# Chapter 2

# Analytic Functions

Complex Variables is the study of functions of a complex variable – specifically, analytic functions of a complex variable. These are functions which have a complex derivative in a sense that we shall define shortly. Analytic functions have amazing properties, and deriving these properties will be the main focus of the text. The amazing properties of analytic functions stem from the fact that the existence of a complex derivative is a much more powerful condition than the existence of the real derivative for a function of a real variable.

We have already seen several examples of analytic functions. The exponential function and the sine and cosine functions are analytic functions on the entire plane. The log and $n$th root functions are analytic except on the cut line for the branch of the log function that is used in their definitions.

We must prove one difficult theorem about analytic functions (the Cauchy Integral Theorem), and then a wealth of amazing theorems with simple proofs will follow.

We begin this chapter with a review of the basic facts about continuous functions in the plane. We then define analytic functions and prove some elementary theorems concerning them. Next we introduce contour integration and prepare for our assault on Cauchy's Theorem. Finally, in sections 2.5 and 2.6 we prove two limited versions of this theorem. This provides us enough of the power of Cauchy's Theorem to allow us to move on to Chapter 3, where we develop a wide range of applications. However, we put off proving the most general version of Cauchy's Theorem until Chapter 4.

## 2.1 Continuous Functions

Since, geometrically, $\mathbb{C}$ is just $\mathbb{R}^2$, notions of distance, convergence of sequences, and continuity of functions are the same for $\mathbb{C}$ as for $\mathbb{R}^2$. In particular, a complex valued function of a complex variable may be regarded as a function of two real variables with values in $\mathbb{R}^2$. The student who has had a course in multivariable calculus already knows what it means for such a function to be continuous.

Still, we will review the basics of continuity in this context, partly in order to set terminology and notation that is peculiar to the complex variables context.

## Open and Closed Sets

Recall that the open disc $D_r(z_0)$ and closed disc $\overline{D}_r(z_0)$, centered at $z_0$, with radius $r > 0$, are defined by

$$D_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\} \quad \text{and} \quad \overline{D}_r(z_0) = \{z \in \mathbb{C} : |z - z_0| \leq r\}.$$

Open intervals and closed intervals on the real line play an important part in calculus in one variable. Open and closed discs are the direct analogues in $\mathbb{C}$ of open and closed intervals on the line. However, the geometry of the plane is much more complicated than that of the line. We will need the concepts of open and closed for sets that are far more complicated than discs. This leads to the following definition.

**Definition 2.1.1.** If $W$ is a subset of $\mathbb{C}$, we will say that $W$ is *open* if, for each point $w \in W$, there is an open disc centered at $w$ which is contained in $W$. We will say that a subset of $\mathbb{C}$ is *closed* if its complement is open. A *neighborhood* of a point $z \in \mathbb{C}$ is any open set which contains $z$.

It might seem obvious that open discs are open sets and closed discs are closed sets. However, that is only because we have chosen to call them *open* discs and *closed* discs. We actually have to prove that they satisfy the conditions of the preceding definition. We do this in the next theorem.

**Theorem 2.1.2.** *We have:*

*(a) the empty set $\emptyset$ is both open and closed;*

*(b) the whole space $\mathbb{C}$ is both open and closed;*

*(c) each open disc is open;*

*(d) each closed disc is closed.*

*Proof.* The empty set $\emptyset$ is open because it has no points, and so the condition that a set be open, stated in Definition 2.1.1, is vacuously satisfied. The set $\mathbb{C}$ is open because it contains any open disc centered at any of its points. Thus, $\emptyset$ and $\mathbb{C}$ are both open. Since they are complements of one another, they are also both closed.

To prove (c), we suppose $D_r(z_0)$ is an open disc and $w$ is one of its points. Then $|w - z_0| < r$ and so, if we set $s = r - |w - z_0|$, then $s > 0$. Also, if $z \in D_s(w)$, then $|z - w| < s$ and so

$$|z - z_0| \leq |z - w| + |w - z_0| < s + |w - z_0| = r,$$

and so $z \in D_r(z_0)$ (see Figure 2.1). Thus, we have shown that, for each $w \in D_r(z_0)$, there is an open disc, $D_s(w)$, centered at $w$, which is contained in

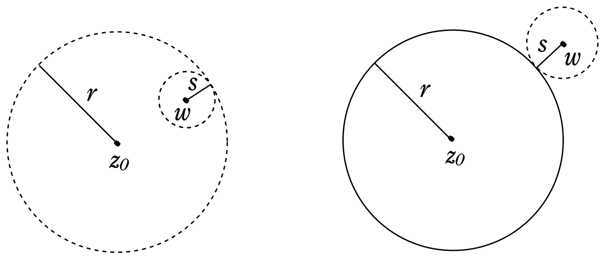


Figure 2.1: Proving Theorem 2.1.2 (c) and (d).

$D_r(z_0)$. By definition, this means that $D_r(z_0)$ is open. This completes the proof of (c).

To prove (d), we consider a closed disc $\overline{D}_r(z_0)$. To prove that it is a closed set, we must show its complement is open. Suppose $w$ is a point in its complement. This means $w \in \mathbb{C}$ but $w \notin \overline{D}_r(z_0)$, and so $|w - z_0| > r$. This time we set $s = |w - z_0| - r$ and we claim that the open disc $D_s(w)$ is contained in the complement of $\overline{D}_r(z_0)$. In fact, if $z \in D_s(w)$, then $|z - w| < s$ and so, by the second form of the triangle inequality (Exercise 1.2.2)

$$|z - z_0| \geq |w - z_0| - |z - w| > |w - z_0| - s = r$$

which means $z$ is in the complement of $\overline{D}_r(z_0)$. Thus, we have proved that each point of the complement of $D_r(z_0)$ is the center of an open disc contained in the complement of $\overline{D}_r(z_0)$. This proves that this complement is open, hence, that $\overline{D}_r(z_0)$ is closed. □

The next theorem shows that the collection of all open subsets of $\mathbb{C}$ forms what is called a *topology* for $\mathbb{C}$. It says that the collection of open subsets is closed under arbitrary unions and finite intersections.

**Theorem 2.1.3.** *The union of an arbitrary collection of open sets is open, while the intersection of any finite collection of open sets is open. On the other hand, the intersection of an arbitrary collection of closed sets is closed, while the union of any finite collection of closed sets is closed.*

*Proof.* If $z_0 \in \mathbb{C}$, $\mathcal{V}$ is an arbitrary collection of open sets, and $U = \bigcup \mathcal{V}$ is its union, then $z_0$ is in $U$ if and only if it is in at least one of the sets in $\mathcal{V}$. Suppose, it is in $V \in \mathcal{V}$. Then, since $V$ is open, there is a disc $D_r(z_0)$, centered at $z_0$, which is contained in $V$. Then this disc is also contained in $U$. This proves that $U$ is open.

Now suppose $\{V_1, V_2, \cdots, V_k\}$ is a finite collection of open sets and

$$z_0 \in U = V_1 \cap V_2 \cap \cdots \cap V_n.$$

Then, since each $V_k$ is open, there exists for each $k$ a radius $r_k$ such that $D_{r_k}(z_0) \subset V_k$. If $r = \min\{r_1, r_2, \cdots, r_n\}$, then $D_r(z_0) \subset V_k$ for every $k$, which implies that $D_r(z_0) \subset U$. It follows that $U$ is open.

The proofs of the statements for closed sets follow from those for open sets by taking complements. We leave the details to Exercise 2.1.1. □

An easy consequence of the above theorem is that if $U$ is open and $K$ is closed, then the set theoretic difference $U \setminus K$ is open. Similarly, $K \setminus U$ is closed (Exercise 2.1.2).

**Example 2.1.4.** If $0 < r < R$, prove that the annulus

$$A = \{z \in \mathbb{C} : r < |z| < R\},$$

is open.

**Solution:** The disc $D_R(0)$ is open, the disc $\overline{D}_r(0)$ is closed, and $A$ is the set theoretic difference $D_R(0) \setminus \overline{D}_r(0)$. Thus, by the previous remark, $A$ is open.

## Interior, Closure, and Boundary

If $E$ is a subset of $\mathbb{C}$, then $E$ contains a largest open subset, meaning an open subset of $E$ that contains all other open subsets of $E$. In fact, the union of all open subsets of $E$ is open, by Theorem 2.1.3, and is a subset of $E$ which contains all open subsets of $E$. Similarly, the intersection of all closed sets containing $E$ is the smallest closed set containing $E$. Thus, the following definition makes sense.

**Definition 2.1.5.** Let $E$ be a subset of $\mathbb{C}$. Then:

(a) the largest open subset of $E$ is called the *interior* of $E$ and is denoted $E^\circ$;

(b) the smallest closed set containing $E$ is called the *closure* of $E$ and is denoted $\overline{E}$;

(c) the set $\overline{E} \setminus E^\circ$ is called the boundary of $E$ and is denoted $\partial E$.

Recall that a neighborhood of a point $z_0 \in \mathbb{C}$ is any open set containing $z_0$. The proof of the following theorem is elementary and is left to the exercises.

**Theorem 2.1.6.** *Let $E$ be a subset of $\mathbb{C}$ and $z$ an element of $\mathbb{C}$. Then:*

*(a) $z \in E^\circ$ if and only if there is a neighborhood of $z$ that is contained in $E$;*

*(b) $z \in \overline{E}$ if and only if every neighborhood of $z$ contains a point of $E$;*

*(c) $z \in \partial E$ if and only if every neighborhood of $z$ contains points of $E$ and points of the complement of $E$.*

**Example 2.1.7.** Find the interior, closure and boundary for the set

$$E = \{z \in \mathbb{C} : |z| < 1,\ \mathrm{Im}\,(z) \geq 0\} \cup \{-iy : y \in [0, 1]\}.$$

**Solution:** It is immediate from the previous theorem that

$$\begin{aligned} E^\circ &= \{z \in \mathbb{C} : |z| < 1,\ \mathrm{Im}\,(z) > 0, \} \\ \overline{E} &= \{z \in \mathbb{C} : |z| \leq 1,\ \mathrm{Im}\,(z) \geq 0\} \cup \{-iy : y \in [0, 1]\}, \\ \partial E &= \{z \in \mathbb{C} : |z| = 1,\ \mathrm{Im}\,(z) \geq 0\} \cup [-1, 1] \cup \{-iy : y \in [0, 1]\}. \end{aligned}$$

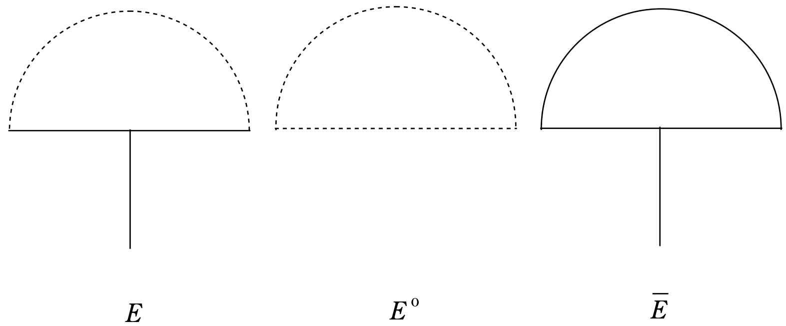


Figure 2.2: The Set $E$ of Example 2.1.7, its Interior $E^o$, and Closure $\overline{E}$.

## Limits

We will be primarily concerned with complex valued functions defined on open subsets of $\mathbb{C}$. However, we will also need to deal with functions whose domain is some other kind of subset of $\mathbb{C}$. For example, in the next section we will be dealing with curves in $\mathbb{C}$. A curve in $\mathbb{C}$ is a continuous complex valued function whose domain is an interval on the real line. We will also deal eventually with functions defined on a closed disc or a circle or some other closed subset of $\mathbb{C}$. The definitions of limit and continuity we adopt here are general enough to handle all these situations. However, as a result, they are very much domain dependent. That is, the limit of a function at a point and whether or not the function is continuous at a point depend very much on which set is considered to be the domain of the function.

The definition of limit is the familiar one from calculus except for extra care about the domain of the function and a condition concerning isolated points, as defined below.

If $E$ is a subset of $\mathbb{C}$, then an *isolated point* of $E$ is an element $a \in E$ such that there is a neighborhood of $a$ which contains no other points of $E$.

**Definition 2.1.8.** If $f$ is a complex valued function with domain $E$, and $a \in \overline{E}$ but is not an isolated point of $\overline{E}$, then we say $\lim_{z \to a} f(z) = L$ if, for every $\epsilon > 0$, there is a $\delta > 0$ such that $0 < |z - a| < \delta$ and $z \in E$ imply $|f(z) - L| < \epsilon$.

The condition "$z \in E$" in this definition means that whether or not the limit exists may depend on $E$. That is, given a function $f$ defined on a set $E$ and a subset $D$ of $E$, we can always consider a new function $f|_D$ which is $f$ restricted to the new domain $D$. It may be that the limit as $z \to a$ does not exist for $f$ as defined on its original domain $E$, but the limit of $f|_D$ does exist.

**Example 2.1.9.** Show that the function $f(z) = z/|z|$ with domain $D_1(0) \setminus \{0\}$ does not have a limit as $z \to 0$, but if this function is restricted to the domain $(0, 1) \subset \mathbb{R}$ then its limit as $z \to 0$ is 1.

**Solution:** If $f$ is considered as a function with domain $D_1(0) \setminus 0$, then for every $\delta > 0$ and every $u$ on the unit circle, there are points $z$ with $0 < |z| < \delta$ and $f(z) = u$. So there is certainly no one number that $f(z)$ is approaching as $z \to 0$. On the other hand, suppose $f$ is restricted to $(0, 1) \subset \mathbb{R}$. On this interval, $f$ is identically 1 and so its limit as $z \to 0$ is also 1.

A *deleted neighborhood* of $a \in \mathbb{C}$ is a set of the form $V \setminus \{a\}$ where $V$ is neighborhood of $a$. Using neighborhoods and deleted neighborhoods, the limit concept may be rephrased as follows.

**Theorem 2.1.10.** *If $f$ is a complex valued function with domain $E$, and if $a \in \overline{E}$, then $\lim_{z\to a} f(z) = L$ if and only if for every neighborhood $W$ of $L$ there is a deleted neighborhood $V$ of $a$ such that*

$$f(E \cap V) \subset W. \tag{2.1.1}$$

*Proof.* If $\lim_{z\to a} f(z) = L$ and $W$ is a neighborhood of $L$, then, since $W$ is open and contains $L$, there is an $\epsilon > 0$ such that $D_\epsilon(L) \subset W$. By the definition of limit, there is a $\delta > 0$ such that $z \in E$ and $0 < |z - a| < \delta$ imply $|f(z) - L| < \epsilon$. This implication is equivalent to the statement

$$f(E \cap D_\delta(a) \setminus \{a\}) \subset D_\epsilon(L) \subset W.$$

Thus, if we choose $V = D_\delta(a) \setminus \{a\}$, then $V$ is a deleted neighborhood of $a$ satisfying (2.1.1). This proves the "only if" part of the theorem.

To prove the "if" part, we assume that for every neighborhood $W$ of $L$ there is a deleted neighborhood $V$ of $a$ such that $f(E \cap V) \subset W$. Then, given $\epsilon > 0$, let $W = D_\epsilon(L)$ and let $V$ be a deleted neighborhood of $a$ satisfying (2.1.1). Since $V \cup \{a\}$ is open, there exists a $\delta > 0$ such that $D_\delta(a) \subset V \cup \{a\}$. Then

$$f(E \cap D_\delta(a) \setminus \{a\}) \subset f(E \cap V) \subset W = D_\epsilon(L).$$

This is equivalent to the statement that $0 < |z - a| < \delta$ and $z \in U$ imply $|f(z) - w| < \epsilon$. Thus, by definition, $\lim_{z\to a} f(z) = L$. □

## Continuity

The definition of a continuous function should also be familiar from calculus.

**Definition 2.1.11.** A function $f$ with domain $E$ is said to be continuous at $a \in E$ if

$$\lim_{z\to a} f(z) = f(a).$$

If $f$ is continuous at every point of $E$, then we say that $f$ is continous on $E$. The set of all functions with domain $E$, which are continuous on $E$, will be denoted $\mathcal{C}(E)$.

**Example 2.1.12.** Show that $|z|$ is a continuous function of $z$ on all of $\mathbb{C}$.

**Solution:** Let $z$ and $a$ be elements of $\mathbb{C}$. By the second form of the triangle inequality, we have

$$||z| - |a|| \leq |z - a|.$$

It follows from this that $\lim_{z \to a} |z| = |a|$ and, hence, that $|z|$ is continuous at $z$. In fact, given $\epsilon > 0$, it suffices to choose $\delta = \epsilon$ in the definition of limit, since $|z - a| < \epsilon$ implies $||z| - |a|| < \epsilon$.

The definition of continuous function is also very domain dependent, for example, if the function $f(z) = z/|z|$ of Example 2.1.9 is given the value 1 at 0, then it is continuous at 0 as a function with domain $[0, 1)$, but if we consider its domain to be $D_1(0)$, then it is not continuous.

The next result characterizes functions which are defined and continuous on an open set $U$ as those functions $f$ on $U$ for which $f^{-1}$ preserves open sets.

**Theorem 2.1.13.** *If $f$ is a complex valued function defined on an open set $U \subset \mathbb{C}$, then $f$ is continuous on $U$ if and only if $f^{-1}(W)$ is open for every open set $W \subset \mathbb{C}$.*

*Proof.* Suppose $f$ is continous on $U$ and $W$ is an open subset of $\mathbb{C}$. If $a \in f^{-1}(W)$, then $\lim_{z \to a} f(z) = f(a) \in W$. Since $W$ is a neighborhood of $f(a)$, it follows from Theorem 2.1.10 that there is a neighborhood $V$ of $a$ such that $f(U \cap V) \subset W$ (actually, Theorem 2.1.10 guarantees a *deleted* neighborhood $V$ with this property, but since $f(a) \in W$ it doesn't change things to put $a$ back in the deleted neighborhood and claim there is an actual neighborhood $V$ as above), Thus, $U \cap V$ is an open subset of $f^{-1}(W)$ containing $a$. We have now proved that every element of $f^{-1}(W)$ is contained in an open set contained in $f^{-1}(W)$. The union of all such open sets is open and is equal to $f^{-1}(W)$. Thus, $f^{-1}(W)$ is open.

To prove the converse, we suppose that $f^{-1}(W)$ is open for every open set $W$. It follows that for each $a \in U$ and each neighborhood $W$ of $f(a)$, we have $f^{-1}(W)$ is a neighborhood of $a$. By Theorem 2.1.10, $\lim_{z \to a} f(z) = f(a)$ and so $f$ is continuous at $a$. Since $a$ was any point of $U$, $f$ is continuous on $U$. □

If $f$ and $g$ are functions with domain a set $E$ and they are both continuous at $a \in E$, then $f + g$ and $fg$ are continuous at $a$, and $f/g$ is continuous at $a$ provided $g(a) \neq 0$. The proofs of these facts for complex valued functions of a complex variable are no different than the proofs, given in calculus, of the corresponding facts for functions of a real variable. Thus, we will accept them without further comment.

Obviously, constants are continuous everywhere, as is the function $z$ itself. It follows that polynomials in $z$ are also continuous everywhere.

If $g$ with domain $E$ is continuous at $a$ and $f$ with domain $D$ is continuous at $b = f(a)$, and if $g(E) \subset D$ then the composite function $f \circ g$, with domain $E$, is continuous at $a$. This is another fact whose proof for functions of a complex variable is no different than the proof from calculus of the analogous result for functions of a real variable.

**Example 2.1.14.** Prove that if $g$ is a continuous non-vanishing function on an open subset $U \subset \mathbb{C}$, then $\log |g|$ is continous on $U$.

**Solution:** The function $|g(z)|$ is continuous on $U$ because it is the composition of the function $g$, which is continuous on $U$, and the function $|\cdot|$, which is continous everywhere. Also, $|g(z)|$ has its values in the positive reals, since $g$ is non-vanishing on $U$. The function log is continuous on the positive reals and so the composition $\log |g|$ is continuous on $U$.

## Exercise Set 2.1

1. Prove the second statement of Theorem 2.1.3. That is prove that the intersection of an arbitrary collection of closed sets is closed, while the union of any finite collection of closed sets is closed.

2. Prove that if $U$ is open and $K$ is closed, then the set theoretic difference $U \setminus K$ is open, while $K \setminus U$ is closed.

3. Prove Theorem 2.1.6.

4. Show that the set $A = \{z \in \mathbb{C} : \mathrm{Re}\,(z) > 0\}$ is open. Hint: you must show that each point $w \in A$ is the center of an open disc which is entirely contained in $A$.

5. Tell which of the following sets are open subsets of $\mathbb{C}$, which are closed, and which are neither (no proof required):

   (a) $\{z \in \mathbb{C} : 1 < |z| < 2\}$;
   
   (b) $\{z \in \mathbb{C} : \mathrm{Im}\,(z) = 0, 0 < \mathrm{Re}\,(z) < 1\}$;
   
   (c) $\{z \in \mathbb{C} : -1 \leq \mathrm{Re}\,(z) \leq 1\}, \ -1 \leq \mathrm{Im}\,(z) \leq 1\}$.

6. Find the interior, closure, and boundary for the set $\{z \in \mathbb{C} : 1 \leq |z| < 2\}$ (no proof required).

7. Prove that $w \in \mathbb{C}$ is in the closure of a set $E \subset \mathbb{C}$ if and only if there is a sequence $\{z_n\} \subset E$ such that $\lim z_n = w$. Thus, a set $E$ is closed if and only if it contains all limits of convergent sequences of points in $E$.

8. Does $\lim_{z\to 0} f(z)$ exist if $f(z) = \dfrac{|z - \overline{z}|}{|z|}$ with domain $\mathbb{C} \setminus \{0\}$? How about if the domain is restricted to be just $\mathbb{R} \setminus \{0\}$?

9. Prove that $\mathrm{Re}\,(z)$, $\mathrm{Im}\,(z)$, and $\overline{z}$ are continuous functions of $z$.

10. At what points of $\mathbb{C}$ is the function $(1 - z^4)^{-1}$ continuous.

11. Prove that $\arg_I$ is continuous except on its cut line.

12. Use the result of the preceding exercise to prove that a branch of the log function is continuous except on its cut line.

13. Use Theorem 2.1.13 to prove that if $f$ and $g$ are continuous functions with open domains $U_f$ and $U_g$ and if $g(U_g) \subset U_f$, then $f \circ g$ is continuous on $U_g$.

14. Prove that if $f$ is a continuous function defined on an open subset $U$ of $\mathbb{C}$, then sets of the form $\{z \in U : |f(z)| < r\}$ and $\{z \in U : \mathrm{Re}\,(f(z)) < r\}$ are open.

15. Use the result of the preceding exercise to come up with an open subset of $\mathbb{C}$ that has not been previously described in this text.

16. Prove that a function $f$ with open domain $U$ is continuous at a point $a \in U$ if and only if whenever $\{z_n\} \subset U$ is a sequence converging to $a$, the sequence $\{f(z_n)\}$ converges to $f(a)$.

## 2.2 The Complex Derivative

There is nothing surprising about the definition of the derivative of a function of a complex variable – it looks just like the definition of the derivative of a function of a real variable. What is surprising are the consequences of a function having a derivative in this sense.

**Definition 2.2.1.** Let $f$ be a function defined on a neighborhood of $z \in \mathbb{C}$. If

$$\lim_{w \to z} \frac{f(w) - f(z)}{w - z}$$

exists, then we denote it by $f'(z)$, and we say $f$ is differentiable at $z$ with derivative $f'(z)$. If $f$ is defined and differentiable at every point of an open set $U$, then we say that $f$ is analytic on $U$.

**Remark 2.2.2.** When convenient, we will make the change of variables $\lambda = w - z$ and write the derivative in the form

$$f'(z) = \lim_{\lambda \to 0} \frac{f(z+\lambda) - f(z)}{\lambda} \tag{2.2.1}$$

Clearly constant functions are differentiable and have derivative 0, since the difference quotient in Definition 2.2.1 is identically 0 for such a function.

The first hint that there is something fundamentally different about this notion of derivative is in the following example.

**Example 2.2.3.** Show that the function $f(z) = z$ is differentiable everywhere on $\mathbb{C}$ with derivative 1 and, hence, is analytic on $\mathbb{C}$, but the function $f(z) = \overline{z}$ is differentiable nowhere.

**Solution:** For $f(z) = z$, the difference quotient in (2.2.1) is

$$\frac{\lambda}{\lambda} = 1,$$

which clearly has limit 1 as $\lambda \to 0$ for every $z$. On the other hand, if $f(z) = \overline{z}$, then the difference quotient is

$$\frac{\overline{\lambda}}{\lambda} = \mathrm{e}^{-2i\theta},$$

if $\lambda = r\mathrm{e}^{i\theta}$ in polar form. The limit of this function as $\lambda \to 0$ clearly does not exist, since it has a different fixed value along each ray eminating from 0. This is true no matter what $z$ is, and so $\overline{z}$ is nowhere differentiable.

What makes this example so surprising, at first, is that, as a function of the two real variables $x$ and $y$, $\overline{z} = x - iy$ is of class $\mathrm{C}^\infty$ – meaning that its partial derivatives of all orders exist and are continuous – and yet, its complex derivative does not exist. Thus, existence of the complex derivative involves more than just smoothness of the function.

We will soon prove that a function which has a power series expansion that converges on an open disc is analytic on that disc. This would imply that the exponential function, for example, is analytic on all of $\mathbb{C}$. We don't have to wait, however, to prove this fact. There is an elementary proof that $e^z$ is analytic on $\mathbb{C}$.

**Example 2.2.4.** Prove that $\mathrm{e}^z$ is an analytic function of $z$ on the entire complex plane and show that it is its own derivative.

**Solution:** Given an arbitrary point $z \in \mathbb{C}$, we will show that $e^z$ has derivative $e^z$ at $z$. By the law of exponents

$$\frac{e^{z+\lambda} - e^z}{\lambda} = e^z \frac{\mathrm{e}^\lambda - 1}{\lambda}.$$

Thus, to show that the derivative of $e^z$ is $\mathrm{e}^z$ we need only show that

$$\lim_{\lambda \to 0} \frac{\mathrm{e}^\lambda - 1}{\lambda} = 1. \tag{2.2.2}$$

However, if $t = |\lambda|$, inspection of the power series for $\mathrm{e}^\lambda$ and $\mathrm{e}^t$ shows that

$$\left| \frac{\mathrm{e}^\lambda - 1}{\lambda} - 1 \right| = \left| \frac{\mathrm{e}^\lambda - 1 - \lambda}{\lambda} \right| \le \frac{\mathrm{e}^t - 1 - t}{t}. \tag{2.2.3}$$

Now to show that the expression on the left has limit zero and, thus, verify (2.2.2), we simply apply L'Hôpital's rule to the expression on the right.

## Elementary Properties of the Derivative

A simple result about derivatives of functions of a real variable that also holds in the context of complex derivatives is the following. The proof is elementary and is left to the exercises.

**Theorem 2.2.5.** *If the complex derivative $f'$ of $f$ exists at $a \in \mathbb{C}$, then $f$ is continuous at $a$.*

The complex derivative has all of the familiar properties in relation to sums, products, and quotients of functions. The proofs of these are in no way different from the proofs of the corresponding results for functions of a real variable. In the following theorem, Part (a) is trivial and we leave Parts (b) and (c) to the exercises.

**Theorem 2.2.6.** *If $f$ and $g$ are functions of a complex variable which are differentiable at $z \in \mathbb{C}$, then*

*(a) $f + g$ is differentiable at $z$ and $(f+g)'(z) = f'(z) + g'(z)$;*

*(b) $fg$ is differentiable at $z$ and $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$;*

*(c) if $g(z) \neq 0$, $1/g$ is differentiable at $z$ and $(1/g)'(z) = -g'(z)/g^2(z)$.*

Parts (a) and (b) of this theorem and the fact that constant functions and the function $z$ are analytic on $\mathbb{C}$ imply that every polynomial in $z$ is analytic on $\mathbb{C}$. Of course, since $\overline{z}$ is not analytic, we can't expect mixed polynomials that contain powers of both $z$ and $\overline{z}$ to be analytic.

Parts (b) and (c) of the theorem imply that $f/g$ is differentiable at $z$ if $f$ and $g$ are and if $g(z) \neq 0$. They also imply the quotient rule

$$\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - g'(z)f(z)}{g^2(z)}.$$

The chain rule also holds for the complex derivative.

**Theorem 2.2.7.** *If $g$ is differentiable at $a$ and $f$ is differentiable at $b = g(a)$, then $f \circ g$ is differentiable at $a$ and*

$$(f \circ g)'(a) = f'(g(a))g'(a)$$

*Proof.* Let $U$ be a neighborhood of $b$ on which $f$ is defined. We define a function $h(w)$ on $U$ in the following way

$$h(w) = \left\{ \begin{array}{cc} \dfrac{f(w) - f(b)}{w - b} & w \neq b \\ f'(b) & w = b \end{array} \right\}.$$

Then $h$ is continuous at $b$, since

$$f'(b) = \lim_{w \to b} \frac{f(w) - f(b)}{w - b}.$$

Also,

$$\frac{f \circ g(z) - f \circ g(a)}{z - a} = h(g(z))\frac{g(z) - g(a)}{z - a} \tag{2.2.4}$$

for all $z$ in the neighborhood $V = g^{-1}(U)$ of $a$. If we take the limit of both sides of (2.2.4) and use the fact that $f$ is continuous at $b$ and $g$ is continuous at $a$, we conclude that $(f \circ g)'(a) = f'(g(a))g'(a)$, as required. □

**Example 2.2.8.** Suppose $p(z)$ is a polynomial in $z$. Where is the function $\mathrm{e}^{p(z)}$ analytic and what is its derivative?

**Solution** Since $e^z$ and $p(z)$ both are differentiable everywhere, so is the composition $\mathrm{e}^{p(z)}$, by Theorem 2.2.7, and the derivative is

$$\left(\mathrm{e}^{p(z)}\right)' = p'(z)\mathrm{e}^{p(z)}.$$

## The Cauchy-Riemann Equations

Since a function $f$ of a complex variable may be regarded as a complex valued function on a subset of $\mathbb{R}^2$, we can write it in the form

$$f(x+iy) = u(x,y) + iv(x,y) \tag{2.2.5}$$

where $u$ and $v$ are the real and imaginary parts of $f$, regarded as functions defined on a subset of $\mathbb{R}^2$. It is natural to ask what the existence of a complex derivative for $f$ implies about the functions $u$ and $v$ as functions of the two real variables $x$ and $y$. It is easy to see that it implies the existence of the partial derivatives $u_x$, $u_y$, $v_x$ and $v_y$. In fact, it implies much more as the following discussion will show.

Recall that a function $g$ of two real variables is said to be *differentiable* at $(x,y)$ if there are numbers $A$ and $B$ such that

$$g(x+h, y+k) - g(x,y) = Ah + Bk + \epsilon(h,k),$$

where $\epsilon(h,k)/|(h,k)| \to 0$ as $(h,k) \to (0,0)$. If $g$ is differentiable at $(x,y)$, then the numbers $A$ and $B$ are the partial derivatives $g_x$ and $g_y$ at $(x,y)$.

Suppose $f$ is a complex valued function defined in a neighborhood of $z \in C$. If $M = f'(z)$ exists, then we may write

$$f(z+\lambda) - f(z) = M\lambda + \epsilon(\lambda) \tag{2.2.6}$$

where $\epsilon(\lambda)/\lambda \to 0$ as $\lambda \to 0$. In fact, $\epsilon(\lambda)$ is given by

$$\epsilon(\lambda) = f(z+\lambda) - f(z) - M\lambda,$$

and so, the fact that $\epsilon(\lambda)/\lambda \to 0$ as $\lambda \to 0$, is equivalent to the statement that $f'(z)$ exists and is equal to $M$.

If we write $f, M, z, \lambda$, and $\epsilon$ in terms of their real and imaginary parts: $f = u + iv, M = C + iD, z = x + iy, \lambda = h + ik$, and $\epsilon = \epsilon_r + i\epsilon_i$ then (2.2.6) becomes

$$\begin{aligned} u(x+h, y+k) + iv(x+h,y+k) - u(x,y) - iv(x,y)& \\ = (C+iD)(h+ik) + \epsilon_r(h,k) + i\epsilon_i(h,k).& \end{aligned} \tag{2.2.7}$$

On equating real and imaginary parts, this leads to the two equations

$$\begin{aligned} u(x+h,y+k) - u(x,y) &= Ch - Dk + \epsilon_r(h,k) \\ v(x+h,y+k) - v(x,y) &= Dh + Ck + \epsilon_i(h,k). \end{aligned} \tag{2.2.8}$$

The condition that $\epsilon(\lambda)/\lambda \to 0$ as $\lambda \to 0$ implies that $\epsilon_r(h,k)/|(h,k)| \to 0$ and $\epsilon_i(h,k)/|(h,k)| \to 0$ (note that $|(h,k)| = \sqrt{h^2+k^2} = |\lambda|$). Thus, we can draw two conclusions from the existence of $f'(z)$: (1) $u$ and $v$ are differentiable at $(x,y)$, and (2) the partial derivatives of $u$ and $v$ at $(x,y)$ are given by

$$\begin{aligned} u_x(x,y) &= C, \quad u_y(x,y) = -D, \\ v_x(x,y) &= D, \quad v_y(x,y) = C. \end{aligned} \tag{2.2.9}$$

A surprising consequence of this is that if $f'$ exists at $z = x+iy$, then

$$\begin{aligned} u_x &= v_y, \\ u_y &= -v_x. \end{aligned} \tag{2.2.10}$$

at $(x,y)$. Equations 2.2.10 are the *Cauchy-Riemann equations*. Equations (2.2.9) also show that if $f'$ exists at $z$, then $f'(z) = C + iD = u_x + iv_x = -i(u_y + iv_y)$. If we set $f_x = u_x + iv_x$ and $f_y = u_y + iv_y$, then this can be written as $f' = f_x = -if_y$ wherever $f'$ exists.

The above discussion shows that, at any point where $f$ has a complex derivative, its real and imaginary parts are differentiable functions and satisfy the Cauchy-Riemann equations. The converse is also true: If the real and imaginary parts of $f$ are differentiable and satisfy the Cauchy-Riemann equations at a point $z = x+iy$, then $f'(z)$ exists. The proof of this is a matter of working backwards through the above discussion, beginning with the assumption that $u$ and $v$ are differentiable at $(x,y)$, with partial derivatives that satisfy $u_x = v_y = C$ and $u_y = -v_x = -D$. This leads to (2.2.8), which eventually leads back to the conclusion that $C+iD$ is the derivative of $f$ at $z = x+iy$. We leave the details to the exercises. The result is the following theorem.

**Theorem 2.2.9.** *If $f = u + iv$ is a complex valued function defined in a neighborhood of $z \in \mathbb{C}$, with real and imaginary parts $u$ and $v$, then $f$ has a complex derivative at $z$ if and only if $u$ and $v$ are differentiable and satisfy the Cauchy-Riemann Equations (2.2.10) at $z = x+iy$. In this case,*

$$f' = f_x = -if_y.$$

**Example 2.2.10.** We already know that $e^z$ is analytic everywhere. However, give a different proof of this by showing $e^z$ satisfies the Cauchy-Riemann Equations.

**Solution:** With $z = x+iy$, we write $\mathrm{e}^z = \mathrm{e}^x(\cos y + i\sin y)$. The real and imaginary parts of $\mathrm{e}^z$ are $u(x,y) = \mathrm{e}^x \cos y$ and $v(x,y) = \mathrm{e}^x \sin y$. Thus,

$$\begin{aligned} u_x(x,y) &= \mathrm{e}^x \cos y = v_y, \quad \text{and} \\ u_y(x,y) &= -\mathrm{e}^x \sin y = -v_x. \end{aligned}$$

**Example 2.2.11.** Use the Cauchy-Riemann equations to prove that, for each branch of the log function, $\log(z)$ is analytic everywhere except on its cut line and has derivative $1/z$.

**Solution:** We first prove that the principal branch of the log function is analytic on the right half plane $H = \{z \in \mathbb{C} : \mathrm{Re}\,(z) > 0\}$. For $z \in H$ we have $z = x + iy = re^{i\theta}$ where

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \tan^{-1}(y/x).$$

Thus, the principal branch of log on $H$ is

$$\log(x + iy) = (1/2)\ln(x^2 + y^2) + i\tan^{-1}(y/x).$$

Taking partial derivatives yields

$$\begin{aligned}
\frac{\partial}{\partial x}(1/2)\ln(x^2 + y^2) &= \frac{x}{x^2 + y^2}, \\
\frac{\partial}{\partial x}\tan^{-1}(y/x) &= \frac{-y/x^2}{1 + (y/x)^2} = \frac{-y}{x^2 + y^2}, \\
\frac{\partial}{\partial y}(1/2)\ln(x^2 + y^2) &= \frac{y}{x^2 + y^2}, \\
\frac{\partial}{\partial y}\tan^{-1}(y/x) &= \frac{1/x}{1 + (y/x)^2} = \frac{x}{x^2 + y^2}.
\end{aligned} \tag{2.2.11}$$

Thus, the Cauchy-Riemann equations are satisfied by the principal branch of the log function on $H$. Furthermore

$$(\log z)' = \frac{\partial}{\partial x}\log(x + iy) = \frac{x - iy}{x^2 + y^2} = \frac{1}{z}.$$

Now if $z$ is any point not on the negative real axis and not in $H$, then we simply rotate $z$ into $H$. That is, we choose $\alpha = \pm\pi/2$ such that $e^{i\alpha}z \in H$. Then

$$\log z = \log(e^{i\alpha}z) - i\alpha.$$

Since log has derivative $1/w$ at $w = e^{i\alpha}z$, it follows from the chain rule that log has derivative $e^{i\alpha}/(e^{i\alpha}z) = 1/z$ at $z$. Thus, the principal branch of the log function is analytic with derivative $1/z$ at any point $z$ not on its cut line.

The analogous statement for other branches of the log function follows from the fact that if $z$ is not on the cut line of the function, then there is an open set containing $z$ on which the function is differs from the principal branch of the log function by a constant.

## Harmonic Functions

In the next chapter, we will prove that analytic functions are $\mathcal{C}^\infty$– that is, they have continuous complex derivatives of all orders. This, in particular, implies that analytic functions have continuous partial derivatives of all orders with respect to $x$ and $y$. Assuming this result for the moment, we have

**Theorem 2.2.12.** *The real and imaginary parts of an analytic function on $U$ are harmonic functions on $U$, meaning they satisfy Laplace's equation*

$$u_{xx} + u_{yy} = 0.$$

*Proof.* If $f = u + iv$ is an analytic function, then $u$ and $v$ satisfy the Cauchy-Riemann equations and so

$$u_{xx} = (u_x)_x = (v_y)_x = (v_x)_y = (-u_y)_y = -u_{yy}.$$

This shows that the real part of $f$ satisfy's Laplaces equation. Since $v$ is the real part of the analytic function $-if$, it follows that $v$ is also harmonic. Thus, both real and imaginary parts of an analytic function are harmonic. □

If $u$ and $v$ are harmonic functions such that the function $f = u + iv$ is analytic, then we say $u$ and $v$ are *harmonic conjugates* of one another.

**Example 2.2.13.** Prove that $u(x, y) = \mathrm{e}^x \cos y$ is a harmonic function on all of $\mathbb{R}^2$ and find a harmonic conjugate for it.

**Solution:** The function $u$ is the real part of $f(z) = \mathrm{e}^z$ and is, therefore, harmonic by the previous theorem. The imaginary part of $f$ is $v(x, y) = \mathrm{e}^x \sin y$ and so this function $v$ is a harmonic conjugate of $u$.

## Exercise Set 2.2

1. Fill in the details in Example 2.2.4 by verifying the inequality (2.2.3) and showing that the limit of the expression on the right is 0.

2. Prove Theorem 2.2.5.

3. Prove Part (b) of Theorem 2.2.6.

4. Prove Part (c) of Theorem 2.2.6.

5. Use induction and Theorem 2.2.6 to show that $(z^n)' = nz^{n-1}$ if $n$ is a non-negative integer.

6. Find the derivative of $z^7 + 5z^4 - 2z^3 + z^2 - 1$. Which results from this section are used in this calculation?

7. Find the derivative of $\mathrm{e}^{z^3}$.

8. If we use the principal branch of the log function, at which points of $\mathbb{C}$ does $\dfrac{\log z}{z}$ have a complex derivative? What is its derivative at these points?

9. Finish the proof of Theorem 2.2.9 by showing that if $f = u + iv$, $u$ and $v$ are differentiable at $z$, and $u$ and $v$ satisfy the Cauchy-Riemann equations at $z$, then $f'(z)$ exists.

10. Use the Cauchy-Riemann equations to verify that the function $f(z) = z^2$ is analytic everywhere.

11. Describe all real valued functions which are analytic on $\mathbb{C}$.

12. Derive the Cauchy-Riemann equations in polar coordinates:

$$u_r = r^{-1}v_\theta$$
$$u_\theta = -rv_r.$$

by using the change of variable formulas $x = r\cos\theta$, $y = r\sin\theta$ and the chain rule.

13. We showed in Example 2.2.11 that each branch of the log function is analytic on the complex plane with its cut line removed. Use the Cauchy-Riemann equations in polar form (previous problem) to give another proof of this fact.

14. Assuming each branch of the log function is analytic, use the chain rule to give another prove that each such function has derivative $1/z$.

15. Use the Cauchy-Riemann equations to prove that if $f$ is analytic on an open set $U$, then the function $g$ defined by $g(z) = \overline{f(\overline{z})}$ is analytic on the set $\{\overline{z} : z \in U\}$.

16. Verify that the function $\log|z|$ is harmonic on $\mathbb{C}\setminus\{0\}$ and find a harmonic conjugate for it on the set consisting of $\mathbb{C}$ with the non-positive real axis removed.

## 2.3 Contour Integrals

Integration plays a key role in this subject – specifically, integration along curves in $\mathbb{C}$. A *curve* or *contour* in the plane $\mathbb{C}$ is a continuous function $\gamma$ from an interval on the line into $\mathbb{C}$. Such an object is sometimes called a parameterized curve and the interval $I$ is called the parameter interval. We will be interested in a particular kind of curve, one whose parameter interval is a closed bounded interval which can be subdivided into finitely many subintervals, on each of which $\gamma$ is continuously differentiable.

### Smooth Curves

Let $I = [a, b]$ be a closed interval on the real line and let $\gamma : I \to \mathbb{C}$ be a complex valued function on $I$. If $c \in I$, then the derivative $\gamma'(c)$ of $\gamma$ at $c$ is defined in the usual way:

$$\gamma'(c) = \lim_{t\to c}\frac{\gamma(t)-\gamma(c)}{t-c}. \tag{2.3.1}$$

Of course, $\gamma$ is complex valued and so this limit should be interpreted as the type of limit discussed in section 2.1. It can be calculated by expressing $\gamma$ in terms of its real and imaginary parts, that is, by writing $\gamma(t) = x(t) + iy(t)$

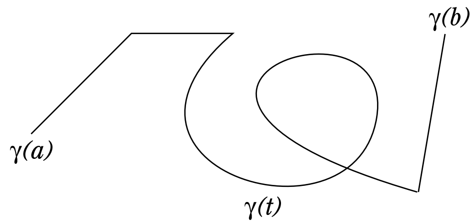


Figure 2.3: A Path in the Plane.

where $x(t)$ and $y(t)$ are real valued functions on $I$. Then $\gamma'(t) = x'(t) + iy'(t)$ (Exercise 2.3.6).

What about the endpoints $a$ and $b$ of the interval $I$? Should we either not talk about the derivative at the endpoints or, perhaps, use one sided derivatives defined in terms of one sided limits (limit from the right at $a$ and limit from the left at $b$)? Actually, there is no need to do anything special at $a$ and $b$ or to exclude them. If the domain of $\gamma$ is $[a, b]$, then our domain dependent definition of limit takes care of the problem. If $c = a$, the limit as $t \to a$ in (2.3.1) only involves values of $t$ to the right of $a$, since only those are in the domain of the difference quotient that appears in this limit. Similarly, if $c = b$, the limit as $t \to b$ involves only points to the left of $b$. Thus, the derivatives at $a$ and $b$ that our definition leads to are what in calculus would be called the right derivative at $a$ and the left derivative at $b$.

The function $\gamma$ is *differentiable* at $c$ if the limit defining $\gamma'(c)$ exists. It is *continuously differentiable* or *smooth* on $I$ if it is differentiable at every point of $I$ and if the derivative is a continuous function on $I$. In this case we will write $\gamma \in \mathcal{C}^1(I)$.

**Definition 2.3.1.** A curve $\gamma : [a, b] \to \mathbb{C}$ in $\mathbb{C}$ is called *piecewise smooth* if there is a partition $a = a_0 < a_1 < \cdots < a_n = b$ of $[a, b]$ such that the restriction of $\gamma$ to $[a_{j-1}, a_j]$ is smooth for each $j = 1, \cdots, n$. A curve which is piecewise smooth will be called a *path*.

With appropriate choices of parameterization, familiar geometric objects in $\mathbb{C}$ can be described as the image of a path.

**Example 2.3.2.** Find a path $\gamma$ that traces once around the circle of radius $r$, centered at 0, in the counterclockwise direction. Describe $\gamma'$.

**Solution:** The smooth path $\gamma(t) = re^{it}$, $t \in [0, 2\pi]$ does the job. Its derivative may be obtained by writing it as $r(\cos t + i \sin t)$ and differentiating the real and imaginary parts. The result is $\gamma'(t) = r(-\sin t + i \cos t) = ire^{it}$.

**Example 2.3.3.** Let $z$ and $w$ be two points in $\mathbb{C}$. Find a path which traces the straight line from $z$ to $w$ and find its derivative.

**Solution:** The path $\gamma$, with parameter interval $[0, 1]$, defined by

$$\gamma(t) = (1 - t)z + tw = z + t(w - z)$$

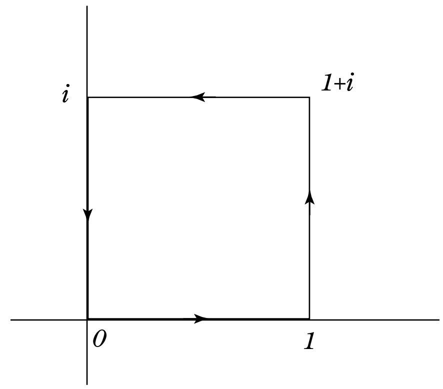


Figure 2.4: The Path of Example 2.3.4.

satisfies $\gamma(0) = z$ and $\gamma(1) = w$. It is a parametric form of a straight line in the plane and its derivative is $\gamma'(t) = w - z$.

**Example 2.3.4.** Find a path that traces once around the square with vertices $0, 1, 1 + i, i$ in the counterclockwise direction. Find $\gamma'(t)$ on the subintervals where $\gamma$ is smooth.

**Solution:** We choose $[0, 1]$ as the parameter interval and define a path $\gamma$ as follows (see Figure 2.4):

$$\gamma(t) = \left\{ \begin{matrix} 4t & 0 \le t \le 1/4 \\ 1 + (4t - 1)i & 1/4 \le t \le 1/2 \\ 3 - 4t + i & 1/2 \le t \le 3/4 \\ (4 - 4t)i & 3/4 \le t \le 1 \end{matrix} \right\}$$

This is continuous on $[0, 1]$ and smooth on each subinterval in the partition $0 < 1/4 < 1/2 < 3/4 < 1$. It traces each side of the square in succession, moving in the counterclockwise direction. On the first interval, $\gamma'$ is the constant 4, on the second it is $4i$, on the third it is $-4$, and on the fourth it is $-4i$.

## Riemann Integral of Complex Valued Functions

The integral of a function along a path will be defined in terms of the Riemann integral on an interval. This is the familiar Riemann integral from calculus, except that the functions being integrated will be complex valued. This difference requires a few comments.

If $f(t) = g(t) + ih(t)$ is a complex valued function on an interval $[a, b]$, where $g$ and $h$ are real valued, then we will say that $f$ is *Riemann Integrable* on $[a, b]$ if both $g$ and $h$ are Riemann Integrable on $[a, b]$ as real valued functions. We then define the integral of $f$ on $[a, b]$ by

$$\int_a^b f(t)\,dt = \int_a^b g(t)\,dt + i\int_a^b h(t)\,dt. \tag{2.3.2}$$

This Riemann integral for complex valued functions has the properties one would expect given knowledge of the Riemann integral for real valued functions. The next three theorems cover some of these properties.

**Theorem 2.3.5.** *Let $f_1$ and $f_2$ be Riemann integrable functions on $[a, b]$ and $\alpha$ and $\beta$ complex numbers. Then, $\alpha f_1 + \beta f_2$ is integrable on $[a, b]$, and*

$$\int_a^b (\alpha f_1(t) + \beta f_2(t))\,dt = \alpha\int_a^b f_1(t)\,dt + \beta\int_a^b f_2(t)\,dt.$$

*Proof.* That this is true if the constants $\alpha$ and $\beta$ are real follows directly from expressing $f_1$ and $f_2$ in terms of their real and imaginary parts. Thus, to prove the theorem we just need to show that $\int_a^b if(t)\,dt = i\int_a^b f(t)\,dt$ if $f = g + ih$ is an integrable function on $[a, b]$. However,

$$\begin{aligned}\int_a^b i(g(t) + ih(t))\,dt &= \int_a^b (-h(t) + ig(t))\,dt \\ &= -\int_a^b h(t)\,dt + i\int_a^b g(t)\,dt = i\left(\int_a^b (g(t) + ih(t))\,dt\right).\end{aligned}$$

This completes the proof. □

**Theorem 2.3.6.** *If $f$ is a function defined on $[a, b]$ and $c \in (a, b)$, then then $f$ is integrable on $[a, b]$ if and only if it is integrable on $[a, c]$ and $[c, b]$. In this case*

$$\int_a^b f(t)\,dt = \int_a^c f(t)\,dt + \int_c^b f(t)\,dt.$$

*Proof.* This follows from the fact that the same things are true of the integrals of the real and imaginary parts $g$ and $h$ of $f$. □

**Theorem 2.3.7.** *If $f$ is an integrable function on $[a, b]$, then*

$$\left|\int_a^b f(t)\,dt\right| \le \int_a^b |f(t)|\,dt.$$

*Proof.* This is proved using a trick. We set $w = \int_a^b f(t)\,dt$. If $w = 0$, there is nothing to prove. If $w \neq 0$, let $u = \overline{w}/|w|$. Then $uw = |w|$ and so

$$\left|\int_a^b f(t)\,dt\right| = u\int_a^b f(t)\,dt = \int_a^b uf(t)\,dt.$$

Since this is a real number, the integral of the imaginary part of $uf$ is zero and we have

$$\left|\int_a^b f(t)\,dt\right| = \int_a^b \mathrm{Re}\,(uf(t))\;dt \le \int_a^b |uf(t)|\,dt = \int_a^b |f(t)|\,dt.$$

□

A complex valued function which is defined and continuous on an interval $[a,b]$ is clearly Riemann integrable on $[a,b]$, since its real and imaginary parts are continuous and continous real valued functions on closed bounded intervals are Riemann integrable.

## Integration Along a Path

If $\gamma$ is a path, then $\gamma'$ exists and is continuous on each interval $[a_{i-1},a_i]$ in a partition $a=a_0<a_1<\cdots<a_n=b$ of the parameter interval $[a,b]$. At the points $a_1,a_2,\cdots,a_{n-1}$ the definition of $\gamma'$ is ambiguous – $\gamma'(a_j)$ has one value from the derivative of $\gamma$ on $[a_{j-1},a_j]$ and another from the derivative of $\gamma$ on $[a_j,a_{j+1}]$. In order to remove this ambiguity, we choose to define $\gamma'$ so as to be left continuous at these points. That is, at $a_j$, we choose the value for $\gamma'$ that comes from its definition on $[a_{j-1},a_j]$. Then $\gamma'$ is well defined on $I=[a,b]$.

If $f$ is a complex valued function defined and continuous on a set $E$ containing $\gamma(I)$, then the function $f(\gamma(t))\gamma'(t)$ is a well defined function on $I$ which is piecewise continuous in the following sense: It is continuous everywhere on $[a,b]$ except at the partition points $a_1,a_2,\cdots,a_{n-1}$. It is left continuous at these points and the limit from the right exists and is finite at these points as well. In other words, this function is continuous from the left everywhere on $[a,b]$ and continuous except at finitely many points where it has simple jump discontinuities.

A function of this type is Riemann integrable on $[a,b]$. To see this, first observe that it is Riemann integrable on each subinterval $[a_{j-1},a_j]$ because, on such an interval, the function agrees with a continuous function except at one point – $a_{j-1}$. A continuous function on a closed interval is Riemann integrable and changing its value at one point does not effect this fact or the value of the integral. Furthermore, by Theorem 2.3.6. if a function is Riemann integrable on two contiguous intervals, then it is integrable on their union. It follows that a function which is integrable on each subinterval in a partition of $[a,b]$ will be integrable on $[a,b]$.

The above discussion settles the question of the Riemann integrability of the integrand in the following definition.

**Definition 2.3.8.** Let $\gamma:[a,b]\to\mathbb{C}$ be a path and let $f$ be a function which is defined and continuous on a set $E$ which contains $\gamma([a,b])$. Then we define the integral of $f$ over $\gamma$ to be

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt. \tag{2.3.3}$$

One may think of this definition in the following way: the contour integral on the left in (2.3.3) is defined to be the Riemann integral obtained by replacing $z$ by $\gamma(t)$ and $dz$ by $\gamma'(t)dt$ and integrating over the parameter interval for $\gamma$.

In practice, we will calculate contour integrals by breaking the path up into its smooth sections, calculating the integrals over these sections and then adding the results. That this is legitimate follows from the fact that the Riemann integral of a function over the union of two contiguous intervals on the line is the sum of the integrals over the two intervals.

## Examples

**Example 2.3.9.** Find $\int_\gamma z\,dz$ if $\gamma$ is the circular path defined in Example 2.3.2.

**Solution:** By Example 2.3.2, we have $\gamma(t) = re^{it}$ for $0 \le t \le 2\pi$ and $\gamma'(t) = ire^{it}$. Thus,

$$\int_\gamma z\,dz = \int_0^{2\pi} re^{it} ire^{it}\,dt = ir^2 \int e^{2it}\,dt = ir^2 \int_0^{2\pi} (\cos 2t + i\sin 2t)\,dt$$
$$= ir^2 \int_0^{2\pi} \cos 2t\,dt - r^2 \int_0^{2\pi} \sin 2t\,dt = 0.$$

**Example 2.3.10.** Find a path $\gamma$ which traces the straight line from 0 to $i$ followed by the straight line from $i$ to $i+1$. Then calculate $\int_\gamma z^2\,dz$ for this path $\gamma$.

**Solution:** We may choose $\gamma$ to be the path parameterized on $[0,2]$ as follows:

$$\gamma(t) = \left\{ \begin{array}{cc} it & \text{if} \quad 0 \le t \le 1 \\ i+t-1 & \text{if} \quad 1 \le t \le 2 \end{array} \right\}.$$

We calculate the integrals over each of the two smooth sections of the path. On $[0,1]$ we have $(\gamma(t))^2 = -t^2$ and $\gamma'(t) = i$. Thus, the integral over the first section of the path is

$$\int_0^1 (\gamma(t))^2 \gamma'(t)\,dt = \int_0^1 -t^2 i\,dt = -t^3 i/3\Big|_0^1 = -i/3.$$

On $[1,2]$ we have $(\gamma(t))^2 = t^2 - 2t + 2(t-1)i$ and $\gamma'(t) = 1$. Thus, the integral over the second section of the path is

$$\int_1^2 (\gamma(t))^2 \gamma'(t)\,dt = \int_0^1 (t^2 - 2t + 2(t-1)i)\,dt = (t^3/3 - t^2 + (t^2 - 2t)i)\Big|_1^2 = -2/3 + i.$$

Thus, $\displaystyle\int_\gamma z^2\,dz = -i/3 - 2/3 + i = -2/3 + 2i/3.$

**Example 2.3.11.** Find a path $\gamma$ which traces once around the triangle with vertices $0, 1, i$ in the counter-clockwise direction, starting at 0. For this path $\gamma$, find $\int_\gamma \overline{z}\,dz$.

**Solution:** A path $\gamma$ with the required properties has parameter interval $[0,3]$ and is given by

$$\gamma(t) = \begin{Bmatrix} t & \text{if} & 0 \le t \le 1 \\ 2 - t + (t-1)i & \text{if} & 1 \le t \le 2 \\ (3-t)i & \text{if} & 2 \le t \le 3 \end{Bmatrix}.$$

On the interval $[0,1]$, we have $\overline{\gamma(t)} = t$ and $\gamma'(t) = 1$. Hence,

$$\int_0^1 \overline{\gamma(t)}\gamma'(t)\,dt = \int_0^1 t\,dt = 1/2.$$

On the interval $[1,2]$, we have $\overline{\gamma(t)} = 2 - t - (t-1)i$ and $\gamma'(t) = -1 + i$. Hence,

$$\int_1^2 \overline{\gamma(t)}\gamma'(t)\,dt = \int_1^2 (3-i)\,dt = 3 - i.$$

On the interval $[2,3]$, we have $\overline{\gamma(t)} = t - 3$ and $\gamma'(t) = -i$. Hence,

$$\int_0^1 \overline{\gamma(t)}\gamma'(t)\,dt = \int_2^3 (3-t)i\,dt = i/2.$$

If we add the contributions of each of these three intervals, the result is

$$\int_\gamma \overline{z}\,dz = 1/2 + 3 - i + i/2 = 7/2 - i/2.$$

## Exercise Set 2.3

1. Find $\int_0^\pi \mathrm{e}^{it}\,dt$.
2. Find $\int_0^1 \sin(it)\,dt$.
3. Find $\int_0^{2\pi} \mathrm{e}^{in}\mathrm{e}^{mi}\,dt$ for all integers $n$ and $m$.
4. Find a path which traces the straight line joining $2-i$ to $-1+3i$.
5. If $z_0 \in \mathbb{C}$, find a path which traces the circle of radius $r$, centered at $z_0$, (a) once in the counter-clockwise direction, (b) once in the clockwise direction, (c) three times in the counter-clockwise direction.
6. Prove that if $\gamma(t) = x(t) + iy(t)$ is a curve defined on an interval $I$, with real and imaginary parts $x(t)$ and $y(t)$, and if $c \in I$, then $\gamma'(c)$ exists if and only if $x'(c)$ and $y'(c)$ exist and, in this case, $\gamma'(c) = x'(c) + iy'(c)$.
7. Show that if $f$ is a smooth complex valued function on an interval $[a,b]$, then $\int_a^b f'(t)\,dt = f(b) - f(a)$.
8. Suppose $\gamma$ is a path with parameter interval $[a,b]$. Use the result of the previous exercise to show that $\int_\gamma 1\,dz = \gamma(b) - \gamma(a)$.

9. Find $\int_\gamma z^2\,dz$ if $\gamma$ traces a straight line from 0 to $w$.
10. Find $\int_\gamma 1/z\,dz$ and $\int_\gamma \overline{z}\,dz$ for the circular path $\gamma(t) = 3\mathrm{e}^{it}$, $0 \le t \le 2\pi$.
11. Find $\int_\gamma \mathrm{Re}\,(z)\ dz$ if $\gamma$ is the path of example 2.3.11.
12. With $\gamma$ as in the previous exercise, find $\int_\gamma \mathrm{Im}\left(z^2\right)\ dz$.
13. Is it generally true that $\mathrm{Re}\left(\int_\gamma f(z)\,dz\right) = \int_\gamma \mathrm{Re}\,(f(z))\ dz$?

## 2.4 Properties of Contour Integrals

We begin this section with the question of parameter independence. To what extent does the integral of a function along a path depend on how the path is parameterized? The same geometric figure $\gamma(I)$ may be parameterized in many ways. For example, the top third of the unit circle may be parameterized by

$$\begin{aligned} \gamma_1(t) &= -t + i\sqrt{1-t^2}, \quad -\sqrt{3}/2 \le t \le \sqrt{3}/2, \quad \text{or} \\ \gamma_2(t) &= \mathrm{e}^{it} = \cos t + i \sin t, \quad \pi/6 \le t \le 5\pi/6, \end{aligned} \tag{2.4.1}$$

and these are only two of infinitely many possibilities. Does the integral of a function over the upper third of the unit circle depend on which of these parmeterizations is chosen?

### Parameter Changes that Change the Integral

The following example shows that some changes of parameterization do change the integral.

**Example 2.4.1.** Find $\int_{\gamma_1} 1/z\,dz$ if $\gamma_1(t) = r\mathrm{e}^{it}$ on $[0, 2\pi]$ is the circular path of Example 2.3.2. Does the answer change if the circle is traversed in the clockwise direction instead, using the path $\gamma_2(t) = r\mathrm{e}^{-it}$ on $[0, 2\pi]$?

**Solution:** From Example 2.3.2 we know that the path $\gamma_1(t) = r\mathrm{e}^{it}$ has $\gamma_1'(t) = ir\mathrm{e}^{it}$ and so the given integral is

$$\int_{\gamma_1} \frac{dz}{z} = \int_0^{2\pi} \frac{(r\mathrm{e}^{it})'}{r\mathrm{e}^{it}} dt = \int_0^{2\pi} i\,dt = 2\pi i.$$

On the other hand, the derivative of $\gamma_2 = \mathrm{e}^{-it}$ is $-ir\mathrm{e}^{-it}$ and so

$$\int_{\gamma_2} \frac{dz}{z} \int_0^{2\pi} -i\,dt = -2\pi i.$$

This example shows that the integral along a path depends not only on the geometric figure that is the image $\gamma(I)$ of the path, but also on the direction the path is traversed (at the very least).

Also, traversing a portion of the curve more than once may affect the integral. For example, if we were to go around the circle twice in Example 2.4.1, by choosing $\gamma(t) = \mathrm{e}^{2it}$ on $[0, 2\pi]$, the result would be $4\pi i$ instead of $2\pi i$.

## The Independence of Parameterization Theorem

There is a degree to which the integral is independent of the parameterization. Certain ways of changing the parameterization do not effect the integral, as the following theorem shows.

**Theorem 2.4.2.** *Let* $\gamma_1 : [a,b] \to \mathbb{C}$ *be a path and* $\alpha : [c,d] \to [a,b]$ *a smooth function with* $\alpha(c) = a$ *and* $\alpha(d) = b$*. If* $\gamma_2$ *is the path with parameter interval* $[c,d]$ *defined by* $\gamma_2(t) = \gamma_1(\alpha(t))$*, then*

$$\int_{\gamma_2} f(z)dz = \int_{\gamma_1} f(z)dz$$

*for every function* $f$ *defined and continuous on a set* $E$ *containing* $\gamma_1([a,b]) = \gamma_2([c,d])$*.*

*Proof.* We have $\gamma_2(t) = \gamma_1(\alpha(t))$ and, by the chain rule,

$$\gamma_2'(t) = \gamma_1'(\alpha(t))\alpha'(t).$$

Thus,

$$\begin{aligned}
\int_{\gamma_2} f(z)\,dz &= \int_c^d f(\gamma_2(t))\gamma_2'(t)\,dt \\
&= \int_c^d f(\gamma_1(\alpha(t)))\gamma_1'(\alpha(t))\alpha'(t)\,dt \\
&= \int_a^b f(\gamma_1(s))\gamma_1'(s)\,ds \\
&= \int_{\gamma_1} f(z)\,dz,
\end{aligned}$$

where the third equality follows from the substitution $s = \alpha(t)$. This completes the proof. ☐

Note that the condition that $\alpha(c) = a$ and $\alpha(d) = b$ is essential in the above theorem. It says that $\alpha$ takes the endpoints of the parameter inverval $[c,d]$ to the endpoints of the parameter interval $[a,b]$ in an order preserving fashion.

**Example 2.4.3.** Are the integrals of a continuous function over the two paths in (2.4.1) necessarily the same?

**Solution:** Yes. If we set $\alpha(t) = -\cos t$, then $\alpha$ is a smooth function mapping the parameter interval $[\pi/6, 5\pi/6]$ to the parameter interval $[-\sqrt{3}/2, \sqrt{3}/2]$ in an order preserving fashion. Furthermore, $\gamma_2 = \gamma_1 \circ \alpha$. Thus, the above theorem insures that the integral of a continuous function over $\gamma_1$ is the same as its integral over $\gamma_2$.

Doesn't Example 2.4.1 contradict Theorem 2.4.2? After all, if $\gamma_1(t) = \mathrm{e}^{it}$ on $[0, 2\pi]$ and $\alpha : [0, 2\pi] \to [0, 2\pi]$ is defined by $\alpha(t) = 2\pi - t$, then $\gamma_2(t) =$

$\gamma_1(\alpha(t)) = e^{-it}$. By Example 2.4.1 the integrals of $1/z$ over these two curves are different. Doesn't Theorem 2.4.2 say they should be the same? No. The conditions $\alpha(a) = c$ and $\alpha(b) = d$ are not satisfied by this choice of $\alpha$, since $\alpha(0) = 2\pi$ and $\alpha(2\pi) = 0$. In other words, this choice of $\alpha$ reverses the order of the endpoints of the parameter interval rather than preserving that order.

In general, the conditions $\alpha(a) = c$ and $\alpha(b) = d$ guarantee that, overall, $\gamma_2$ traverses the curve in the same direction as $\gamma_1$. If $\alpha'$ were positive on the entire interval, then $\alpha$ would be increasing on this interval and $\gamma_1$ and $\gamma_2$ would be moving in the same direction at each point of the curve. If $\alpha'$ is not positive on all of $[c, d]$, then there may be intervals where one path reverses direction and backtracks, while the other path does not. These things don't affect the integral, because if a curve does backtrack for a time, it has to turn around and recover the same ground in order to catch up to the other curve in the end. This is an intuitive explanation; the actual proof that the integral is unaffected is in the proof of the above theorem.

Theorem 2.4.2 leads to a strategy which, for some paths $\gamma_1$ and $\gamma_2$ with the same image, yields a proof that they determine the same integral: Suppose that the parameter intervals for the two paths can each be partitioned into $n$ subintervals in such a way that for $j = 1, \cdots, n$, $\gamma_1$ on its $j$th subinterval and $\gamma_2$ on its $j$th subinterval are related by a smooth function $\alpha_j$, as in Theorem 2.4.2. If this can be done, then it clearly follows that $\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$ for any function $f$ which is continuous on a set containing $\gamma_1(I)$. For this reason, Theorem 2.4.2 is sometimes called the *independence of parameterization theorem.*

**Remark 2.4.4.** Since path integrals are essentially independent of the way the path is parameterized, we will often describe a path without specifying a parameterization. Instead, we will just give a description of the geometric object that is traced, the direction, and how many times. For example, we may describe a path as tracing once around the unit circle in the counterclockwise direction, or tracing once around the boundary $\partial\Delta$ of a given triangle $\Delta$ in the counterclockwise direction, or as tracing the straight line path from a complex number $w_1$ to a complex number $w_2$. In the first two cases we may simply write

$$\int_{|z|=1} f(z)\,dz \quad \text{or} \quad \int_{\partial\Delta} f(z)\,dz$$

for the corresponding path integral. In the latter case, we may write

$$\int_{w_1}^{w_2} f(z)\,dz$$

for the path integral along the straight line from $w_1$ to $w_2$.

## Closed Curves

The curves in examples 2.3.2 and 2.3.4 both have the property that they begin and end at the same point – that is, they are closed paths. A closed path $\gamma$ on a parameter interval $[a, b]$ is one that satisfies $\gamma(a) = \gamma(b)$.

The famous integral theorem of Cauchy states that the integral of an analytic function $f$ around a closed path is 0, provided there is an appropriate relationship between the curve $\gamma$ and the domain $U$ on which $f$ is analytic (roughly speaking, the curve should lie in $U$ but not go around any *holes* in $U$). Since the function $f(z) = z$ is analytic on $\mathbb{C}$ (as is any polynomial in $z$) the next example illustrates this phenomenon.

**Example 2.4.5.** Find $\int_\gamma z\,dz$ if $\gamma$ is the path of Example 2.3.4.

**Solution:** From Example 2.3.4 we know that the path $\gamma(t)$ has values $4t, 1+(4t-1)i, 3-4t+i, (4-4t)i$ and derivatives $4, 4i, -4, and -4i$ on the four subintervals of the partition $0 < 1/4 < 1/2 < 3/4 < 1$. Thus, the integrals over the four smooth pieces of our curve are

$$\int_0^{1/4} 4t \cdot 4\,dt = 8t^2\Big|_0^{1/4} = 1/2$$

$$\int_{1/4}^{1/2} (1+(4t-1)i) \cdot 4i\,dt = (4ti - 8t^2 + 4t)\Big|_{1/4}^{1/2} = i - 1/2$$

$$\int_{1/2}^{3/4} (3-4t+i) \cdot (-4)\,dt = (-12t + 8t^2 - 4ti)\Big|_{1/2}^{3/4} = -1/2 - i$$

$$\int_{3/4}^{1} (4-4t)i \cdot (-4i)\,dt = (+16t - 8t^2)\Big|_{3/4}^{1} = 1/2.$$

Since these add up to 0, we have $\displaystyle\int_\gamma z\,dz = 0$.

The function $1/z$ is also analytic, except at $z = 0$. The circular path of Example 2.4.1 is closed and lies in the domain where $1/z$ is analytic. So why is the integral not 0? Because the path goes around a hole in the domain of $1/z$ – it goes around 0.

## Additivity Properties of Contour Integrals

If $\gamma$ is a path with parameter interval $[a, b]$, then we can use Theorem 2.4.2 to change the parameter interval to any other interval $[c, d]$ with $c < d$, in a way that does not affect the image of $\gamma$ or integrals over $\gamma$. In fact, if we set

$$\alpha(t) = a + \frac{b-a}{d-c}(t-c),$$

then $\alpha$ is smooth, $\alpha([c, d]) = [a, b]$, $\alpha(c) = a$ and $\alpha(d) = b$. Thus, $\gamma_1(t) = \gamma(\alpha(t))$ defines a path $\gamma_1$ with the same image as $\gamma$ and, by Theorem 2.4.2, a path which determines the same integral for continuous functions on its image. Thus, without loss of generality, we may always assume that the parameter interval for a path is any interval we choose.

If $\gamma_1$ and $\gamma_2$ are two paths so that $\gamma_1$ ends where $\gamma_2$ begins, then we can join the two paths to form a single new path $\gamma_1 + \gamma_2$. We do this as follows: If $\gamma_1$

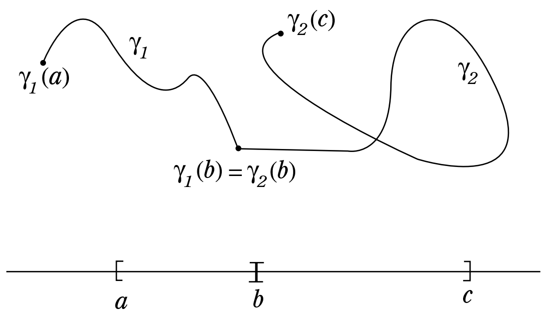


Figure 2.5: The Join of Two Paths.

has parameter interval $[a, b]$, we choose a parameter interval of the form $[b, c]$ for $\gamma_2$. The fact that $\gamma_2$ begins where $\gamma_1$ ends means that $\gamma_1(b) = \gamma_2(b)$. We define $\gamma_1 + \gamma_2$ on $[a, c]$ by

$$(\gamma_1 + \gamma_2)(t) = \begin{Bmatrix} \gamma_1(t) & t \in [a, b] \\ \gamma_2(t) & t \in [b, c] \end{Bmatrix}. \tag{2.4.2}$$

The path $\gamma_1 + \gamma_2$ is called the *join* of $\gamma_1$ and $\gamma_2$.

In Example 2.4.1, changing the path from one tracing the circle counterclockwise to one tracing the circle clockwise had the effect of changing the sign of the integral. As we shall see, this always happens. If $\gamma : [a, b] :\rightarrow \mathbb{C}$ is a path, denote by $-\gamma$ the path defined by

$$-\gamma(t) = \gamma(a + b - t).$$

Then $-\gamma(a) = \gamma(b)$ and $-\gamma(b) = \gamma(a)$. In fact, $-\gamma$ traces the same geometric figure as $\gamma$, but it does so in the opposite direction.

For some closed curves, such as circles, and boundaries of rectangles, triangles, etc., there is clearly a clockwise direction around the curve and a counterclockwise direction. If such a curve is parameterized so that that it is traversed in the counterclockwise direction, we will say the resulting closed path has *positive orientation.* If it is traversed in the clockwise direction, we will say it has *negative orientation.* Clearly, if $\gamma$ has positive orientation, then $-\gamma$ has negative orientation.

The next theorem states the elementary properties of path integrals having to do with linearity and path additivity. Part (b) follows immediately from the corresponding additivity property of the Riemann integral on the line and we have already used it several times. We leave the proofs of (a) and (c) to the exercises (Exercise 2.4.5).

**Theorem 2.4.6.** *Let $\gamma, \gamma_1, \gamma_2$ be paths with $\gamma_1$ ending where $\gamma_2$ begins, $f$ and $g$ two functions which are continuous on a set $E$ containing the images of these paths, and $a$ and $b$ complex numbers. Then* *(a)* $\displaystyle\int_\gamma (af(z) + bg(z))\, dz = a \int_\gamma f(z)\, dz + b \int_\gamma g(z)\, dz;$

*(b)* $\displaystyle\int_{\gamma_1+\gamma_2} f(z)\, dz = \int_{\gamma_1} f(z)\, dz + \int_{\gamma_2} f(z)\, dz;$

*(c)* $\displaystyle\int_{-\gamma} f(z)\, dz = -\int_\gamma f(z)\, dz.$

Part (a) of this theorem says that a path integral is a linear function of the integrand, Part (b) says that it is an additive function of the path, while Part (c) shows why the notation $-\gamma$ is appropriate for the curve that is $\gamma$ traversed in the opposite direction.

## Length of a Path

We define the length $\ell(\gamma)$ of a path $\gamma$ in $\mathbb{C}$ in the same way the length of a curve in $\mathbb{R}^2$ is defined in calculus.

**Definition 2.4.7.** If $\gamma(t) = x(t) + iy(t)$ is a path in $\mathbb{C}$ with parameter interval $[a, b]$, then the length $\ell(\gamma)$ of $\gamma$ is defined to be

$$\ell(\gamma) = \int_a^b |\gamma'(t)|\, dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}, dt.$$

**Example 2.4.8.** Prove that the above definition of length yields the correct length for a path which traces once around a circle of radius r.

**Solution:** The path is $\gamma(t) = r\mathrm{e}^{it}$, with parameter interval $[0, 2\pi]$. The derivative of $\gamma$ is $\gamma'(t) = ir\mathrm{e}^{it}$ and so $|\gamma'(t)| = r$. Thus, $\ell(\gamma) = \int_0^{2\pi} r\, dt = 2\pi r$.

It will be important in coming sections to be able to obtain good upper bounds on the absolute value of a path integral. The key theorem that produces such upper bounds is the following.

**Theorem 2.4.9.** *Let $\gamma$ be a path in $\mathbb{C}$ and $f$ a function continuous on a set containing $\gamma(I)$. If $|f(z)| \le M$ for all $z \in \gamma(I)$, then*

$$\left|\int_\gamma f(z)\, dz\right| \le M\ell(\gamma).$$

*Proof.* If the parameter interval for $\gamma$ is $[a, b]$, then

$$\left|\int_\gamma f(z)\, dz\right| = \left|\int_a^b f(\gamma(t))\gamma'(t)\, dt\right| \le \int_a^b |f(\gamma(t))\gamma'(t)|\, dt$$
$$\le \int_a^b M|\gamma'(t)|\, dt = M\int_a^b |\gamma'(t)|\, dt = M\ell(\gamma).$$

□

The next example is a typical application of this theorem.

**Example 2.4.10.** Show that if $f$ is a bounded continuous function on $\mathbb{C}$, and $\gamma_R$ is the path $\gamma_R(z) = Re^{it}$ for $t \in [0, 2\pi]$, then

$$\lim_{R\to\infty} \int_{\gamma_R} \frac{f(z)}{(z-w)^2}\, dz = 0. \tag{2.4.3}$$

for each $w \in \mathbb{C}$.

**Solution:** The statement that $f$ is bounded means there is an upper bound $M$ for $|f|$. That is, $|f(z)| \le M$ for all $z \in \mathbb{C}$. We also have $|z - w| \ge |z| - |w|$ by the second form of the triangle inequality. If $z \in \gamma(I)$, then $|z| = R$ and so $|z - w| \ge R - |w|$, which implies $|z - w|^{-2} \le (R - |w|)^{-2}$. Thus, for $z \in \gamma(I)$, we have the following bound on the integrand of (2.4.3):

$$\left|\frac{f(z)}{z-w}\right| \le \frac{M}{(R-|w|)^2}.$$

Since $\ell(\gamma_R) = 2\pi R$, Theorem 2.4.9 implies that

$$\left|\int_{\gamma_R} \frac{f(z)}{(z-w)^2}\, dz\right| \le \frac{2\pi MR}{(R-|w|)^2}.$$

The right side of this inequality has limit 0 as $R \to \infty$ and this implies (2.4.3).

## Exercise Set 2.4

1. Compute $\int_\gamma z^2\, dz$ if $\gamma$ is any path which traces once around the circle of radius one in the counterclockwise direction.

2. Compute $\int_\gamma 1/z\, dz$ if $\gamma$ is any path which traces twice around the circle of radius one, centered at 0, in the counterclockwise direction.

3. If $z_0$ and $w_0$ are two points of $\mathbb{C}$, compute $\int_\gamma z\, dz$ if $\gamma$ is any path which traces the straight line from $z_0$ to $w_0$ once.

4. Compute the integral of the previous exercise for any smooth path $\gamma$ which begins at $z_0$ and ends at $w_0$.

5. Prove Parts (a) and (c) of Theorem 2.4.6.

6. Describe a smooth, order preserving function $\alpha$ which takes the parameter interval $[0, 1]$ to the parameter interval $[2, 5]$.

7. Prove that a parameter change $\gamma \to \gamma \circ \alpha$, like the one in Theorem 2.4.2, does not change the length of a path provided $\alpha$ is an non-decreasing function (has a non-negative derivative).

8. Show that $\left|\int_\gamma \frac{\cos z}{z}\, dz\right| \le 2\pi e$ if $\gamma$ is a path that traces the unit circle once. Hint: show that $|\cos z| \le e$ if $|z| = 1$.

9. Show that if $\Delta$ is a triangle in the plane of diameter $d$ (length of its longest side), and if $f$ is a continuous function on $\Delta$ with $|f|$ bounded by $M$ on $\Delta$, then
$$\left|\int_{\partial\Delta} f(z)\,dz\right| \leq 3Md.$$

10. Prove that $\int_\gamma p(z)\,dz = 0$ if $\gamma(t) = \mathrm{e}^{it}$, $0 \leq t \leq 2\pi$, and $p(z)$ is any polynomial in $z$ (this is a special case of Cauchy's Theorem, but don't assume Cauchy's Theorem in your proof).

11. Let $R(z)$ be the remainder after $n$ terms in the power series for $\mathrm{e}^z$. That is,
$$R(z) = \mathrm{e}^z - \sum_{k=1}^{n} \frac{z^k}{k!} = \sum_{k=n+1}^{\infty} \frac{z^k}{k!}.$$
Prove that $|R(z)| \leq \dfrac{e-1}{(n+1)!}$ if $|z| \leq 1$.

12. Prove that $\int_\gamma \mathrm{e}^z\,dz = 0$ if $\gamma(t) = \mathrm{e}^{it}$, $0 \leq t \leq 2\pi$ using the previous exercise and Exercise 10.

## 2.5 Cauchy's Theorem for a Triangle

The core material of any beginning Complex Variables text is the proof of Cauchy's Theorem and the exploration of its consequences. Roughly speaking, Cauchy's Theorem states that the integral of an analytic function around a closed path is zero, provided the path is contained in the open set $U$ on which the function is analytic and does not go around any "holes" in $U$. Part of the problem here is to make sense of the idea of a "hole" in an open set and to decide what it means for a path to go around such a hole.

We take a first step toward the proof of Cauchy's Theorem in this section by proving it in the case where the path is the boundary of a triangle and the function is analytic on an open set containing the triangle.

The proof of this result will make essential use of a couple of properties of compact sets. Thus, we will precede the proof with a discussion of compact sets.

### Compact Sets

A subset $K$ of $\mathbb{R}^n$ is called *compact* if every open cover of $K$ has a finite subcover. Here, by an open cover of $K$, we mean a collection of open sets whose union contains $K$. A finite subcover is then a finite subcollection of this collection which also has union containing $K$.

Here we will state without proof a number of facts about compact sets. The proofs can be found in any text on Advanced Calculus or undergraduate Real Analysis.

**Theorem 2.5.1. (Heine-Borel Theorem)** *A subset of* $\mathbb{R}^n$ *is compact if and only if it is closed and bounded.*

Another characterization of compact sets is obtained by using complementation to turn the statement about open covers in the definition into a statement about collections of closed sets. The result is the following:

A set $K$ is compact if and only if for every collection of closed subsets of $K$ with empty intersection, there is a finite subcollection with empty intersection. The same thing stated somewhat differently is:

**Theorem 2.5.2.** *A set* $K$ *is compact if and only if, whenever a collection of closed subsets of* $K$ *has the property that each finite subcollection has non-empty intersection, then the full collection also has non-empty intersection.*

A consequence of the previous theorem is the following:

**Corollary 2.5.3.** *If*

$$A_1 \supset A_2 \supset \cdots \supset A_n \supset \cdots$$

*is a nested sequence of non-empty compact subsets of* $\mathbb{R}^n$*, then* $\bigcap A_n \neq \emptyset$.

This is one of the properties of compact sets that we shall need in our proof of Cauchy's Theorem on a triangle. The others are as follows:

**Theorem 2.5.4.** *If* $f$ *is a continuous function, defined and continuous on a compact subset* $K$ *of* $\mathbb{R}^n$*, with values in* $\mathbb{R}^m$*, then* $f(K)$ *is also compact.*

A non-empty compact subset of $\mathbb{R}$ contains both a maximal element and a minimal element. If $f$ is a real valued function, defined and continuous on a compact set $K$, then $f(K)$ is a compact subset of the line and, hence, has maximal and minimal elements. This proves the following corollary of the previous theorem.

**Corollary 2.5.5.** *A continuous real valued function on a compact set has a maximal value and a minimal value.*

## Antiderivatives

There is one case in which it is very easy to prove that the integral of a continuous function $f$ around a closed path is 0. This is the case where the function $f$ has an antiderivative.

As with functions of a real variable, an antiderivative for a function $f$, defined on an open set $U$, is a function $g$ such that $g' = f$ on $U$.

**Theorem 2.5.6.** *If* $f$ *is a continuous function defined on an open set* $U$*, and if* $f$ *has an antiderivative* $g$ *on* $U$*, then*

$$\int_\gamma f(z)\,dz = g(\gamma(b)) - g(\gamma(a))$$

*if* $\gamma$ *is any path in* $U$ *with parameter interval* $[a, b]$*. If* $\gamma$ *is a closed path, then this integral is* $0$*.*

*Proof.* Since $g' = f$ on $U$, we have

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt = \int_a^b g'(\gamma(t))\gamma'(t)\,dt.$$

There is a version of the chain rule which holds for the composition of an analytic function with a path (Exercise 2.5.4). It tells us that

$$(g(\gamma(t)))' = g'(\gamma(t))\gamma'(t).$$

Thus,

$$\int_\gamma f(z)\,dz = \int_a^b (g(\gamma(t)))'\,dt = g(\gamma(b)) - g(\gamma(a)),$$

by the complex version of the Fundamental Theorem of Calculus (Exercise 2.3.7). If the path is closed, then $\gamma(b) = \gamma(a)$ and the integral is 0. □

## Cauchy's Theorem

The fact that a function $f$ has a complex derivative at $w$ means that, near $w$, it can be approximated by a linear function of $z$ (more precisely, by a polynomial of degree one in $z$). Such a function has a complex antiderivative, as does any polynomial, and so its integral around a closed curve is zero. Thus, a function with a complex derivative at $w$ can be approximated near $w$ by a function whose integral around any closed path is zero. This is the basis for an argument that the integral of a function with a complex derivative at $w$, around a small triangle containing $w$, is much smaller than one would predict (using, for example, the estimate given in Exercise 2.3.9). This is made precise in the next lemma, which is the basis for the proof of Cauchy's Theorem for triangles.

**Lemma 2.5.7.** *Let $f$ be a function which is continuous on a neighborhood of $w \in \mathbb{C}$ and which has a complex derivative at $w$. Then for every $\epsilon > 0$, there is a $\delta > 0$ such that*

$$\left|\int_{\partial\Delta} f(z)\,dz\right| < \epsilon d^2$$

*if $\Delta$ is any triangle containing $w$, of diameter $d \le \delta$.*

*Proof.* Since $f$ is continuous on a neighborhood of $w$, we may choose an $r > 0$ such that $f$ is continuous on the open disc $D_r(w)$.

Since $f$ has a complex derivative at $w$,

$$\lim_{z\to w} \frac{f(z) - f(w)}{z - w} = f'(w)$$

exists. Thus, we may choose a positive $\delta < r$ such that $|z - w| < \delta$ implies that

$$\left|\frac{f(z) - f(w)}{z - w} - f'(w)\right| < \frac{\epsilon}{3}. \tag{2.5.1}$$

If we multiply (2.5.1) by $|z - w|$, the result is

$$|f(z) - f(w) - f'(w)(z - w)| < \frac{\epsilon}{3}|z - w| \quad \text{for all} \quad z \in D_\delta(w). \tag{2.5.2}$$

If $\Delta$ is a triangle of diameter $d \le \delta$, containing $w$, we set

$$I = \int_{\partial\Delta} f(z)dz.$$

Then

$$I = \int_{\partial\Delta} (f(w) + f'(w)(z - w))dz + \int_{\partial\Delta} (f(z) - f(w) - f'(w)(z - w))dz.$$

However,

$$\int_{\partial\Delta} (f(w) + f'(w)(z - w))dz = 0,$$

because the integrand has a complex anti-derivative: $f(w)z + f'(w)(z^2/2 - wz)$ (remember $w$ is a constant – the variable is $z$), and so

$$I = \int_{\partial\Delta} (f(z) - f(w) - f'(w)(z - w))dz. \tag{2.5.3}$$

Then (2.5.2) implies that

$$|I| < d^2\epsilon,$$

since the modulus of the integrand in (2.5.3) is bounded by $\epsilon d/3$ and the length of the path $\partial\Delta$ is no more than $3d$. This completes the proof. □

**Theorem 2.5.8.** *Let $f$ be a function which is analytic in an open set $U$, and suppose that $\Delta$ is a triangle contained in $U$. Let $\partial\Delta$ denote the boundary of $\Delta$, considered as a closed path with positive orientation. Then*

$$\int_{\partial\Delta} f(z)\, dz = 0.$$

*Proof.* We set

$$I = \int_{\partial\Delta} f(z)\, dz.$$

Our objective is to prove that $I = 0$. We will do this by showing that $|I| < \epsilon$ for every positive number $\epsilon$. Thus, let $\epsilon$ be an arbitrary positive number.

We subdivide the triangle $\Delta$ into four smaller triangles by joining the midpoints of the sides of $\Delta$. The resulting four triangles are all similar to $\Delta$ with sides exactly half as long as the corresponding sides of $\Delta$ (see Figure 2.6).

We now apply Parts(b) and (c) of Theorem 2.4.6. The sum of the integrals around each of these smaller triangles is a sum of integrals along their edges, with each of the edges interior to the original triangle occurring twice – once going one direction and once going the opposite direction. Thus, the contributions of these interior edges cancel, leaving only the contributions from the edges which

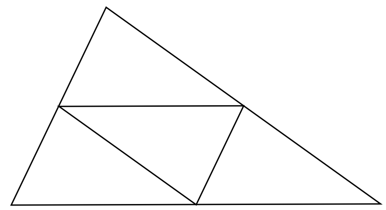

Figure 2.6: Subdividing the Triangle $\Delta$ in Theorem 2.5.8.

lie along the boundary of the original triangle. It follows that the sum of the integrals of $f$ around the boundaries of these four smaller triangles is $I$, and so one of these integrals must have modulus at least $|I|/4$. Let $\Delta_1$ denote the corresponding triangle. In other words, $\Delta_1$ is chosen from the four subtriangles so that

$$|I_1| \geq |I|/4 \quad \text{where} \quad I_1 = \int_{\partial\Delta_1} f(z)\,dz.$$

Note also that, if $h$ is the diameter of $\Delta$ (which is the length of its longest side), then $\Delta_1$ has diameter $h_1 = h/2$.

We now repeat the above construction with $\Delta$ replaced by $\Delta_1$. That is we subdivide $\Delta_1$ into four similar triangles and choose one of them, call it $\Delta_2$, with the property that

$$|I_2| \geq |I_1|/4 \geq |I|/4^2 \quad \text{where} \quad I_2 = \int_{\partial\Delta_2} f(z)\,dz,$$

and with diameter $h_2 = h_1/2 = h/2^2$.

Proceeding by induction, we may choose for each $n$ a triangle $\Delta_n$, of diameter $h_n$, so that

$$\Delta_n \subset \Delta_{n-1},$$

$$|I_n| \geq |I|/4^n \quad \text{where} \quad I_n = \int_{\partial\Delta_n} f(z)dz \tag{2.5.4}$$

and

$$h_n = h/2^n. \tag{2.5.5}$$

The collection $\{\Delta_n\}$ is a nested sequence of closed bounded non-empty sets in the plane (see Figure 2.7) and, hence, by Corollary 2.5.3, there is a point $w$ in the intersection $\bigcap_n \Delta_n$.

We now apply the previous lemma with $\epsilon/h^2$ replacing the $\epsilon$ of the lemma. We conclude there is a $\delta > 0$ such that the integral of $f$ around any triangle containing $w$, of diameter $d \leq \delta$, is less than $d^2\epsilon/h^2$.

We may then choose $n$ large enough that $h_n = h/2^n < \delta$. Since $h_n$ is the diameter of $\Delta_n$ and $w \in \Delta_n$, we have

$$|I_n| < \frac{h_n^2}{h^2}\epsilon.$$

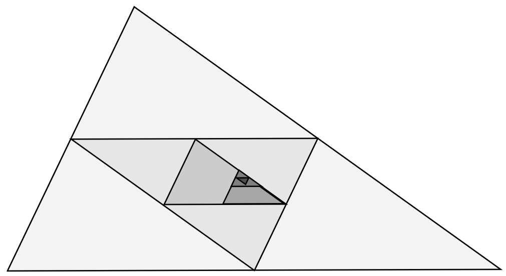

Figure 2.7: The Nested Sequence of Triangles in Theorem 2.5.8.

Putting this together with (2.5.5) we conclude

$$|I_n| < \frac{\epsilon}{4^n}.$$

Combined with (2.5.4), this yields

$$|I| \leq 4^n |I_n| < \epsilon.$$

Since $\epsilon$ was an arbitrary positive number, we conclude that $I = 0$. This completes the proof. □

We will need a slightly stronger version of this theorem in which we allow the possibility that there is one point in $\Delta$ where $f$ may not have a complex derivative, but where $f$ is continuous.

**Theorem 2.5.9.** *Let $f, U$, and $\Delta$ be as in the previous theorem except that we assume that $f$ is continuous on $U$ and analytic on $U \setminus \{c\}$ for some exceptional point $c \in \Delta$. Then we still have*

$$\int_{\partial\Delta} f(z)dz = 0.$$

*Proof.* If $c$ is a vertex of $\Delta$, then, given $\epsilon > 0$, we may subdivide $\Delta$ into smaller triangles in such a way that the one containing $c$ has circumference less than $\epsilon/M$, where $M$ is the maximum value of $|f|$ on $\Delta$ (Figure 2.8(a)). The integral of $f$ over the boundary of this triangle will then be less than or equal to $\epsilon$. The integrals of $f$ over the boundaries of other triangles in the subdivision are all 0 by the previous theorem (none of them contain $c$). As before, the integral of $f$ over $\partial\Delta$ is the sum of the integrals over the boudaries of the triangles in the subdivision and, hence, has modulus less than or equal to $\epsilon$. Since $\epsilon$ was arbitrary, we conclude that the integral of $f$ over $\partial\Delta$ is zero.

If $c$ is in $\Delta$ but is not a vertex, then the triangle can be subdivided into triangles which do contain $c$ as a vertex (Figure 2.8(b)). The integral around the boundary of each of these is zero, so the integral around $\partial\Delta$ is also 0. This completes the proof. □

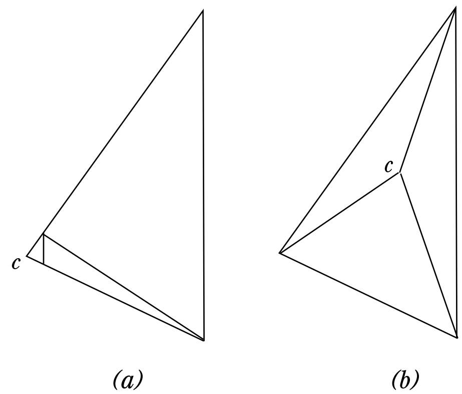


Figure 2.8: Dealing with an Exceptional Point $c$.

## Exercise Set 2.5

1. Prove the Bolzano-Weierstrass Theorem: If $K$ is a compact subset of $\mathbb{R}^n$, then every sequence in $K$ has a subsequence which converges to an element of $K$.

2. Use Corollary 2.5.5 to show that if $K$ is a compact subset of $\mathbb{C}$ and $f$ is a continuous complex valued function on $K$, then the modulus $|f(z)|$ of $f$ takes on a maximal value at some point of $K$.

3. Show that if $K$ is a compact subset of $\mathbb{C}$, then there is a point $z_0 \in K$ of minimum modulus – that is, a point $z_0 \in K$ such that
$$|z_0| \leq |z| \quad \text{for all} \quad z \in K.$$

4. Prove that if $g$ is analytic on an open subset $U$ of $\mathbb{C}$ and $\gamma : [a, b] \to U$ is a path in $U$, then
$$(g(\gamma(t)))' = g'(\gamma(t))\gamma'(t)$$
for $t \in [a, b]$. Hint: the proof is very similar to the proof of Theorem 2.2.7.

5. Calculate $\int_\gamma z^n\, dz$ if $n$ is a non-negative integer and $\gamma$ is a path in the plane joining the point $z_0$ to the point $w_0$. Hint: Use Theorem 2.5.6.

6. Show that $\int_\gamma p(z)\, dz = 0$ if $\gamma$ is any closed path in the plane and $p$ is any polynomial.

7. Calculate $\int_\gamma 1/z\, dz$ if $\gamma$ is any path joining $-i$ to $i$ which lies in the right half plane $\{z \in \mathbb{C} : \mathrm{Re}\,(z) \geq 0\}$ . Hint: use the result of Example 2.2.11 and Theorem 2.5.6.

8. Using the same hint as in the previous exercise, show that

$$\int_\gamma \frac{1}{z}\,dz = 0$$

if $\gamma$ is any closed path contained in the complement of the set of non-negative real numbers. Compare this with Example 2.4.1.

9. If $\sqrt{z}$ is defined by $\sqrt{z} = \mathrm{e}^{(\log z)/2}$ for the branch of the log function defined by the condition $-\pi/2 \le \arg(z) \le 3\pi/2$, find an antiderivative for $\sqrt{z}$ and then find $\int_\gamma \sqrt{z}\,dz$ where $\gamma$ is any path from $-1$ to 1 which lies in the upper half plane.

10. Prove that if $f$ is analytic in an open set containing a rectangle $R$, then the path integral of $f$ around the boundary of this rectangle is 0.

11. Let $\gamma$ be the path which traces the straight line from 1 to $1+i$, then the straight line from $1+i$ to $i$ and then the straight line from $i$ to 0. Calculate $\int_\gamma z^n\,dz$.

12. Let $\Delta$ be the triangle with vertices $1-i, i, \text{and} -1-i$ and $S$ be the square with vertices $1-i, 1+i, -1+i, \text{and} -1-i$. If $f$ is any function which is analytic on $\mathbb{C}\setminus\{0\}$, prove that

$$\int_{\partial\Delta} f(z)\,dz = \int_{\partial S} f(z)\,dz.$$

where $\partial\Delta$ and $\partial S$ are traversed in the counterclockwise direction.

13. For any pair of points $a, b$ in $\mathbb{C}$, denote the integral of a function $f$ along the straight line segment joining $a$ to $b$ by $\int_a^b f(z)\,dz$, as in Remark 2.4.4. Suppose $f$ is analytic in an open set containing the triangle with vertices $a, b, c$. Show that

$$\int_a^c f(z)\,dz - \int_a^b f(z)\,dz = \int_b^c f(z)\,dz.$$

14. Show that Theorem 2.5.9 can be strengthend to conclude that the integral of $f$ around any triangle in $U$ is 0 if $f$ is continuous on $U$ and analytic on $U \setminus I$ where $I$ is an interval contained in $U$. Hint: first consider the case where one side of the triangle lies along the interval $I$.

15. If $f$ is analytic on an open set $U$, then the integral of $f$ around the boundary of any triangle in $U$ is 0 (Theorem 2.5.8), as is its integral around the boundary of any rectangle in $U$ (Exercise 2.5.10). What other geometric figures have this property? What is the most general theorem along these lines you can think of?

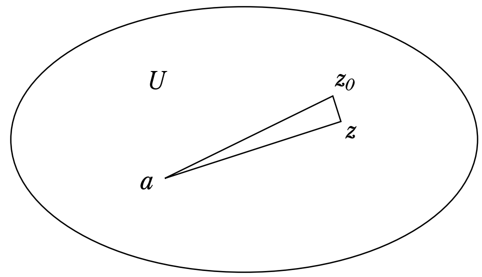


Figure 2.9: The Triangle $\Delta$ of Theorem 2.6.1.

## 2.6 Cauchy's Theorem for a Convex Set

A convex set $C$ in $\mathbb{C}$ is a set with the property that if $a$ and $b$ are points in $C$ then the line segment joining $a$ and $b$ is also contained in $C$.

### Existence of Antiderivatives

The strategy for proving Cauchy's Theorem for convex sets is to prove that every analytic function on a convex set has an antiderivative and then apply Theorem 2.5.6. The first step in this program is accomplished with the following theorem

**Theorem 2.6.1.** *Let $U$ be a convex open set and suppose $f$ is a function which is continuous on $U$ and has the property that its integral around the boundary of any triangle in $U$ is zero. If $a \in U$ is fixed and $F(z)$ is defined for all $z \in U$ by*

$$F(z) = \int_a^z f(w)\, dw,$$

*then $F'(z) = f(z)$ for all $z \in U$.*

*Proof.* Let $[a, z]$ denote the line segment joining $a$ to $z$, considered as a path from $a$ to $z$. For $z \in U$, this line segment lies entirely in $U$ and so we may define a function $F(z)$ by

$$F(z) = \int_a^z f(w)\, dw,$$

where by this we mean the path integral of $f$ along the path $[a, z]$, as in Remark 2.4.4. We will show that $F'(z) = f(z)$. To do this, we let $z$ and $z_0$ be points of $U$ and consider the triangle $\Delta$ with vertices $a, z, z_0$ (see Figure 2.9). The fact that $U$ is convex implies that $\Delta \subset U$.

Let $\partial\Delta$ denote the boundary of $\Delta$, considered as a contour which goes from $a$ to $z$ to $z_0$ and then to $a$ again. Since, by hypothesis, the integral of $f$ around the boundary of any triangle in $U$ is 0, we have

$$\begin{aligned} 0 &= \int_{\partial\Delta} f(w)\,dw \\ &= \int_a^z f(w)\,dw + \int_z^{z_0} f(w)\,dw + \int_{z_0}^a f(w)\,dw \\ &= F(z) - F(z_0) - \int_{z_0}^z f(w)\,dw. \end{aligned}$$

We conclude that

$$F(z) - F(z_0) = \int_{z_0}^z f(w)\,dw.$$

If we add and subtract the number $f(z_0)$ in the integrand of this integral, we obtain

$$\begin{aligned} F(z) - F(z_0) &= \int_{z_0}^z f(z_0)\,dw + \int_{z_0}^z (f(w) - f(z_0))\,dw \\ &= f(z_0)(z - z_0) + \int_{z_0}^z (f(w) - f(z_0))\,dw. \end{aligned}$$

If we divide by $z - z_0$ and subtract the first term on the right from both sides we get

$$\frac{F(z) - F(z_0)}{z - z_0} - f(z_0) = \frac{1}{z - z_0} \int_{z_0}^z (f(w) - f(z_0))\,dw. \tag{2.6.1}$$

Thus, to finish the proof that $F'(z_0) = f(z_0)$, we just need to show that the expression on the right in (2.6.1) has limit 0 as $z \to z_0$.

Given $\epsilon > 0$, we may choose a $\delta > 0$ such that $|f(w) - f(z_0)| < \epsilon$ when $|w - z_0| < \delta$. This follows from the fact that $f$ is continuous on $U$. If $|z - z_0| < \delta$ then $|w - z_0| < \delta$ for every $w$ on the line segment $[z_0, z]$ and so $|f(w) - f(z_0)| < \epsilon$ for every $w$ on this line segment. Then

$$\left| \int_{z_0}^z (f(w) - f(z_0))\,dw \right| < \epsilon |z - z_0|$$

and so

$$\left| \frac{1}{z - z_0} \int_{z_0}^z (f(w) - f(z_0))\,dw \right| < \epsilon.$$

This shows that

$$\lim_{z \to z_0} \frac{1}{z - z_0} \int_{z_0}^z (f(w) - f(z_0))\,dw = 0,$$

which completes the proof. □

## Cauchy's Integral Theorem

We now have all the tools in place to prove Cauchy's Integral Theorem for convex sets. This is not the most general form of the theorem – that will come later – but it is sufficiently general to allow us to derive a wealth of surprising consequences.

**Theorem 2.6.2.** *Let $U$ be a convex open set and suppose $f$ is a function which is analytic on $U$, except possibly at one point, where it is at least continuous. Then*

$$\int_\gamma f(z)\,dz = 0$$

*for every closed path in $U$.*

*Proof.* By the previous theorem, and Theorem 2.5.8, $f$ has a complex antiderivative $F$ in $U$. Then Theorem 2.5.6 implies that the integral of $f$ around any closed path is zero. □

One of the obvious applications of Cauchy's Theorem is in proving independence of path results for path integrals.

**Corollary 2.6.3.** *If $U$ is a convex set and $f$ is analytic on $U$, and $a, b \in U$, then $\int_\gamma f(z)\,dz$ is the same for all paths $\gamma$ in $U$ which begin at $a$ and end at $b$.*

The proof is left as an exercise. Another kind of path independence is illustrated by the following example.

**Example 2.6.4.** Without doing any calculating, prove that the integral of $1/z$ around any positively oriented ellipse with 0 inside is $2\pi i$.

**Solution:** From Example 2.4.1 we know that the integral of $1/z$ around a positively oriented circle centered at 0 is $2\pi i$. Choose such a circle, with radius small enough that the circle lies inside the ellipse. Then join the circle to the ellipse with four line segments, two of which lie along the $x$-axis, and two of which lie along the $y$-axis. This creates four closed paths, each of which consists of a path along a piece on the circle followed by a line segment, followed by a piece of the ellipse followed by a line segment leading back to the original point. Each of these closed paths is contained in a convex open set on which $1/z$ is analytic and so the integral of $1/z$ around each of them is 0. However, the sum of these integrals is also the difference between the integral of $1/z$ around the circle and its integral around the ellipse, because the contributions of the line segments cancel. Thus the integral of $1/z$ around the circle is the same as that around the ellipse and we conclude that the latter is $2\pi i$.

## Index of a Path around a Point

Having proved Cauchy's Integral Theorem for a convex set, we can now prove a companion result – Cauchy's Integral Formula on a convex set. There are several versions of this result. The one we will present here allows the integral to take

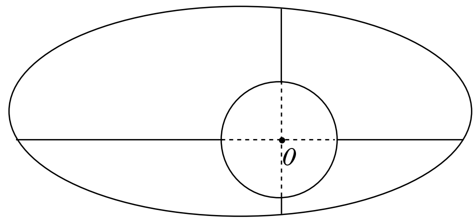


Figure 2.10: The Picture for Example 2.6.4.

place over a very general path, but requires that we know how many times the path goes around a given point. The tool that measures this is described in the following definition.

**Definition 2.6.5.** Let $\gamma : I \to \mathbb{C}$ be any closed path in $\mathbb{C}$ and let $z$ be a point of $\mathbb{C}$ which does not lie on $\gamma(I)$. We set

$$\mathrm{Ind}_\gamma(z) = \frac{1}{2\pi i}\int_\gamma \frac{dw}{w-z}.$$

This is called the *index* of $z$ with respect to $\gamma$. It is also sometimes called the *winding number* of $\gamma$ around $z$.

**Theorem 2.6.6.** *If $\gamma$ is a closed path in $\mathbb{C}$ with parameter interval $I = [a, b]$, then $Ind_\gamma(z)$ is an integer valued function of $z$ defined on the complement of $\gamma(I)$.*

*Proof.* Let $z_0$ a point in the complement of $\gamma(I)$. Since $\gamma$ is a closed path, we have $\gamma(a) = \gamma(b)$. We define a complex valued function $\lambda(t)$ on the interval $a \le t \le b$ by

$$\lambda(t) = \int_a^t \frac{\gamma'(s)}{\gamma(s) - z_0}\, ds.$$

Then $\lambda(a) = 0$ and $\lambda(b) = 2\pi i\,\mathrm{Ind}_\gamma(z)$. If we can show that $\mathrm{e}^{\lambda(b)} = 1$, then the proof will be complete, since the only numbers $w$ with $e^w = 1$ are the numbers $2\pi i\, n$ where $n$ is an integer.

By the Fundamental Theorem of Calculus,

$$\lambda'(t) = \frac{\gamma'(t)}{\gamma(t) - z_0},$$

while the derivative of $\mathrm{e}^{\lambda(t)}$ is

$$(\mathrm{e}^{\lambda(t)})' = e^{\lambda(t)}\lambda'(t) = \mathrm{e}^{\lambda(t)}\frac{\gamma'(t)}{\gamma(t) - z_0}.$$

It follows that

$$\left(\frac{e^{\lambda(t)}}{\gamma(t) - z_0}\right)' = \frac{1}{(\gamma(t) - z_0)^2}\left(\mathrm{e}^{\lambda(t)}\frac{\gamma'(t)}{\gamma(t) - z_0}(\gamma(t) - z_0) - e^{\lambda(t)}\gamma'(t)\right) = 0.$$

Hence, $e^{\lambda(t)}/(\gamma(t)-z_0)$ is a constant. We conclude that

$$\frac{e^{\lambda(t)}}{\gamma(t)-z_0} = \frac{e^{\lambda(a)}}{\gamma(a)-z_0} = \frac{1}{\gamma(a)-z_0}$$

for every $t \in [a,b]$. If we set $t = b$, this gives us

$$e^{\lambda(b)} = \frac{\gamma(b)-z_0}{\gamma(a)-z_0}.$$

Since $\gamma(a) = \gamma(b)$, it follows that $e^{\lambda(b)} = 1$. This completes the proof. □

## Cauchy's Integral Formula for Convex Sets

**Theorem 2.6.7.** *Let $U$ be a convex open set, $f$ a function which is analytic on $U$ and $\gamma : I \to U$ a closed path in $U$. Then*

$$Ind_\gamma(z)f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}dw,$$

*for every point $z \in U$ which does not lie on $\gamma(I)$.*

*Proof.* Let $z$ be a point of $\mathbb{C}$ which is not on $\gamma(I)$. Consider the function $g(z,w)$ defined for $z, w \in U$ by

$$g(z,w) = \begin{cases} \dfrac{f(w)-f(z)}{w-z}, & \text{if } w \neq z \\ f'(z), & \text{otherwise.} \end{cases}$$

For each fixed $z$, this function is analytic in $w$ everywhere on $U$ except possibly at $w = z$, but it is at least continuous at $w = z$. Since Theorem 2.6.2 holds even if the function is not analytic at some point but is continuous there, it follows that

$$\begin{aligned} 0 = \int_\gamma g(z,w)dw &= \int_\gamma \frac{f(w)}{w-z}dw - \int_\gamma \frac{f(z)}{w-z}dw \\ &= \int_\gamma \frac{f(w)}{w-z}dw - 2\pi i \operatorname{Ind}_\gamma(z)f(z), \end{aligned}$$

as long as $z$ is not on the contour $\gamma$ (note that this is required in order to write the integral in the first line above as the difference of two integrals, since otherwise these two integrals might not exist individually). We conclude that

$$\operatorname{Ind}_\gamma(z)f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}dw,$$

as required. This completes the proof. □

This is a striking result, for it says that the values of an analytic function at points "inside" a closed path are determined by its values at points on the path. Here, a point is considered inside the path if the path has non-zero index at the point.

**Corollary 2.6.8.** *If $U$ is a convex open set, $z \in U$ and $\gamma$ a closed path in $U$ with $Ind_\gamma(z) = 1$, then*

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw.$$

*for every function $f$ analytic on $U$.*

Intuitively, the meaning of the hypothesis $\mathrm{Ind}_\gamma(z) = 1$ in the above corollary is that the closed path $\gamma$ goes around $z$ once and does so in the positive direction.

Cauchy's Integral Theorem and Cauchy's Integral Formula have a wealth of applications. We will begin exploring these in the next chapter.

However, in order for Cauchy's Integral Theorem, in the above form, to be usable, we need to be able to easily compute the index of a curve around a given point. The last section of this chapter is devoted to developing the essential properties of the index function which make this possible.

## Exercise Set 2.6

1. Prove that a function which has complex derivative identically 0 on a convex open set $U$ is constant on $U$.

2. Calculate $\int_\gamma (z^2-4)^{-1}\,dz$ if $\gamma$ is the unit circle traversed once in the positive direction.

3. Calculate $\int_\gamma (1-\mathrm{e}^z)^{-1}\,dz$ if $\gamma$ is the circle $\gamma(t) = 2i + \mathrm{e}^{it}$.

4. Calculate $\int_\gamma 1/z\,dz$ if $\gamma$ is any circle which does not pass through 0. Note that the answer depends on $\gamma$.

5. Find $\int_\gamma 1/z^2\,dz$ if $\gamma$ is any closed path in $\mathbb{C} \setminus \{0\}$.

6. Show that the principal branch of the log function can be described by the formula $\log(z) = \int_1^z 1/w\,dw$ for $z \notin (-\infty, 0]$.

7. Prove Corollary 2.6.3.

8. Without doing any calculating, show that the integral of $1/z$ around the boundary of the triangle with vertices $i, 1-i, -1-i$ is $2\pi i$.

9. Let $f$ be a function which is analytic on $\mathbb{C} \setminus \{z_0\}$. Show that the contour integral of $f$ around a circle of radius $r > 0$, centered at $z_0$, is independent of $r$.

10. Calculate $\mathrm{Ind}_\gamma(z_0)$ if $\gamma = z_0 + \mathrm{e}^{int}$ and $n$ is any integer.

11. Calculate $\mathrm{Ind}_\gamma(1+i)$ if $\gamma$ is the path which traces the line from 0 to 2, then proceeds counterclockwis around the circle $|z| = 2$ from 2 to $2i$ and then traces the line from $2i$ to 0. What is the answer if this path is traversed in the opposite direction?

12. Use Cauchy's Integral Formula to calculate $\displaystyle\int_{|z|=1} \frac{e^z}{z}\, dz$.

13. Use Cauchy's Formula to show that

$$\int_{|z-1|=1} \frac{1}{z^2-1}\, dz = \pi i, \quad \int_{|z+1|=1} \frac{1}{z^2-1}\, dz = -\pi i.$$

Hint: use a partial fractions decomposition of the integrand.

14. Show that

$$\int_{|z|=3} \frac{1}{z^2-1}\, dz = 0.$$

15. Use Cauchy's Integral Formula to prove that if $f$ is a function which is analytic in an open set containing the closed unit disc $\overline{D}_1(0)$, and if $T = \{z : |z| = 1\}$ is the unit circle, then $|f(0)| \le M$, where $M$ is the maximum value of $|f|$ on $T$.

16. Show that if $\gamma$ is a path from $z_1$ to $z_2$ which does not pass through the point $z_0$, then

$$\int_\gamma \frac{1}{w-z_0}\, dw = \log\left(\frac{z_2-z_0}{z_1-z_0}\right),$$

for some branch of the log function. Note that, in the case where $z_1 = z_2$, this is just Theorem 2.6.6.

## 2.7 Properties of the Index Function

If $\gamma$ is a closed path, then removing $\gamma(I)$ from the plane results in a set which is divided into number of connected pieces. These are open sets called the *connected components* of the complement of $\gamma(I)$. We will prove that $\mathrm{Ind}_\gamma(z)$ is constant on each of these components. Thus, to calculate $\mathrm{Ind}_\gamma(z)$ on a given component, one only needs to calculate it at one point of the component.

Before proving this, we need to have a firm idea of what a connected component is. This leads to a discussion of connected sets.

### Connected Sets

**Definition 2.7.1.** A set $E \subset \mathbb{C}$ is *separated* if there exists a pair $A, B$ of open subsets of $\mathbb{C}$ such that $E \subset A \cup B$, $A \cap E \neq \emptyset$, $B \cap E \neq \emptyset$ and $A \cap B = \emptyset$. The pair $A, B$ is then said to *separate* $E$. If $E$ is not separated, then it is said to be *connected*.

The union of a family of connected sets with a point in common is also connected (Exercise 2.7.1). It follows that, if $z \in E$, then the union of all connected subsets of $E$ containing $z$ is itself connected. This implies that each point of $E$ is contained in a maximal connected subset of $E$. A maximal connected subset of $E$ is called a *connected component* of $E$ or simply a *component* of $E$. Two components of $E$ are either disjoint or are identical, since, otherwise, their union would be a connected set larger than one of them. Thus, the components of $E$ form a pairwise disjoint family of subsets of $E$ whose union is $E$.

In this section we are primarily concerned with open sets and their components. If $E$ is open, then the sets $U \cap E$ and $V \cap E$ of Definition 2.7.1 are open subsets of $E$. It follows that an open set $E$ is separated if and only if it is the union of two disjoint non-empty open subsets of itself. It is connected if this is not the case. The next theorem states the essential facts regarding connected open sets that we will need in this section.

An open set $U$ is said to be *path connected* if every two points in $U$ can be connected by a path which lies entirely in $U$.

**Theorem 2.7.2.** *Let $U$ be an open subset of $\mathbb{C}$. Then*

*(a) each component of $U$ is also open;*

*(b) $U$ is connected if and only if it is path connected.*

*Proof.* We prove (b). The proof of (a) is left as an exercise.

Suppose $U$ is connected. Given $z \in U$, let $V_z$ be the set of points of $U$ that are connected to $z$ by path in $U$, and let $w$ be some other point of $U$. There is an open disc $D$, centered at $w$, which is contained in $U$. Since any two points in $D$ are connected by a line segment, either all points of $D$ are in $V_z$ or all points of $D$ are in $U \setminus V_z$. Hence, $V_z$ and $U \setminus V_z$ are open subsets of $U$ whose union is $U$. Since $U$ is connected, one of them must be empty. Since, it contains $z$, $V_z$ is not empty, and so $U \setminus V_z$ must be empty. This means $V_z = U$ and every point of $U$ is connected to $z$ by a path in $U$. Hence, $U$ is path connected.

Conversely, suppose $U$ is path connected. If $U = A \cup B$, where $A$ and $B$ are disjoint non-empty open sets, then the function $f$ which is 1 on $A$ and 0 on $B$ is a continuous function on $U$, since the inverse image of any open subset of $\mathbb{R}$ under $f$ is $A$, $B$, $U$, or $\emptyset$ and these are all open. Now since $U$ is path connected, there is a path $\gamma$ connecting a point of $A$ to a point of $B$. Then $f \circ \gamma$ is a continuous function on an interval $I$ which takes on the values 1 and 0 and only these values. This is impossible, by the Intermediate Value Theorem. The resulting contradiction shows that there is no pair $A$, $B$, as above, and so $U$ is connected. □

By the above theorem, if $K$ is a compact subset of $\mathbb{C}$ (such as the image $\gamma(I)$ of a closed path $\gamma$), then $\mathbb{C} \setminus K$ is the union of its connected components, each of which is an open, path connected set.

**Theorem 2.7.3.** *If $K$ is a compact subset of $\mathbb{C}$, then $\mathbb{C} \setminus K$ has exactly one unbounded component.*

*Proof.* If $K$ is a compact set, then $K$ is closed and bounded. Since it is bounded, it is contained in some closed disc $\overline{D}_r(0)$. Then its complement $\mathbb{C}\setminus K$ is open and contains the complement of $\overline{D}_r(0)$. Since the complement of $\overline{D}_r(0)$ is connected, it is contained in one of the components of $\mathbb{C} \setminus K$. This means all the other components of $\mathbb{C} \setminus K$ are contained in $\overline{D}_r(0)$ and are, hence, bounded. □

**Example 2.7.4.** What are the components of $\mathbb{C} \setminus T$, where $T$ is the unit circle?

**Solution:** Clearly the sets $\{z \in \mathbb{C} : |z| < 1\}$ and $\{z \in \mathbb{C} : |z| > 1\}$ are path connected and, hence, connected. These two connected open sets have $\mathbb{C} \setminus T$ as union and so they must be the components of $\mathbb{C} \setminus T$. One of them is bounded and the other is unbounded.

## Index is Constant on Components

We can now establish the result alluded to at the beginning of this section.

**Theorem 2.7.5.** *If $\gamma : I \to \mathbb{C}$ is a closed path, then $\mathrm{Ind}_\gamma(z)$ is constant on each component of $\mathbb{C} \setminus \gamma(I)$, and is zero in the unbounded component of $\mathbb{C} \setminus \gamma(I)$.*

*Proof.* The set $\gamma(I)$ is the image of a compact set under a continuous function and so it is compact, hence, closed. Its complement $\mathbb{C} \setminus \gamma(I)$ is, therefore, open. Thus, if $z_0 \in \mathbb{C} \setminus \gamma(I)$, there is an open disc, centered at $z_0$, and contained in $\mathbb{C} \setminus \gamma(I)$. Let $R$ be the radius of one such disc. We will show that, on some smaller disc, centered at $z_0$, $\mathrm{Ind}_\gamma(z)$ is constant.

Suppose $r$ is a positive number less than $R$ and $z \in D_r(z_0)$. Then

$$\begin{aligned} \mathrm{Ind}_\gamma(z) - \mathrm{Ind}_\gamma(z_0) &= \frac{1}{2\pi i}\int_\gamma \frac{dw}{w-z} - \frac{1}{2\pi i}\int_\gamma \frac{dw}{w-z_0} \\ &= \frac{1}{2\pi i}\int_\gamma \frac{z-z_0}{(w-z)(w-z_0)}dw. \end{aligned} \tag{2.7.1}$$

Furthermore, every point $w$ of $\gamma(I)$ is at least a distance $R$ away from $z_0$ and a distance $R - r$ away from $z$. That is,

$$|w - z_0| \geq R \quad \text{and} \quad |w - z| \geq R - r.$$

Since $|z - z_0| < r$, this implies that the integrand of the last integral in (2.7.1) is less than or equal to $r/R(R - r)$ and, hence, that

$$|\mathrm{Ind}_\gamma(z) - \mathrm{Ind}_\gamma(z_0)| \leq \frac{r\ell(\gamma)}{2\pi R(R-r)}.$$

We can make the right side of this inequality as small as we want by choosing $r$ sufficiently small. In particular, we can make it less than 1. However, $\mathrm{Ind}_\gamma(z)$ and $\mathrm{Ind}_\gamma(z_0)$ are both integers. If they differ by less than 1, then they are the same. Thus, if $r$ is chosen small enough, $\mathrm{Ind}_\gamma(z)$ is constant on $D_r(z_0)$.

Let $A$ be a component of $\mathbb{C} \setminus \gamma(I)$, and for each integer $n$ let $V_n$ be the set of points of $A$ on which $\mathrm{Ind}_\gamma(z) = n$. The above argument shows that each $V_n$ is an open subset of $A$. So is the union of all the sets $V_m$ for which $m \neq n$. These two sets separate $A$ unless one of them is empty. Since $A$ is connected, one of them must be empty. This means that if $V_n$ is not empty, then it is all of $A$. Thus, only one of the sets $V_n$ can be non-empty and this means that $\text{Ind}_\gamma(z)$ is constant on $A$.

It remains to show that $\text{Ind}_\gamma(z) = 0$ on the unbounded component of $\mathbb{C}\setminus\gamma(I)$. Let $D$ be an open disc containing $\gamma(I)$ and let $z_0$ be a point outside this disc. Then $z_0$ is in the unbounded component of $\mathbb{C} \setminus \gamma(I)$. Furthermore,

$$\text{Ind}_\gamma(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{dw}{w - z_0} = 0$$

by Cauchy's Integral Theorem, since $D$ is a convex set containing $\gamma(I)$ and $\dfrac{1}{w - z_0}$ is analytic on $D$. Since $\text{Ind}_\gamma(z)$ is constant on each component of $\mathbb{C}\setminus\gamma(I)$, it must be identically 0 on the unbounded component. □

**Example 2.7.6.** Calculate $\text{Ind}_\gamma(z)$ for the path $\gamma$ which traces $n$ times around a circle of radius $r$ centered at $z_0$, where, if $n$ is positive, this means trace the circle in the counterclockwise direction, and, if $n$ is negative, it means trace the circle in the clockwise direction .

**Solution:** A parameterization for such a $\gamma$ is $\gamma(t) = z_0 + re^{int}$ with parameter interval $I = [0, 2\pi]$. Here $\gamma'(t) = irne^{int}$ and so

$$\text{Ind}_\gamma(z_0) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{\gamma'(t)dt}{\gamma(t) - z_0} = \frac{1}{2\pi i}\int_0^{2\pi} in\, dt = n.$$

We use the previous theorem to find the index at other points $z$. Since the interior of the circle traced by $\gamma$ is a component of $\mathbb{C} \setminus \gamma(I)$, $\text{Ind}_\gamma(z)$ must be constant on it. Therefore, it has the value $n$ at every $z$ with $|z - z_0| < r$. The other component of $\mathbb{C} \setminus \gamma(I)$ is the unbounded component $\{z : |z - z_0| > r\}$. On it, $\text{Ind}_\gamma(z) = 0$.

Thus, in this example, $\text{Ind}_\gamma(z)$ is an integer $n$ which is the number of times the path goes around $z$ if the path has positive orientation (goes in the counterclockwise direction), and is the negative of this number if the path has negative orientation. This includes the case where the path doesn't go around $z$ at all, because $z$ lies outside the circle. Then $n = 0$.

## Crossing a Path

For most paths, the index function can easily be computed using the principle that if a path is crossed from right to left at a "simple" point of the path, then the index increases by 1. We will make this statement precise below and outline its proof. Most of the details are left to the exercises.

**Definition 2.7.7.** Let $\gamma$ be a path with parameter interval $I = [a, b]$ and let $D$ be an open disc in the plane. We will say that $\gamma$ *simply splits* $D$ if

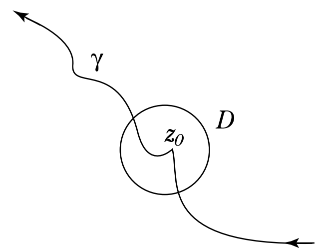


Figure 2.11: A Disc Simply Split by a Path.

(a) $J = \gamma^{-1}(D)$ is a non-empty open subinterval of $I$, or (in the case where the path is closed and $\gamma(a) = \gamma(b) \in D$) a union of two half open subintervals $[a, c)$ and $(d, b]$; and

(b) $D \setminus \gamma(J)$ has exactly two components.

Roughly speaking, a disc $D$ is simply split by a path if the path passes just once through $D$ and cuts it into exactly two connected components.

In this situation, there is a way to make sense of which of the two components is to the left of the path and which is to the right (Exercise 2.7.3).

**Theorem 2.7.8.** *Let $\gamma$ be a closed path which simply splits a disc $D$. Then*

$$Ind_\gamma(z) = 1 + Ind_\gamma(w)$$

*if $z$ is in the left and $w$ in the right component of $D \setminus \gamma(J)$.*

*Thus, if $z_0$ is a point of $\gamma$ which is the center of a disc which is simply split by $\gamma$, then $Ind_\gamma(z)$ increases by 1 as $z$ crosses $\gamma$ from right to left at $z_0$.*

The proof is left to the exercises.

The next example illustrates how to use this to compute $\text{Ind}_\gamma(z)$ in specific situations.

**Example 2.7.9.** Suppose $\gamma$ is a path which traces the circle $|z| = 2$ once in the counterclockwise direction beginning at 2 and then traces the circle $|z - 1| = 1$ once in the counterclockwise direction. Find $\text{Ind}_\gamma(z)$ for each $z$ that is not in $\gamma(I)$.

**Solution:** The components of $\mathbb{C} \setminus \{\gamma(I)\}$ are

$$A = \{z \in \mathbb{C} : |z| > 2\};$$
$$B = \{z \in \mathbb{C} : |z| < 2,\ |z - 1| > 1\};$$
$$C = \{z \in \mathbb{C} : |z - 1| < 1\}.$$

Since $A$ is the unbounded component, $\text{Ind}_\gamma(z) = 0$ if $z \in A$. Crossing $\gamma$ from right to left at $2i$ takes us from points of $A$ to points of $B$. Hence, by the preceding theorem, $\text{Ind}_\gamma(z) = 1$ for $z \in B$. Crossing $\gamma$ from right to left at $1 + i$ takes us from points of $B$ to points of $C$ and so, again by the preceding theorem, $\text{Ind}_\gamma(z) = 2$ if $z \in C$.

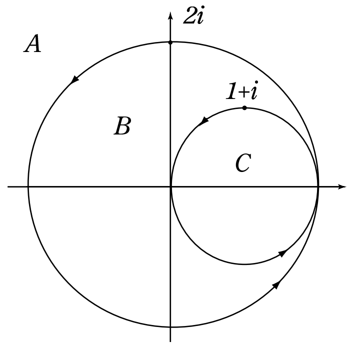


Figure 2.12: The Path for Example 2.7.9

If $\gamma$ is a path with parameter interval $I = [a, b]$, and $t_0 \in (a, b)$, then the derivatives of $\gamma$ as a function on $[a, t_0]$ and $\gamma$ as a function on $[t_0, b]$ both exist at $t_0$ (see Definition 2.3.1 and the discussion preceding it). They are equal if $t_0$ is a smooth point of the path. If $t_0$ is not a smooth point, then $\gamma$ has two derivatives at $t_0$ – a left derivative $D_\ell \gamma$ and a right derivative $D_r \gamma$. At the endpoint $a$ of $I$, $\gamma$ has only a right derivative, while at the endpoint $b$ it has only a left derivative.

**Definition 2.7.10.** If $\gamma$ is a path with parameter interval $I = [a, b]$, a point $z_0$ on $\gamma(I)$ is said to be a *simple point* if $z_0 = \gamma(t_0)$ for exactly one parameter value $t_0 \in (a, b)$ or for the two values $a$ and $b$ (in this case, $\gamma$ is closed), and if the left and right derivatives of $\gamma$ at $t_0$ (at $a$ and $b$ if $z_0 = \gamma(z) = \gamma(b)$) are both non-zero.

In other words, a point is a simple point of a path if the path passes through the point just once and it both approaches the point from a definite direction with a positive speed and leaves the point in a definite direction with a positive speed. This leads to a very useful criterion for a point on a path to be the center of a disc that is simply split by the path.

**Theorem 2.7.11.** *If $z_0$ is a simple point of the path $\gamma$, then there is an open disc, centered at $z_0$, which is simply split by $\gamma$.*

Exercises 2.7.11 through 2.7.14 are devoted to proving this theorem in the case where the point $z_0$ is actually a smooth simple point of $\gamma$. In Exercise 2.7.15, the reader is asked to modify this argument so as to prove the theorem in general.

Note that in attempting to prove the above theorem, we may assume that $z_0 = \gamma(t_0)$ for some interior point $t_0$ of the parameter interval (otherwise we can just reparameterize to make this the case). Also, we may assume that $z_0 = 0$ since, otherwise, we can just translate the curve to make this the case. Both of these assumptions are made in Exercises 2.7.11 to 2.7.15.

Theorems 2.7.11 and 2.7.8 imply that if a closed path $\gamma$ is crossed from right to left at a simple point, then $\text{Ind}_\gamma$ increases by 1. For most of the paths $\gamma$ that we shall encounter, all but finitely many points of $\gamma$ are simple points. Exceptions to this rule are paths which retrace parts of themselves, cross themselves infinitely often, or come to a dead stop over some segment of the parameter interval.

## Exercise Set 2.7

1. Prove that the union of a family of connected sets with a point in common is also a connected set.

2. Prove Part (a) of Theorem 2.7.2. That is, prove that each component of an open set is open.

3. Let $D$ be an open disc and $\gamma : I \to \mathbb{C}$ be a closed path which simply splits $D$. Argue that, if we think of the positive direction along the curve through $D$ as being "up", then it makes sense to think of one of the components into which $\gamma$ splits $D$ as the "left" one and the other as the "right" one. Describe how to tell which is which.

4. Suppose a closed path $\gamma$ simply splits a disc $D$ as in Figure 2.11. Define two new paths $\gamma_1$ and $\gamma_2$ as follows: The curve $\gamma_1$ agrees with $\gamma$ until $\gamma$ enters $D$. It then departs from $\gamma$ and instead traces the boundary of $D$ in the clockwise direction until it rejoins $\gamma$. It agrees with $\gamma$ from that point on. The path $\gamma_2$ does the same thing except it traces the boundary of $D$ in the clockwise direction. For which points $z$ inside $D$ does $\text{Ind}_\gamma(z) = \text{Ind}_{\gamma_1}(z)$? For which points $z$ inside $D$ does $\text{Ind}_\gamma(z) = \text{Ind}_{\gamma_2}(z)$? What is $\text{Ind}_{\gamma_1}(z) - \text{Ind}_{\gamma_2}(z)$ if $z$ is any point inside $D$? Hint: use Cauchy's Integral Theorem and Example 2.7.6.

5. Use the results of the previous exercise to prove Theorem 2.7.8.

6. Prove that if $\gamma$ is a closed path whose complement has just two components and if $\gamma$ has at least one simple point, then $\text{Ind}_\gamma(z) = \pm 1$ on the bounded component.

7. In Example 2.7.9 how would the answers differ if the inner circle is traced in the clockwise direction rather than the counterclockwise direction?

8. If a path $\gamma$ traces a figure eight once, what are the possibilities for $\text{Ind}_\gamma(z)$ in the two bounded components of the complement of the figure eight.

9. Determine the value of $\text{Ind}_\gamma(z)$ in each of the components of $\mathbb{C} \setminus \gamma(I)$ if $\gamma$ is the curve of Figure 2.13.

10. Suppose $\gamma_1$ and $\gamma_2$ are closed paths and $z$ is not on either path. Show that $\text{Ind}_{\gamma_1+\gamma_2}(z) = \text{Ind}_{\gamma_1}(z) + \text{Ind}_{\gamma_2}(z)$ and $\text{Ind}_{-\gamma_1} = -\text{Ind}_{\gamma_1}(z)$. Hint: use Theorem 2.4.6.

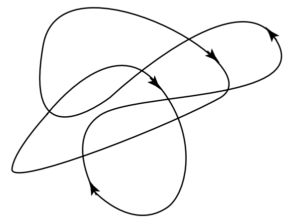

Figure 2.13: The Path for Exercise 2.7.9.

11. Let $\gamma(t) = x(t) + iy(t)$ be a path with parameter interval $I = [a, b]$ and $t_0$ a point in $(a, b)$ at which the path is smooth and simple. For further simplicity, assume $\gamma(t_0) = 0$ (we can always achieve this by translating the path). Prove that, even though $\gamma''(t_0)$ may not exist, the second derivative of the function
$$h(t) = |\gamma(t)|^2 = x^2(t) + y^2(t)$$
does exist at $t_0$ and equals $2|\gamma'(t_0)|^2$.

12. Let $\gamma$ and $t_0$ be as in the previous exercise. Prove that there is an interval $(c, d) \subset (a, b)$, containing $t_0$, such that $|\gamma(t)|$ is strictly decreasing on $[c, t_0]$ and strictly increasing on $[t_0, d]$. Show that this implies that, if $\delta = \min\{|\gamma(c)|, |\gamma(d)|\}$, then $\gamma(t)$ crosses each circle of radius less than $\delta$, centered at 0, exactly once for $t \in (c, t_0)$ and exactly once for $t \in (t_0, d)$.

13. Let $\gamma$, $t_0$ and $(c, d)$ be as in the previous exercise, prove that there is an open disc $D$, centered at 0, such that $\gamma^{-1}(D)$ is an open subinterval $J$ of $(c, d)$. Hint: begin by showing you can choose $D$ small enough that it contains no points $\gamma(t)$ for $t \notin (c, d)$; then use the result of the previous exercise.

14. With $\gamma$, $t_0$, $(c, d)$, and $D$ as in the previous exercise, prove that $D$ is simply split by $\gamma$. This proves Theorem 2.7.11 in the case of a smooth simple point.

15. How would the argument outlined in the previous four exercises need to be modified to prove Theorem 2.7.11 for a point which is a simple point of $\gamma$, but not a smooth point – note that, in this case, $\gamma(t)$ will have two derivatives at $t_0$ – one from the left and one from the right.