f is strictly monotone, for each $y \in J$ there is exactly one $x \in I$ such that $f(x) = y$. We set $g(y) = x$. Then, by the choice of x , we have $f(g(y)) = f(x) = y$ and $g(f(x)) = g(y) = x$.

The function g is strictly monotone because f is strictly monotone. Furthermore, the range of g is I . By the previous theorem, this implies that g is continuous. \square

Exercise Set 3.2

1. Find the maximum and minimum values of the function $f(x) = x^2 - 2x$ on the interval $[0, 3]$.
2. Prove that if f is a continuous function on a closed bounded interval I and if $f(x)$ is never 0 for $x \in I$, then there is a number $m > 0$ such that $f(x) \geq m$ for all $x \in I$ or $f(x) \leq -m$ for all $x \in I$.
3. Prove that if f is a continuous function on a closed bounded interval $[a, b]$ and if (x_0, y_0) is any point in the plane, then there is a closest point to (x_0, y_0) on the graph of f .
4. Find an example of a function which is continuous on a bounded (but not closed) interval I , but is not bounded. Then find an example of a function which is continuous and bounded on a bounded interval I , but does not have a maximum value.
5. Find an example of a function which is continuous on a closed (but not bounded) interval I , but is not bounded. Then find an example of a function which is continuous and bounded on a closed interval I , but does not have a maximum value.
6. Give an example of a function defined on the interval $[0, 1]$, which does not take on every value between $f(0)$ and $f(1)$.
7. Show that if f and g are continuous functions on the interval $[a, b]$ such that $f(a) < g(a)$ and $g(b) < f(b)$, then there is a number $c \in (a, b)$ such that $f(c) = g(c)$.
8. Let f be a continuous function from $[0, 1]$ to $[0, 1]$. Prove there is a point $c \in [0, 1]$ such that $f(c) = c$ – that is, show that f has a *fixed point*. Hint: apply the Intermediate Value Theorem to the function $g(x) = f(x) - x$.

9. Use the Intermediate Value Theorem to prove that, if n is a natural number, then every positive number a has a positive n th root.
10. Prove that a polynomial of odd degree has at least one real root.
11. Use the Intermediate Value Theorem to prove that if f is a continuous function on an interval $[a, b]$ and if $f(x) \leq m$ for every $x \in [a, b]$, then $f(b) \leq m$.
12. Prove that if f is strictly increasing on $[a, b]$, then its inverse function is strictly increasing on $[f(a), f(b)]$.

3.3 Uniform Continuity

Compare the definition of continuity given in Definition 3.1.1 with the following definition.

Definition 3.3.1. If f is a function with domain D , then f is said to be *uniformly continuous* on D if for each $\epsilon > 0$ there is a $\delta > 0$ such that

$$|f(x) - f(a)| < \epsilon \quad \text{whenever} \quad x, a \in D \text{ and } |x - a| < \delta. \quad (3.3.1)$$

By contrast, Definition 3.1.1 tells us that f is continuous on D if for each $a \in D$ and each $\epsilon > 0$ there is a $\delta > 0$ such that

$$|f(x) - f(a)| < \epsilon \quad \text{whenever} \quad x \in D \text{ and } |x - a| < \delta.$$

These two definitions appear to be identical until one examines them closely. The difference is subtle but extremely important. In the definition of uniform continuity, given ϵ , a single δ must be chosen that works for all points $a \in D$, while in the definition of continuity, δ is allowed to depend on a .

Example 3.3.2. Find a function which is continuous on its domain, but not uniformly continuous.

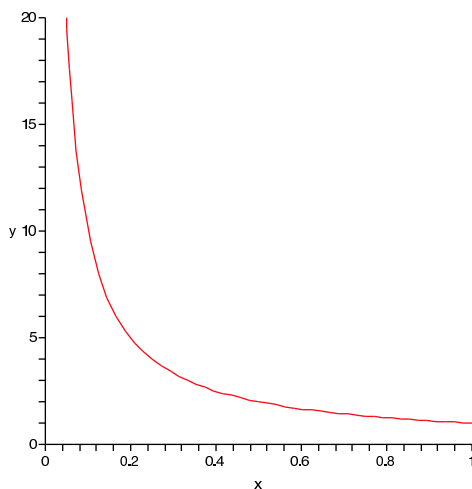
Solution: We claim that the function $f(x) = 1/x$ with domain $(0, 1]$ is continuous but not uniformly continuous on $(0, 1]$.

It is continuous because x is continuous on $(0, 1]$ and is never 0 on this set. Thus, Theorem 3.1.8(d) implies that $1/x$ is continuous at each point of $(0, 1]$.

On the other hand, if we attempt to verify that f is uniformly continuous, we run into trouble. Given $\epsilon > 0$, we try to find a $\delta > 0$ such that (3.3.1) holds. However, if δ is any positive number and x and a are chosen so that $0 < x < a < \delta$, then it will be true that

$$|x - a| < \delta.$$

However, we can make $1/x$ and, hence, $|1/x - 1/a|$ as large as we want by simply keeping $a < \delta$ fixed and choosing $x < a$ small enough. In particular, $|1/x - 1/a|$ can be made larger than ϵ regardless of what ϵ we start with. Thus, $f(x) = 1/x$ is not uniformly continuous on $(0, 1]$

Figure 3.2: The Function $1/x$ on $(0, 1]$.

Example 3.3.3. Prove that $f(x) = 1/x$ is uniformly continuous on any interval of the form $[r, 1]$, where $r > 0$.

Solution: If x and a are in the interval $[r, 1]$, then

$$\left| \frac{1}{x} - \frac{1}{a} \right| = \frac{|x - a|}{ax} \leq \frac{|x - a|}{r^2}.$$

Thus, given $\epsilon > 0$, if we choose $\delta = r^2\epsilon$, then

$$\left| \frac{1}{x} - \frac{1}{a} \right| < \epsilon \quad \text{whenever} \quad |x - a| < \delta.$$

This implies that $f(x) = 1/x$ is uniformly continuous on $[r, 1]$.

Conditions Ensuring Uniform Continuity

In the last example, the domain of the function f was a closed, bounded interval. It turns out that, in this case, continuity implies uniform continuity. This is the main theorem of this section.

Theorem 3.3.4. *If f is a continuous function on a closed, bounded interval I , then f is uniformly continuous on I .*

Proof. We will prove the contrapositive. Suppose f is not uniformly continuous on $[a, b]$. Then there is an $\epsilon > 0$ for which no δ can be found which satisfies (3.3.1). In particular, none of the numbers $1/n$ for $n \in \mathbb{N}$ will suffice for δ . This means that, for each n , there are numbers $x_n, a_n \in I$ such that

$$|x_n - a_n| < 1/n \quad \text{but} \quad |f(x_n) - f(a_n)| \geq \epsilon.$$

By the Bolzano-Weierstrass Theorem, some subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ converges to a point x of I . The inequality $|x_{n_k} - a_{n_k}| < 1/n_k \leq 1/k$ implies that $\{a_{n_k}\}$ converges to the same number. Since $|f(x_{n_k}) - f(a_{n_k})| \geq \epsilon$, the sequences $\{f(x_{n_k})\}$ and $\{f(a_{n_k})\}$ cannot converge to the same number. However, they would both have to converge to $f(x)$ if f were continuous at x , by Theorem 3.1.5. Thus, we conclude that f is not continuous at every point of I . \square

Consequences of Uniform Continuity

Theorem 3.3.5. *If f is uniformly continuous on its domain D , and if $\{x_n\}$ is any Cauchy sequence in D , then $\{f(x_n)\}$ is also a Cauchy sequence.*

Proof. Given $\epsilon > 0$, by uniform continuity there is a $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon \quad \text{whenever} \quad x, y \in D \text{ and } |x - y| < \delta.$$

Since $\{x_n\}$ is Cauchy, there is an N such that

$$|x_n - x_m| < \delta \quad \text{whenever} \quad n, m > N.$$

Combining these two statements tells us that

$$|f(x_n) - f(x_m)| < \epsilon \quad \text{whenever} \quad n, m > N.$$

Thus, $\{f(x_n)\}$ is a Cauchy sequence. \square

An interval may be closed, open or half open. If I is an interval, we denote by \bar{I} the closed interval consisting of I along with any endpoints of I that may be missing from I . If I is a bounded interval, then \bar{I} is a closed, bounded interval.

Given a continuous function f on a bounded interval I that is not closed, it may or may not be possible to extend f to a continuous function on \bar{I} . That is, it may or may not be possible to give f values at the missing endpoint(s) that make the new function continuous. The next theorem tells when this can be done.

Theorem 3.3.6. *If f is a continuous function on a bounded interval I , which may not be closed, then f has a continuous extension to \bar{I} if and only if f is uniformly continuous on I .*

Proof. If f has a continuous extension \tilde{f} to \bar{I} , then \tilde{f} is uniformly continuous on \bar{I} by Theorem 3.3.4. But if a function is uniformly continuous on a set, then it is also uniformly continuous when restricted to any smaller set. Since f is just \tilde{f} restricted to the smaller domain I , f is uniformly continuous on I .

Conversely, suppose f is uniformly continuous on I . Let a be a missing endpoint of I (left or right). There are lots of sequences in I which converge to a . Let $\{a_n\}$ be one of these. Then $\{a_n\}$ is a Cauchy sequence in I and so the previous theorem implies that $\{f(a_n)\}$ is also a Cauchy sequence. Since Cauchy sequences converge, we know that there is a y such that $f(a_n) \rightarrow y$.

We claim that if $\{b_n\}$ is any other sequence in I converging to a , then $\{f(b_n)\}$ converges to the same number y . We prove this by constructing a new sequence $\{c_n\}$ in I , which also converges to a , by interlacing the terms of $\{a_n\}$ and $\{b_n\}$. That is, we set

$$\begin{aligned}c_{2k-1} &= a_k; \\ c_{2k} &= b_k.\end{aligned}$$

Since $c_n \rightarrow a$, we may argue as before, that $\{f(c_n)\}$ converges to some number. But one of its subsequences, $\{f(c_{2k-1})\}$, converges to y . This implies that $\{f(c_n)\}$ must converge to y as must any of its subsequences. In particular $\{c_{2k}\} = \{b_k\}$ converges to y . This proves our claim. That is, the number $y = \lim f(a_n)$ is the same no matter what sequence $\{a_n\}$ in I converging to a is chosen.

We now define a new function \tilde{f} on $I \cup \{a\}$, by setting $\tilde{f}(a) = y$ and $\tilde{f}(x) = f(x)$ for each $x \in I$. It is clear from the construction that \tilde{f} will be continuous at a , since $\tilde{f}(x_n) \rightarrow y = f(a)$ for every sequence $\{x_n\}$ in $I \cup \{a\}$ that converges to a .

This proves that a uniformly continuous function on a bounded interval I can be extended to be continuous on the interval obtained by adjoining one missing endpoint to I . If the other endpoint is also missing, we simply repeat the process to get an extension to all of \bar{I} . \square

This theorem often provides a quick way to see that a function on a bounded interval is not uniformly continuous.

Example 3.3.7. Show that the function $f(x) = \frac{1}{1-x^2}$ is not uniformly continuous on the interval $(-1, 1)$.

Solution: If f is uniformly continuous on this interval, then the previous theorem implies that f has a continuous extension to $[-1, 1]$. However, a continuous function on a closed bounded interval is bounded. The function f is not bounded on $(-1, 1)$, and so no extension of it to $[-1, 1]$ can be bounded. Thus, f is not uniformly continuous.

If the interval I is unbounded, then it is possible for a function on I to be uniformly continuous and yet unbounded.

Example 3.3.8. Show that the function $f(x) = \sqrt{x}$ is uniformly continuous on $[1, +\infty)$.

Solution: If $x, y \in [1, +\infty)$, then

$$|\sqrt{x} - \sqrt{y}| = \frac{|x - y|}{\sqrt{x} + \sqrt{y}} < |x - y|,$$

since $\sqrt{x} \geq 1 > 1/2$ and $\sqrt{y} \geq 1 > 1/2$ if $x, y \in [1, +\infty)$. This clearly implies that f is uniformly continuous on $[1, +\infty)$. In fact, given $\epsilon > 0$, it suffices to choose $\delta = \epsilon$ to obtain

$$|f(x) - f(y)| < \epsilon \quad \text{whenever} \quad x, y \in [1, +\infty) \text{ and } |x - y| < \delta.$$

Exercise Set 3.3

1. Is the function $f(x) = x^2$ uniformly continuous on $(0, 1)$? Justify your answer.
2. Is the function $f(x) = 1/x^2$ uniformly continuous on $(0, 1)$? Justify your answer.
3. Is the function $f(x) = x^2$ uniformly continuous on $(0, +\infty)$? Justify your answer.
4. Using only the ϵ - δ definition of uniform continuity, prove that the function $f(x) = \frac{x}{x+1}$ is uniformly continuous on $[0, \infty)$.
5. In Example 3.3.8 we showed that \sqrt{x} is uniformly continuous on $[1, +\infty)$. Show that it is also uniformly continuous on $[0, 1]$.
6. Prove that if I and J are overlapping intervals in \mathbb{R} ($I \cap J \neq \emptyset$) and f is a function, defined on $I \cup J$, which is uniformly continuous on I and uniformly continuous on J , then it is also uniformly continuous on $I \cup J$. Use this and the previous exercise to prove that \sqrt{x} is uniformly continuous on $[0, +\infty)$.
7. Prove that if I is a bounded interval and f is an unbounded function defined on I , then f cannot be uniformly continuous.
8. Let f be a function defined on an interval I and suppose that there are positive constants K and r such that

$$|f(x) - f(y)| \leq K|x - y|^r \quad \text{for all } x, y \in I.$$

Prove that f is uniformly continuous.

9. Is the function $f(x) = \sin 1/x$ continuous on $(0, 1)$? Is it uniformly continuous on $(0, 1)$. Justify your answers.
10. Is the function $f(x) = x \sin 1/x$ uniformly continuous on $(0, 1)$? Justify your answer.

3.4 Uniform Convergence

Uniform convergence is a subject that is both similar to and very different from uniform continuity. Uniform continuity is a condition on the continuity of a single function, while uniform convergence is a condition on the convergence of a sequence of functions.

Sequences of Functions

In calculus we often encounter sequences of functions as opposed to sequences of numbers. They occur as partial sums of power series, for example. Other examples are the following (note that x is a variable):

1. $\{x/n\}$, $x \in \mathbb{R}$;
2. $\{x^n\}$, $x \in \mathbb{R}$;
3. $\left\{\frac{1}{1+nx}\right\}$, $x > 0$;
4. $\left\{\frac{1-x^n}{1-x}\right\}$, $x \in (-1, 1)$;
5. $\{\sin nx\}$, $x \in [0, 2\pi)$.

It is important to have methods to show that various things are preserved by passing to the limit of a sequence of functions. If the functions in the sequence are all continuous on a certain set D , is the limit continuous on D ? Is the integral of the limit equal to the limit of the integrals if we are integrating over some interval on which all the functions are defined? The answer to both of these questions is “yes” provided the convergence is *uniform*.

Uniform Convergence

Let $\{f_n\}$ be a sequence of functions on a set $D \subset \mathbb{R}$. We say that $\{f_n\}$ converges pointwise to a function f on D if, for each $x \in D$, the sequence of numbers $\{f_n(x)\}$ converges to the number $f(x)$. If we write out what this means in terms of the definition of convergence of a sequence of numbers we get the statement in (a) of the following definition. Statement (b) is the definition of uniform convergence.

Definition 3.4.1. Let $\{f_n\}$ be a sequence of functions on a set $D \subset \mathbb{R}$. Then

- (a) $\{f_n\}$ is said to converge *pointwise* to a function f on D if, for each $x \in D$ and each $\epsilon > 0$, there is an N such that

$$|f(x) - f_n(x)| < \epsilon \quad \text{whenever } n > N.$$

- (b) $\{f_n\}$ is said to converge *uniformly* on D to a function f if, for each $\epsilon > 0$, there is an N such that

$$|f(x) - f_n(x)| < \epsilon \quad \text{whenever } x \in D \text{ and } n > N;$$

As with continuity and uniform continuity, the definitions of pointwise convergence and uniform convergence seem identical until one studies them closely. In fact, they are very different. In the case of pointwise convergence, x is given along with ϵ before N is chosen. Here N may well depend on both ϵ and x . In the case of uniform convergence, only ϵ is given initially; then an N must be chosen which works for all x . That is, N does not depend on x in this case.

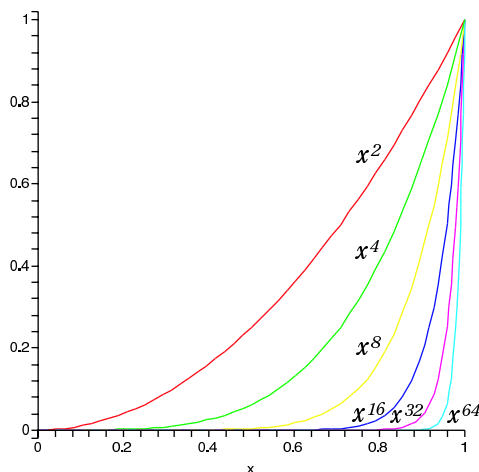


Figure 3.3: The Sequence $\{x^n\}$ does not Converge Uniformly on $[0, 1]$.

Example 3.4.2. Give an example of a sequence of functions defined on $[0, 1]$ which converges pointwise on $[0, 1]$ but not uniformly.

Solution: An example is the sequence $\{f_n\}$ on $[0, 1]$ defined by $f_n(x) = x^n$, which is illustrated in Figure 3.3. This sequence of functions converges to the function f which is 0 if $x < 1$ and 1 if $x = 1$. Since the sequence $\{f_n(x)\}$ converges to $f(x)$ for each value of x , the sequence $\{f_n\}$ converges pointwise to f on $[0, 1]$. However, the convergence is not uniform on $[0, 1]$. In fact,

$$|f_n(x) - f(x)| = x^n \quad \text{if } x \in [0, 1),$$

and so, given $\epsilon > 0$, in order for it to be true that $|f_n(x) - f(x)| < \epsilon$ for all $x \in [0, 1]$ and some n , we would need that

$$x^n < \epsilon \quad \text{for all } x \in [0, 1).$$

However, since x^n is continuous on $[0, 1]$, this would imply that $1 = 1^n \leq \epsilon$ (Exercise 3.3.11). Obviously, there are positive numbers ϵ for which this is not true (any positive $\epsilon < 1$). This shows that the convergence of $\{f_n\}$ on $[0, 1]$ is not uniform.

The problem in the above example is due to what is happening near $x = 1$. If we stay away from 1, the situation improves.

Example 3.4.3. If $0 < r < 1$, prove that the sequence $\{f_n\}$, defined by $f_n(x) = x^n$, converges uniformly to 0 on $[0, r]$.

Solution: We have

$$|x^n - 0| = x^n \leq r^n \quad \text{for all } x \in [0, r]. \quad (3.4.1)$$

Now, given $\epsilon > 0$, we choose N so that

$$r^n < \epsilon \quad \text{whenever} \quad n > N,$$

This is possible because $r^n \rightarrow 0$ if $0 \leq r < 1$. Combining this with (3.4.1) yields

$$|x^n - 0| < \epsilon \quad \text{whenever} \quad x \in [0, r] \text{ and } n > N.$$

This proves that $\{x^n\}$ converges uniformly to 0 on $[0, r]$.

Uniform Convergence and Continuity

Theorem 3.4.4. *Let $\{f_n\}$ be a sequence of functions, all of which are defined and continuous on a set D . If $\{f_n\}$ converges uniformly to a function f on D , then f is continuous on D .*

Proof. If $a \in D$, we will show that f is continuous at a . Given $\epsilon > 0$, we first use the uniform convergence to choose an N such that

$$|f_n(x) - f(x)| < \epsilon/3 \quad \text{whenever} \quad x \in D, n > N.$$

We then fix a natural number $n > N$ and use the fact that each f_n is continuous at a to choose a $\delta > 0$ such that

$$|f_n(x) - f_n(a)| < \epsilon/3 \quad \text{whenever} \quad x \in D \text{ and } |x - a| < \delta.$$

On combining these and using the triangle inequality, we conclude that

$$\begin{aligned} |f(x) - f(a)| &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(a)| + |f_n(a) - f(a)| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon, \end{aligned}$$

whenever $x \in D$ and $|x - a| < \delta$. This proves that f is continuous at a . Since a was an arbitrary point of D , f is continuous on D . \square

Example 3.4.5. Analyze the convergence of the sequence of functions $\{f_n\}$ defined on $[0, \infty)$ by

$$f_n(x) = \frac{1}{1 + nx}$$

Does the sequence converge pointwise? Does it converge uniformly?

Solution: Since $f_n(0) = 1$ for all n , the sequence $\{f_n(x)\}$ converges to 1 at $x = 0$. Since each f_n can be re-written as

$$f_n(x) = \frac{1/n}{1/n + x},$$

and the denominator of this expression converges to x , the sequence $\{f_n(x)\}$ converges to 0 if $x \neq 0$. Thus, $\{f_n(x)\}$ converges pointwise to the function f on $[0, \infty)$ defined by $f(x) = 0$ if $x > 0$ and $f(0) = 1$.

It follows from the previous theorem that the convergence is not uniform, because f is not continuous on $[0, \infty)$ although each of the functions f_n is continuous on this interval.

Tests For Uniform Convergence

A sequence $\{f_n\}$ converges uniformly to f on a set D if and only if $\{|f_n - f|\}$ converges uniformly to 0 on D . Thus, it is useful to have simple tests for when a sequence converges uniformly to 0. We will give two such tests. One gives conditions which guarantee that a sequence converges uniformly to 0 and the other gives a condition, which if not true, guarantees that a sequence does not converge uniformly to 0. Both theorems have very simple proofs which are left to the exercises.

The following theorem is useful for showing that a sequence converges uniformly.

Theorem 3.4.6. *Let $\{f_n\}$ be a sequence of functions defined on a set D . If there is a sequence of numbers b_n , such that $b_n \rightarrow 0$, and*

$$|f_n(x)| \leq b_n \quad \text{for all } x \in D,$$

then $\{f_n\}$ converges uniformly to 0 on D .

The following theorem provides a useful test for proving a sequence does not converge uniformly.

Theorem 3.4.7. *Let $\{f_n\}$ be a sequence of functions defined on a set D . If $\{f_n\}$ converges uniformly to 0 on D , then $\{f_n(x_n)\}$ converges to 0 for every sequence $\{x_n\}$ of points of D .*

Example 3.4.8. If $f_n(x) = \frac{n}{x+n}$, prove that $\{f_n\}$ converges uniformly to 1 on the interval $[0, r]$ for each positive number r , but does not converge uniformly on $[0, \infty)$.

Solution: We have

$$|f_n(x) - 1| = \frac{x}{x+n} \leq \frac{x}{n} \leq \frac{r}{n},$$

if $x \in [0, r]$. Since $r/n \rightarrow 0$, Theorem 3.4.6 implies that $\frac{x}{x+n}$ converges uniformly to 0 on $[0, r]$ and, hence, that $\{f_n\}$ converges uniformly to 1 on $[0, r]$.

On the other hand if we set $x_n = n$, then $\{x_n\}$ is a sequence of numbers in $[0, \infty)$ and $f_n(x_n) = 1/2$. Since $f_n(x_n) - 1$ does not converge to 0, Theorem 3.4.7 implies that $\{f_n - 1\}$ does not converge uniformly to 0 on $[0, \infty)$ and, hence, that $\{f_n\}$ does not converge uniformly to 1 on $[0, \infty)$.

Uniformly Cauchy Sequences

Definition 3.4.9. A sequence of functions $\{f_n\}$ on a set D is said to be *uniformly Cauchy* on D if for each $\epsilon > 0$, there is an N such that

$$|f_n(x) - f_m(x)| < \epsilon \quad \text{whenever } x \in D \text{ and } n, m > N.$$

If $\{f_n\}$ is a uniformly Cauchy sequence, then $\{f_n(x)\}$ is a Cauchy sequence for each $x \in D$. By Theorem 2.5.7, $\{f_n(x)\}$ converges. Thus, $\{f_n\}$ converges pointwise to some function f on D . The next theorem tells us that the convergence is uniform. Its proof is left to the exercises.

Theorem 3.4.10. *A sequence of functions $\{f_n\}$ on D is uniformly convergent on D if and only if it is uniformly Cauchy on D .*

Exercise Set 3.4

1. Prove that the sequence $\{x/n\}$ converges uniformly to 0 on each bounded interval, but does not converge uniformly on \mathbb{R} .
2. Prove that the sequence $\frac{1}{x^2 + n}$ converges uniformly to 0 on \mathbb{R} .
3. Prove that the sequence $\{\sin(x/n)\}$ converges to 0 pointwise on \mathbb{R} , but it does not converge uniformly on \mathbb{R} .
4. Prove that the sequence $\frac{\sin nx}{n}$ converges uniformly to 0 on $[0, 1]$.
5. Prove that $\{x^n(1-x)\}$ converges uniformly to 0 on $[0, 1]$. Hint: find where each of these functions has its maximum on $[0, 1]$.
6. Prove Theorem 3.4.6.
7. Prove Theorem 3.4.7.
8. Prove that if $\{f_n\}$ is a sequence of uniformly continuous functions on a set D and if this sequence converges uniformly to f on D , then f is also uniformly continuous.
9. For $x \in (-1, 1)$ set $s_n(x) = \sum_{k=0}^n x^k$. This is the n th partial sum of a geometric series. Prove that $s_n(x) = \frac{1 - x^{n+1}}{1 - x}$.
10. Prove that the sequence $\{s_n\}$ of the previous exercise converges uniformly to $\frac{1}{1-x}$ on each interval of the form $[-r, r]$ with $r < 1$, but it does not converge uniformly on $(-1, 1)$.
11. Prove Theorem 3.4.10. Hint: use an argument like the one in the proof of Theorem 2.5.7.
12. Prove that if $\{a_k\}$ is a bounded sequence of numbers and a sequence $\{s_n\}$ is defined on $(-1, 1)$ by

$$s_n(x) = \sum_{k=0}^n a_k x^k,$$

then $\{s_n\}$ converges to a continuous function on $(-1, 1)$. Hint: prove this sequence is uniformly Cauchy on each interval $[-r, r]$ for $0 < r < 1$.