Chapter 1

The Real Numbers

This course has two goals: (1) to develop the foundations that underlie calculus and all of post calculus mathematics, and (2) to develop students' ability to understand definitions and proofs and to create proofs of their own – that is, to develop students' *mathematical sophistication*.

The typical freshman and sophomore calculus courses are designed to teach the techniques needed to solve problems using calculus. They are not primarily concerned with proving that these techniques work or teaching why they work. The key theorems of calculus are not really proved, although sometimes proofs are given which rely on other reasonable, but unproved assumptions. Here we will give rigorous proofs of the main theorems of calculus. To do this requires a solid understanding of the real number system and its properties. This first chapter is devoted to developing such an understanding.

Our study of the real number system will follow the historical development of numbers: We first discuss the natural numbers or counting numbers (the positive integers), then the integers, followed by the rational numbers. Finally, we discuss the real number system and the property that sets it apart from the rational number system – the completeness property. The completeness property is the missing ingredient in most calculus courses. It is seldom discussed, but without it, one cannot prove the main theorems of calculus.

The natural numbers can be defined as a set satisfying a very simple list of axioms – Peano's axioms. All of the properties of the natural numbers can be proved using these axioms. Once this is done, the integers, the rational numbers, and the real numbers can be constructed and their properties proved rigorously. To actually carry this out would make for an interesting, but rather tedious course. Fortunately, that is not the purpose of this course. We will not give a rigorous construction of the real number system beginning with Peano's axioms, although we will give a brief outline of how this is done. However, the main purpose of this chapter is to state the properties that characterize the real number system and develop some facility at using them in proofs. The rest of the course will be devoted to using these properties to develop rigorous proofs of the main theorems of calculus.

1.1 Sets and Functions

We precede our study of the real numbers with a brief introduction to sets and functions and their properties. This will give us the opportunity to introduce the set theory notation and terminology that will be used throughout the text.

Sets

A *set* is a collection of objects. These objects are called the *elements* of the set. If x is an element of the set A, then we will also say that x *belongs* to A or x is *in* A. A shorthand notation for this statement that we will use extensively is

$$x \in A.$$

Two sets A and B are the same set if they have the same elements – that is, if every element of A is also an element of B and every element of B is also an element of A. In this case, we write $A = B$.

One way to define a set is to simply list its elements. For example, the statement

$$A = \{1, 2, 3, 4\}$$

defines a set A which has as elements the integers from 1 to 4.

Another way to define a set is to begin with a known set A and define a new set B to be all elements $x \in A$ that satisfy a certain condition $Q(x)$. The condition $Q(x)$ is a statement about the element x which may be true for some values of x and false for others. We will denote the set defined by this condition as follows:

$$B = \{x \in A : Q(x)\}.$$

This is mathematical shorthand for the statement "B is the set of all x in A such that $Q(x)$". For example, if A is the set of all students in this class, then we might define a set B to be the set of all students in this class who are sophomores. In this case, $Q(x)$ is the statement "x is a sophomore". The set B is then defined by

$$B = \{x \in A : x \text{ is a sophomore}\}.$$

Example 1.1.1. Describe the set $(0, 3)$ of all real numbers greater than 0 and less than 3 using set notation.

Solution: In this case the statement $Q(x)$ is the statement "$0 < x < 3$". Thus,

$$(0, 3) = \{x \in \mathbb{R} : 0 < x < 3\}.$$

A set B is a *subset* of a set A if B consists of some of the elements of A – that is, if each element of B is also an element of A. In this case, we use the shorthand notation

$$B \subset A.$$

Of course, A is a subset of itself. We say B is a *proper* subset of A if $B \subset A$ and $B \neq A$.

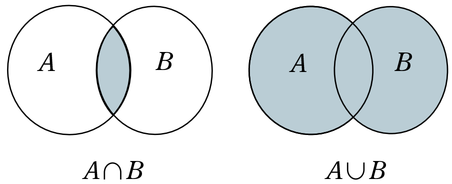

Figure 1.1: Intersection and Union of Two Sets.

For example, the open interval $(0,3)$ of the preceding example is a proper subset of the set $\mathbb{R}$ of real numbers . It is also a proper subset of the half open interval $(0,3]$ – that is, $(0,3) \subset (0,3]$, but the two are not equal because the second contains 3 and the first does not.

There is one special set that is a subset of every set. This is the empty set $\emptyset$. It is the set with no elements. Since it has no elements, the statement that "each of its elements is also an element of A" is true no matter what the set A is. Thus, by the definition of subset,

$$\emptyset \subset A$$

for every set A.

If A and B are sets, then the *intersection* of A and B, denoted $A \cap B$, is the set of all objects that are elements of A and of B. That is,

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

Similarly, the *union* of A and B, denoted $A \cup B$, is the set of objects which are elements of A or elements of B (possibly elements of both). That is,

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

Example 1.1.2. If A is the closed interval $[-1,3]$ and B is the open interval $(1,5)$, describe $A \cap B$ and $A \cup B$.

Solution: $A \cap B = (1,3]$ and $A \cup B = [-1,5)$.

If $\mathcal{A}$ is a (possibly infinite) collection of sets, then the intersection and union of the sets in $\mathcal{A}$ are defined to be

$$\bigcap \mathcal{A} = \{x : x \in A \text{ for all } A \in \mathcal{A}\}$$

and

$$\bigcup \mathcal{A} = \{x : x \in A \text{ for some } A \in \mathcal{A}\}.$$

Note how crucial the distinction between "for all' and "for some' is in these definitions.

The intersection $\bigcap \mathcal{A}$ is also often denoted

$$\bigcap_{A \in \mathcal{A}} A \quad \text{or} \quad \bigcap_{s \in S} A_s$$

if the sets in $\mathcal{A}$ are indexed by some index set S. Similar notation is often used for the union.

Example 1.1.3. If $\mathcal{A}$ is the collection of all intervals of the form $[s, 2]$ where $0 < s < 1$, find $\bigcap \mathcal{A}$ and $\bigcup \mathcal{A}$.

Solution: A number x is in the set

$$\bigcap \mathcal{A} = \bigcap_{s \in (0,1)} [s, 2]$$

if and only if

$$s \le x \le 2 \quad \text{for every positive } s < 1. \tag{1.1.1}$$

Clearly every x in the interval $[1, 2]$ satisfies this condition. We will show that no points outside this interval satisfy (1.1.1).

Certainly an $x > 2$ does not satisfy (1.1.1). If $x < 1$, then $s = x/2 + 1/2$ (the midpoint between x and 1) is a number less than 1 but greater than x, and so such an x also fails to satisfy (1.1.1). This proves that

$$\bigcap \mathcal{A} = [1, 2].$$

A number x is in the set

$$\bigcup \mathcal{A} = \bigcup_{s \in (0,1)} [s, 2]$$

if and only if

$$s \le x \le 2 \quad \text{for some positive } s < 1. \tag{1.1.2}$$

Every such x is in the interval $(0, 2]$. Conversely, we will show that every x in this interval satisfies (1.1.2). In fact, if $x \in [1, 2]$, then x satisfies (1.1.2) for every $s < 1$. If $x \in (0, 1)$, then x satisfies 1.1.2 for $s = x/2$. This proves that

$$\bigcup \mathcal{A} = (0, 2].$$

If $B \subset A$, then the set of all elements of A which are not elements of B is called the *complement* of B in A. This is denoted $A \setminus B$. Thus,

$$A \setminus B = \{x \in A : x \notin B\}.$$

Here, of course, the notation $x \notin B$ is shorthand for the statement "x is not an element of B".

If all the sets in a given discussion are understood to be subsets of a given *universal* set X, then we may use the notation B^c for $X \setminus B$ and call it simply the *complement* of B. This will often be the case in this course, with the universal set being the set $\mathbb{R}$ of real numbers or, in later chapters, real n dimensional space $\mathbb{R}^n$ for some n.

Example 1.1.4. If A is the interval $[-2, 2]$ and B is the interval $[0, 1]$, describe $A \setminus B$ and the complement B^c of B in $\mathbb{R}$.

Solution: We have

$$A \setminus B = [-2, 0) \cup (1, 2] = \{x \in \mathbb{R} : -2 \leq x < 0 \text{ or } 1 < x \leq 2\},$$

while

$$B^c = (-\infty, 0) \cup (1, \infty) = \{x \in \mathbb{R} : x < 0 \text{ or } 1 < x\}.$$

Theorem 1.1.5. *If A and B are subsets of a set X and A^c and B^c are their complements in X. then*

(a) $(A \cup B)^c = A^c \cap B^c$*; and*

(b) $(A \cap B)^c = A^c \cup B^c$.

Proof. We prove (a) first. To show that two sets are equal, we must show that they have the same elements. An element of X belongs to $(A \cup B)^c$ if and only if it is not in $A \cup B$. This is true if and only if it is not in A and it is not in B. By definition this is true if and only if $x \in A^c \cap B^c$. Thus, $(A \cup B)^c$ and $A^c \cap B^c$ have the same elements and, hence, are the same set.

If we apply part (a) with A and B replaced by A^c and B^c and use the fact that $(A^c)^c = A$ and $(B^c)^c = B$, the result is

$$(A^c \cup B^c)^c = A \cap B.$$

Part (b) then follows if we take the complement of both sides of this identity. □

A statement analogous to Theorem 1.1.5 is true for unions and intersections of collections of sets (Exercise 1.1.7).

Two sets A and B are said to be *disjoint* if $A \cap B = \emptyset$. That is, they are disjoint if they have no elements in common. A collection $\mathcal{A}$ of sets is called a *pairwise disjoint* collection if $A \cap B = \emptyset$ for each pair A, B of distinct sets in $\mathcal{A}$.

Functions

A *function* f from a set A to a set B is a rule which assigns to each element $x \in A$ exactly one element $f(x) \in B$. The element $f(x)$ is called the image of x under f or the value of f at x. We will write

$$f : A \to B$$

to indicate that f is a function from A to B. The set A is called the *domain* of f. If E is any subset of A then we write

$$f(E) = \{f(x) : x \in E\}$$

and call $f(E)$ the *image* of E under f.

We don't assume that every element of B is the image of some element of A. The set of elements of B which are images of elements of A is $f(A)$ and is called the *range* of f. If every element of B is the image of some element of A (so that the range of f is B), then we say that f is *onto*.

A function $f : A \to B$ is is said to be *one-to-one* if, whenever $x, y \in A$ and $x \neq y$, then $f(x) \neq f(y)$ – that is, if f takes distinct points to distinct points.

If $g : A \to B$ and $f : B \to C$ are functions, then there is a function $f \circ g : A \to C$, called the *composition* of f and g, defined by

$$f \circ g(x) = f(g(x)).$$

Since $g(x) \in B$ and the domain of f is B, this definition makes sense.

If $f : A \to B$ is a function and $E \subset B$, then the *inverse image* of E under f is the set

$$f^{-1}(E) = \{x \in A : f(x) \in E\}.$$

That is, $f^{-1}(E)$ is the set of all elements of A whose images under f belong to E.

Inverse image behaves very well with respect to the set theory operations, as the following theorem shows.

Theorem 1.1.6. *If $f : A \to B$ is a function and E and F are subsets of B, then*

(a) $f^{-1}(E \cup F) = f^{-1}(E) \cup f^{-1}(F)$*;*

(b) $f^{-1}(E \cap F) = f^{-1}(E) \cap f^{-1}(F)$*; and*

(c) $f^{-1}(E \setminus F) = f^{-1}(E) \setminus f^{-1}(F)$ *if* $F \subset E$*.*

Proof. We will prove (a) and leave the other two parts to the exercises.

To prove (a), we will show that $f^{-1}(E \cup F)$ and $f^{-1}(E) \cup f^{-1}(F)$ have the same elements. If $x \in f^{-1}(E \cup F)$, then $f(x) \in E \cup F$. This means that $f(x)$ is in E or in F. If it is in E, then $x \in f^{-1}(E)$. If it is in F, then $x \in f^{-1}(F)$. In either case, $x \in f^{-1}(E) \cup f^{-1}(F)$. This proves that every element of $f^{-1}(E \cup F)$ is an element of $f^{-1}(E) \cup f^{-1}(F)$.

On the other hand, if $x \in f^{-1}(E) \cup f^{-1}(F)$, then $x \in f^{-1}(E)$, in which case $f(x) \in E$, or $x \in f^{-1}(F)$, in which case $f(x) \in F$. In either case, $f(x) \in E \cup F$, which implies $x \in f^{-1}(E \cup F)$. This proves that every element of $f^{-1}(E) \cup f^{-1}(F)$ is also an element of $f^{-1}(E \cup F)$. Combined with the previous paragraph, this proves that the two sets are equal. □

Image does not behave as well as inverse image with respect the set operations. The best we can say is the following:

Theorem 1.1.7. *If $f : A \to B$ is a function and E and F are subsets of A, then*

(a) $f(E \cup F) = f(E) \cup f(F)$*;*

(b) $f(E \cap F) \subset f(E) \cap f(F)$*;*

(c) $f(E) \setminus f(F) \subset f(E \setminus F)$ *if* $F \subset E$*.*

Proof. We will prove (c) and leave the others to the exercises.

To prove (c), we must show that each element of $f(E) \setminus f(F)$ is also an element of $f(E \setminus F)$. If $y \in f(E) \setminus f(F)$, then $y = f(x)$ for some $x \in E$ and y is not the image of any element of F. In particular, $x \notin F$. This means that $x \in E \setminus F$ and so $y \in f(E \setminus F)$. This completes the proof. □

The above theorem cannot be improved. That is, it is not in general true that $f(E \cap F) = f(E) \cap f(F)$ or that $f(E) \setminus f(F) = f(E \setminus F)$ if $F \subset E$. The first of these facts is shown in the next example. The second is left to the exercises.

Example 1.1.8. Give an example of a function $f : A \to B$ for which there are subsets $E, F \subset A$ with $f(E \cap F) \neq f(E) \cap f(F)$.

Solution: Let A and B both be $\mathbb{R}$ and let $f : A \to B$ be defined by

$$f(x) = x^2.$$

If $E = (0, \infty)$ and $F = (-\infty, 0)$, then $E \cap F = \emptyset$, and so $f(E \cap F)$ is also the empty set. However, $f(E) = f(F) = (0, \infty)$, and so $f(E) \cap f(F) = (0, \infty)$ as well. Clearly $f(E \cap F)$ and $f(E) \cap f(F)$ are not the same in this case.

Exercise Set 1.1

1. If $a, b \in \mathbb{R}$ and $a < b$, give a description in set theory notation for each of the intervals $(a, b), [a, b], [a, b)$, and $(a, b]$ (see Example 1.1.1).

2. If A, B, and C are sets, prove that

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

3. If A and B are two sets, then prove that A is the union of a disjoint pair of sets, one of which is contained in B and one of which is disjoint from B.

4. What is the intersection of all the open intervals containing the closed interval $[0, 1]$.

5. What is the union of all of the closed intervals contained in the open interval $(0, 1)$?

6. What is $\bigcup\{(n, n + 1) : n \text{ is an integer}\}$.

7. If $\mathcal{A}$ is a collection of subsets of a set X, formulate and prove a theorem like Theorem 1.1.5 for the intersection and union of $\mathcal{A}$.

8. Which of the following functions $f : \mathbb{R} \to \mathbb{R}$ are one to one and which ones are onto. Justify your answer.

   (a) $f(x) = x^2$;

(b) $f(x) = x^3$;

(c) $f(x) = e^x$.

9. Prove Part (b) of Theorem 1.1.6.

10. Prove Part (c) of Theorem 1.1.6.

11. Prove Part (a) of Theorem 1.1.7.

12. Prove Part (b) of Theorem 1.1.7.

13. Give an example of a function $f : A \to B$ and subsets $F \subset E$ of A for which $f(E) \setminus f(F) \neq f(E \setminus F)$.

14. Prove that equality holds in Parts (b) and (c) of Theorem 1.1.7 if the function f is one-to-one.

15. Prove that if $f : A \to B$ is a function which is one-to-one and onto, then f has an *inverse function* − that is, there is a function $g : B \to A$ such that $g(f(x)) = x$ for all $x \in A$ and $f(g(y)) = y$ for all $y \in B$.

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1.2 The Natural Numbers

The natural numbers are the numbers we use for counting, and so, naturally, they are also called the *counting numbers*. They are the positive integers $1, 2, 3, \cdots$.

The requirements for a system of numbers we can use for counting are very simple. There should be a first number (the number 1), and for each number there must always be a next number (a successor). After all, we don't want to run out of numbers when counting a large set of objects. This line of thought leads to Peano's axioms which characterize the system of natural numbers $\mathbb{N}$:

N1. there is an element $1 \in \mathbb{N}$;

N2. for each $n \in \mathbb{N}$ there is a successor element $n + 1 \in \mathbb{N}$;

N3. 1 is not the successor of any element of $\mathbb{N}$;

N4. if two elements of $\mathbb{N}$ have the same successor, then they are equal;

N5. if a subset A of $\mathbb{N}$ contains 1 and is closed under succession (meaning $n + 1 \in A$ whenever $n \in A$), then $A = \mathbb{N}$.

Everything we need to know about the natural numbers can be deduced from these axioms. That is, using only these axioms, one can define addition and multiplication of natural numbers and prove that they satisfy the usual arithmetic properties. One can also define the order relation on the natural numbers and prove that it has the appropriate properties. To do all of this is not difficult, but it is tedious and time consuming. We won't do this here, but we will assume that it can be done. In some of the examples at the end of this section and some of the exercises, we will explain a few of the steps that would be involved in such an undertaking.

Our main focus in this section will be on understanding how to use mathematical induction, a powerful technique that is a direct consequence of Axiom **N5**.

Induction

Axiom **N5** above is often called the *induction* axiom, since it is the basis for mathematical induction. Mathematical induction is used in making definitions that involve a sequence of objects to be defined and in proving propositions that involve a sequence of statements to be proved. Here, by a *sequence* we mean a function whose domain is the natural numbers. Thus, a sequence of statements is an assignement of a statement to each $n \in \mathbb{N}$. For example, "$n(n+1)$ is even" is a sequence of statements, one for each $n \in \mathbb{N}$.

The following theorem states the mathematical induction principle as it applies to proving propositions.

Theorem 1.2.1. *Suppose $\{P_n\}$ is a sequence of statements, one for each $n \in \mathbb{N}$. These statements are all true provided*

1. *P_1 is true (the base case is true); and*
2. *whenever P_n is true for some $n \in \mathbb{N}$, then P_{n+1} is also true (the induction step can be carried out).*

Proof. Let A be the subset of $\mathbb{N}$ consisting of those n for which P_n is true. Then hypothesis (1) of the theorem implies that $1 \in A$, while hypothesis (2) implies that $n+1 \in A$ whenever $n \in A$. By Axiom **N5**, $A = \mathbb{N}$, and so P_n is true for every n. □

Example 1.2.2. Prove by induction that every number of the form $5^n - 2^n$, with $n \in \mathbb{N}$ is divisible by 3.

Solution: The proposition P_n is that $5^n - 2^n$ is divisible by 3.

Base case: Since $5 - 2 = 3$, P_1 is true;

Induction step: We need to show that P_{n+1} is true whenever P_n is true. We do this by rewriting the expression $5^{n+1} - 2^{n+1}$ as

$$5^{n+1} - 5 \cdot 2^n + 5 \cdot 2^n - 2^{n+1} = 5(5^n - 2^n) + (5-2)2^n.$$

If P_n is true then the first term on the right is divisible by 3. The second term on the right is also divisible by 3, since $5 - 2 = 3$. This implies that $5^{n+1} - 2^{n+1}$ is divisible by 3 and, hence, that P_{n+1} is true. This completes the induction step.

By induction (that is, by Theorem 1.2.1), P_n is true for all n.

A natural number n is a *prime* if it is not 1 and if its only factors are 1 and n.

Example 1.2.3. Prove that each natural number $n > 1$ is a product of primes.

Solution: Here we understand that a prime number itself is a product of primes – a product with only one factor. Note that if k and m are two numbers which are products of primes, then their product km is also a product of primes.

Let the proposition P_n be that every $m \in \mathbb{N}$, with $1 < m \leq n$, is a product of primes.

Base case: P_1 is true because there is no $m \in \mathbb{N}$ with $1 < m \leq 1$.

Induction step: suppose n is a natural number for which P_n is true. Then each m with $1 < m \leq n$ is a product of primes . Now $n + 1 > 1$ and so it is either a prime, or it factors as a product km with k and m not equal to 1 or $n+1$. In the first case, P_{n+1} is true. In the second case, both $k \leq n$ and $m \leq n$ and so both k and m are products of primes, since P_n is true. This implies that $n + 1 = km$ is also a product of primes and, in turn, this implies that P_{n+1} is true.

By induction, P_n is true for all $n \in \mathbb{N}$ and this means that every natural number $n > 1$ is a product of primes.

Inductive Definitions

Inductive definitions are used to define sequences. The sequence $\{x_n\}$ to be defined is a sequence of elements of some set X, which may or may not be a set of numbers. We wish to define the sequence in such a way that x_1 is a specified element of X and, for each $n \in \mathbb{N}$, x_{n+1} is a certain function of x_n. That is, we are given an element $x \in X$ and a sequence of functions $f_n : X \to X$ and we wish to construct a sequence $\{x_n\}$ such that

$$x_1 = x \quad \text{and} \quad x_{n+1} = f_n(x_n) \text{ for all } n \in \mathbb{N}. \tag{1.2.1}$$

This equation, defining x_{n+1} in terms of x_n, is called a *recursion relation*. Sequences defined in this way occur very often in mathematics. Newton's method from calculus and Euler's method for numerically solving differential equations are two important examples.

Theorem 1.2.4. *Given a set X, an element $x \in X$, and a sequence $\{f_n\}$ of functions from X to X, there is a unique sequence $\{x_n\}$ in X which satisfies (1.2.1).*

Proof. Let A be the subset of $\mathbb{N}$ consisting of all numbers n such that there is a unique partial sequence $\{x_m\}_{m \leq n}$ such that

$$x_1 = x \quad \text{and} \quad x_{m+1} = f_m(x_m) \text{ for all } m < n. \tag{1.2.2}$$

The number 1 belongs to A because we can and must choose $x_1 = x$. We next show that if a number n belongs to A, then so does its successor $n + 1$.

If $n \in A$, we have a unique partial sequence $\{x_m\}_{m \le n}$ satisfying (1.2.2). We want to define a partial sequence $\{x_m\}_{m \le n+1}$ which satisfies (1.2.2) with n replaced by $n+1$. We can do this, but in only one way. We must choose x_m for $m \le n$ as before, because that choice was unique. Then we are forced by the recursion relation to choose $x_{n+1} = f_n(x_n)$. Thus, not only can such a partial sequence be chosen, it is unique. This proves that $n+1 \in A$. We conclude from the induction axiom that $A = \mathbb{N}$. This means that partial sequences satisfying (1.2.2) are uniquely defined for each n. These partial sequences then fit together to define one complete sequence satisfying (1.2.1). □

Example 1.2.5. Define a sequence $\{x_n\}$ of real numbers by setting $x_1 = 1$ and using the recursion relation

$$x_{n+1} = \sqrt{x_n + 1}. \tag{1.2.3}$$

Show that this is an increasing sequence of positive numbers less than 2.

Solution: The function $f(x) = \sqrt{x+1}$ may be regarded as a function from the set of positive real numbers into itself. We can apply the previous theorem, with each of the functions f_n equal to f, to conclude that a sequence $\{x_n\}$ is uniquely defined by setting $x_1 = 1$ and imposing the recursion relation (1.2.3).

Let P_n be the proposition that $x_n < x_{n+1} < 2$. We will prove that P_n is true for all n by induction.

Base Case: P_1 is the statement $x_1 < x_2 < 2$. Since $x_1 = 1$ and $x_2 = \sqrt{2}$, this is true.

Induction Step: Suppose P_n is true for some n. Then $x_n < x_{n+1} < 2$. If we add one and take the square root, this becomes

$$\sqrt{x_n + 1} < \sqrt{x_{n+1} + 1} < \sqrt{3}.$$

Using the recursion relation (1.2.3), this yields

$$x_{n+1} < x_{n+2} < \sqrt{3}$$

Since $\sqrt{3} < 2$, P_{n+1} is true. This completes the induction step.

We conclude that P_n is true for all $n \in \mathbb{N}$.

Binomial Formula

The proof of the binomial formula is an excellent example of the use of induction.

We will use the notation

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

This is the number of ways of choosing k objects from a set of n objects.

Theorem 1.2.6. *If x and y are real numbers and $n \in \mathbb{N}$, then*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Proof. We prove this by induction on n.

Base Case: Since $\binom{1}{0}$ and $\binom{1}{1}$ are both 1, the binomial formula is true when $n = 1$.

Induction Step: If we assume the formula is true for a certain n, then multiplying both sides of this formula by $x + y$ yields

$$\begin{aligned} (x+y)^{n+1} &= x \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} + y \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \\ &= \sum_{k=0}^{n} \binom{n}{k} x^{k+1} y^{n-k} + \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k+1}. \end{aligned} \tag{1.2.4}$$

If we change variables in the first sum on the second line of (1.2.4) by replacing k by $k - 1$, then our expression for $(x + y)^{n+1}$ becomes

$$\begin{aligned} x^{n+1} &+ \sum_{k=1}^{n} \binom{n}{k-1} x^k y^{n-k+1} + \sum_{k=1}^{n} \binom{n}{k} x^k y^{n-k+1} + y^{n+1} \\ &= x^{n+1} + \sum_{k=1}^{n} \left[\binom{n}{k-1} + \binom{n}{k} \right] x^k y^{n+1-k} + y^{n+1}. \end{aligned} \tag{1.2.5}$$

If we use the identity (to be proved in Exercise 1.4.9)

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k},$$

then the right side of equation (1.2.5) becomes

$$x^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} x^k y^{n+1-k} + y^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} x^k y^{n+1-k}.$$

Thus, the binomial formula is true for $n + 1$ if it is true for n. This completes the induction step and the proof of the theorem. □

Using Peano's Axioms to Develop Properties of $\mathbb{N}$

In this subsection, we will demonstrate some of the steps involved in developing the arithmetic and order properties of $\mathbb{N}$ using only Peano's axioms. It is not a complete development, but just a taste of what is involved. We begin with the definition of addition.

Definition 1.2.7. We fix $m \in \mathbb{N}$ and define a sequence $\{m+n\}_{n\in\mathbb{N}}$ by induction on n. The first element of this sequence is defined to be the successor, $m + 1$, to m. If $m + n$ has been defined for some n, then we define $m + (n + 1)$ by the recursion relation

$$m + (n+1) = (m+n) + 1; \tag{1.2.6}$$

that is, $m + (n + 1)$ is defined to be the successor of $m + n$. By Theorem 1.2.4 (which was proved using only the Peano axioms), there is a unique sequence $\{m + n\}_{n\in\mathbb{N}}$ defined by these conditions.

Example 1.2.8. Using the above definition and Peano's axioms, prove the associative law for addition in $\mathbb{N}$. That is, prove

$$m + (n + k) = (m + n) + k \quad \text{for all} \quad k, n, m \in \mathbb{N}.$$

Solution: We fix m and n and, for each $k \in \mathbb{N}$, let P_k be the proposition $m + (n + k) = (m + n) + k$. We prove that P_k is true for all $k \in \mathbb{N}$ by induction on k.

The base case P_1 is just the recursion relation (1.2.6) used in the definition of addition. Thus, it is true by definition.

For the induction step, we assume P_k is true for some k – that is, we assume

$$m + (n + k) = (m + n) + k.$$

We then take the successor of both sides of this equation to obtain

$$(m + (n + k)) + 1 = ((m + n) + k) + 1.$$

If we use (1.2.6) on both sides of this equation, the result is

$$m + ((n + k) + 1) = (m + n) + (k + 1).$$

Using (1.2.6) again, this time on the left side of the equation, leads to

$$m + (n + (k + 1)) = (m + n) + (k + 1).$$

Since this is proposition P_{k+1}, the induction is complete.

Example 1.2.9. Using Definition 1.2.7 and Peano's axioms, prove that $1+n = n + 1$ for every $n \in \mathbb{N}$.

Solution: Let P_n be the statement $1 + n = n + 1$. We prove by induction that P_n is true for every n. It is trivially true in the base case $n = 1$, since P_1 just says $1 + 1 = 1 + 1$.

For the induction step, we assume that P_n is true for some n – that is we assume $1 + n = n + 1$. If we add 1 to both sides of this equation (i.e. take the successor of both sides), we have

$$(1 + n) + 1 = (n + 1) + 1.$$

By Definition 1.2.7, the left side of this equation is equal to $1 + (n + 1)$. Thus,

$$1 + (n + 1) = (n + 1) + 1.$$

Thus, P_{n+1} is true if P_n is true and the induction is complete.

A similar induction, this time on m, with n fixed can be used to prove the commutative law of addition − that is, $m + n = n + m$ for all $n, m \in \mathbb{N}$. The base case for this induction is the statement proved above. The associative law proved in Example 1.2.8 is needed in the proof of the induction step. We leave the details to the exercises.

We leave the definition of multiplication in $\mathbb{N}$ to the exercises. Its definition and the fact that it also satisfies the associative and commutative laws follows a pattern similar to the one above for addition.

The order relation in $\mathbb{N}$ can be defined as follows:

Definition 1.2.10. If $n, m \in \mathbb{N}$, we will say the n is less than m, denoted $n < m$, if there is a $k \in \mathbb{N}$ such that $m = n + k$. We say n is less than or equal to m and write $n \leq m$ if $n < m$ or $n = m$.

Some of the properties of this order relation are worked out in the exercises.

Exercise Set 1.2

1. Using induction, prove that $n^2 + 3n + 3$ is odd for every $n \in \mathbb{N}$;

2. Using induction, prove that $7^n - 2^n$ is divisible by 5 for every $n \in \mathbb{N}$.

3. Using induction, prove that $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ for every $n \in \mathbb{N}$.

4. Using induction, prove that $\displaystyle\sum_{k=1}^{n}(2k-1) = n^2$ for every $n \in \mathbb{N}$.

5. Let a sequence $\{x_n\}$ of numbers be defined recursively by
$$x_1 = 0 \quad \text{and} \quad x_{n+1} = \frac{x_n + 1}{2}.$$
Prove by induction that $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$. Would this conclusion change if we set $x_1 = 2$?

6. Let a sequence $\{x_n\}$ of numbers be defined recursively by
$$x_1 = 1 \quad \text{and} \quad x_{n+1} = \frac{1}{1 + x_n}.$$
Prove by induction that x_{n+2} is between x_n and x_{n+1} for each $n \in \mathbb{N}$.

7. Mathematical induction also works for a sequence $P_k, P_{k+1}, \cdots$ of propositions, indexed by the integers $n \geq k$ for some $k \in \mathbb{N}$. The statement is: If P_k is true and P_{n+1} true whenever P_n is true and $n \geq k$, then P_n is true for all $n \geq k$. Prove this.

8. Use induction in the form stated in the preceding exercise to prove that $n^2 < 2^n$ for all $n \geq 5$.

9. Prove the identity
$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k},$$
which was used in the proof of Theorem 1.2.6.

10. Write out the binomial formula in the case $n = 4$.

11. Prove the commutative law for addition, $n + m = m + n$, holds in $\mathbb{N}$. Use induction on m and Examples 1.2.9 and 1.2.8.

12. Use Peano's axioms and the results we have proved using these axioms to prove that if $n, m \in \mathbb{N}$, then $m + n \neq n$. Hint: use induction on n.

13. Use the preceding exercise to prove that if $n, m \in \mathbb{N}$, $n \leq m$, and $m \leq n$ then $n = m$.

14. Prove that the order relation on $\mathbb{N}$ has the transitive property: If $k \leq n$ and $n \leq m$, then $k \leq m$.

15. Use the preceding exercise and Peano's axioms to prove that if $n \in \mathbb{N}$, then for each element $m \in \mathbb{N}$ either $m \leq n$ or $n \leq m$. Hint: use induction on n.

16. Show how to define the product mn of two natural numbers using only Peano's axioms. You may use the results of any Examples in this section which were done using only Peano's axioms.

17. Prove the well ordering principal for the natural numbers: each non-empty subset S of $\mathbb{N}$ contains a smallest element. Hint: apply the induction axiom to the set
$$T = \{n \in \mathbb{N} : n < m \text{ for all } m \in S\}.$$

18. Use the result of Exercise 1.2.17 to prove the division algorithm: If n and m are natural numbers with $m < n$, and if m does not divide n, then there are natural numbers q and r such that $n = qm + r$ and $r < m$. Hint: consider the set S of all natural numbers s such that $(s + 1)m > n$.

1.3 Integers and Rational Numbers

The need for larger number systems than the natural numbers became apparent early in mathematical history. We need the number 0 in order to describe the number of elements in the empty set. The negative numbers are needed to describe deficits. Also, the operation of subtraction leads to non-positive integers unless $n - m$ is to be defined only for $m < n$.

Beginning with the system of natural numbers $\mathbb{N}$ and its properties derivable from Peano's axioms, the system of integers $\mathbb{Z}$ can easily be constructed. One simply adjoins to $\mathbb{N}$ a new element called 0 and, for each $n \in \mathbb{N}$ a new element called $-n$. Of course, one then has to define addition and multiplication and an order relation "$\leq$" for this new set $\mathbb{Z}$ in a way that is consistent with the existing definitions of these things for $\mathbb{N}$. When addition and multiplication are defined, we want them to have the properties that $0+n=n$, and $n+(-n)=0$. It turns out that these requirements and the commutative, associative and distributive laws (described below) are enough to uniquely determine how addition and multiplication are defined in $\mathbb{Z}$.

When all of this has been carried out, the new set of numbers $\mathbb{Z}$ can be shown to be a *commutative ring*, meaning that it satisfies the axioms listed below.

The Commutative Ring of Integers

A binary operation on a set A is rule which assigns to each ordered pair (a, b) of elements of A a third element of A.

Definition 1.3.1. A *commutative ring* is set R with two binary operations, addition $((a, b) \to a + b)$ and multiplication $((a, b) \to ab)$, that satisfy the following axioms:

A1. (Commutative Law of Addition) $x + y = y + x$ for all $x, y \in R$;

A2. (Associative Law of Addition) $x + (y + z) = (x + y) + z$ for all $x, y, z \in R$;

A3. (Additive Identity) there is an element $0 \in R$ such that $0 + x = x$ for all $x \in R$;

A4. (Additive Inverses) for each $x \in R$, there is an element $-x$ such that $x + (-x) = 0$;

M1. (Commutative Law of Multiplication) $xy = yx$ for all $x, y \in R$;

M2. (Associative Law of Multiplication) $x(yz) = (xy)z$ for all $x, y, z \in R$;

M3. (Multiplicative Identity) there is an element $1 \in R$ such that $1 \neq 0$ and $1x = x$ for all $x \in R$;

D. (Distributive Law) $x(y + z) = xy + xz$ for all $x, y, z \in R$.

A large number of familiar properties of numbers can be proved using these axioms, and this means that these properties hold in any commutative ring. We will prove some of these in the examples and exercises.

Example 1.3.2. If F is a commutative ring and $x, y, z \in F$, prove that

(a) $x + z = y + z$ implies $x = y$;

(b) $x \cdot 0 = 0$;

(c) $(-x)y = -xy$;

Solution: Suppose $x+z=y+z$. On adding $-z$ to both sides, this becomes

$$(x+z)+(-z)=(y+z)+(-z).$$

Applying the associative law of addition (**A2**) yields

$$x+(z+(-z))=y+(z+(-z)).$$

But $(z+(-z))=0$ by **A4** and $x+0=x$ by **A3** and **A1**. Similarly, $y+0=y$. We conclude that $x=y$. This proves (a).

By **A3**, $0+0=0$. By **D** and **A3**,

$$x\cdot 0+x\cdot 0=x\cdot 0=0+x\cdot 0.$$

Using (a) above, we conclude that $x\cdot 0=0$.

To prove (c), we first note that, by definition, $-xy$ is the additive inverse of xy (it follows from (a) that there is only one of these). We will show that $(-x)y$ is also an additive inverse for xy. By **D**, (b), and **A1**,

$$xy+(-x)y=(x+(-x))y=0\cdot y=0.$$

This proves that $(-x)y$ is an additive inverse for xy and, hence, must be $-xy$.

Subtraction in a commutative ring is defined in terms of addition and the additive inverse by setting

$$x-y=x+(-y).$$

The system of integers satisfies all the laws of Definition 1.3.1, and so it is a commutative ring. In fact, it is a commutative ring with an order relation, since the order relation on $\mathbb{N}$ can be used to define a compatible order relation on $\mathbb{Z}$. However, $\mathbb{Z}$ is still inadequate as a number system. This is due to our need to talk about fractional parts of things. This defect is fixed by passing from the integers to the rational numbers.

The Field of Rational Numbers

A *field* is a commutative ring in which division is possible as long as the divisor is not 0. That is,

Definition 1.3.3. A field is a commutative ring satisfying the additional axiom:

M4. (Multiplicative Inverses) for each non-zero element x there is an element x^{-1} such that $x^{-1}x=1$.

In a field, an element y can be divided by any non-zero element x. The result is $x^{-1}y$, which can also be written as $\frac{y}{x}$.

The rational number system $\mathbb{Q}$ is a field that is constructed directly from the integers. The construction begins by considering all symbols of the form $\frac{n}{m}$, with $n,m\in\mathbb{Z}$ and $m\neq 0$. We identify two such symbols $\frac{n}{m}$ and $\frac{p}{q}$ whenever

$nq = mp$. The resulting object is called a fraction. Thus, $\frac{4}{6}$ and $\frac{2}{3}$ represent the same fraction because $4 \cdot 3 = 6 \cdot 2$. The set $\mathbb{Q}$ is then the set of all fractions.

Addition and multiplication in $\mathbb{Q}$ are defined in the familiar way:

$$\frac{n}{m} + \frac{p}{q} = \frac{nq + mp}{mq} \quad \text{and} \quad \frac{n}{m} \cdot \frac{p}{q} = \frac{np}{mq}.$$

A fraction of the form $\frac{n}{1}$ is identified with the integer n. This makes the set of integers $\mathbb{Z}$ a subset of $\mathbb{Q}$.

The above construction yields a system that satisfies **A1** through **A4**, **M1** through **M4** and **D**. It is therefore a field. We call it the field of rational numbers and denote it by $\mathbb{Q}$. We won't prove here that $\mathbb{Q}$ satisfies all of the field axioms, but a few of them will be verified in the examples and exercises of this section. We will also use the examples and exercises to show how the field axioms can be used to prove other standard facts about arithmetic in fields such as $\mathbb{Q}$.

Example 1.3.4. Assuming that $\mathbb{Z}$ satisfies the axioms of a commutative ring , verify that $\mathbb{Q}$ satisfies **A3** and **M3**.

Solution: The additive identity in $\mathbb{Z}$ is the integer 0, which is identified with the fraction $\frac{0}{1}$. If we add this to another fraction $\frac{n}{m}$, the result is

$$\frac{0}{1} + \frac{n}{m} = \frac{0 \cdot m + 1 \cdot n}{1 \cdot m} = \frac{n}{m}.$$

Thus, $0 = \frac{0}{1}$ is an additive identity for $\mathbb{Q}$ and axiom **A3** is satisfied.

The multiplicative identity in $\mathbb{Z}$ is the integer 1 which is identified with the fraction $\frac{1}{1}$. If we multiply this by another fraction $\frac{n}{m}$, the result is

$$\frac{1}{1} \cdot \frac{n}{m} = \frac{1 \cdot n}{1 \cdot m} = \frac{n}{m}.$$

Thus, $1 = \frac{1}{1}$ is a multiplicative identity for $\mathbb{Q}$ and axiom **M3** is satisfied.

Example 1.3.5. Verify that $\mathbb{Q}$ satisfies **M4**.

Solution: We know that the elements of $\mathbb{Q}$ of the form $\frac{0}{m}$ represent the zero element of $\mathbb{Q}$. Thus, each non-zero element is represented by a fraction $\frac{n}{m}$ in which $n \neq 0$. Then $\frac{m}{n}$ is also a fraction, and

$$\frac{m}{n} \cdot \frac{n}{m} = \frac{nm}{nm} = \frac{1}{1} = 1.$$

Thus, $\frac{m}{n}$ is a multiplicative inverse for $\frac{n}{m}$. This proves that **M4** is satisfied in $\mathbb{Q}$.

The Ordered Field of Rational Numbers

Using the order relation on the integers, it is easy to define an order relation on $\mathbb{Q}$. If r is an element of $\mathbb{Q}$, then we declare $r \geq 0$ if r can be represented in the form $\frac{n}{m}$ for integers $n \geq 0$ and $m > 0$. The order relation is then defined by declaring

$$\frac{p}{q} \leq \frac{n}{m} \quad \text{if and only if} \quad \frac{n}{m} - \frac{p}{q} \geq 0.$$

With the order relation defined this way, $\mathbb{Q}$ becomes an ordered field. That is, it satisfies the axioms in the following definition.

Definition 1.3.6. A field F is called an *ordered field* if it has an order relation "$\leq$" such that the following are satisfied for all $x, y, z \in F$:

O1. either $x \leq y$ or $y \leq x$;

O2. if $x \leq y$ and $y \leq x$, then $x = y$:

O3. if $x \leq y$ and $y \leq z$, then $x \leq z$.

O4. if $x \leq y$, then $x + z \leq y + z$;

O5. if $x \leq y$ and $0 \leq z$, then $xz \leq yz$.

Remark 1.3.7. Given an order relation "$\leq$", we don't distinguish between the statements "$x \leq y$ and "$y \geq x$" – they mean the same thing. Also, If $x \leq y$ and $x \neq y$, then we write $x < y$ or, equivalently, $y > x$.

Example 1.3.8. Prove that if F is an ordered field, then

(a) if $x, y \in F$ and $x \leq y$, then $-y \leq -x$;

(b) if $x \in F$, then $x^2 \geq 0$;

(c) $0 < 1$.

Solution: If $x \leq y$, then $0 = x - x \leq y - x$ by **O4**. Using **O4** again, along with **A1** through **A4** yields $-y \leq (y - x) - y = -x$. This completes the proof of (a).

By **O1**, if $x \in F$, then $0 \leq x$ or $x \leq 0$. If $0 \leq x$, then we multiply this inequality by x and use **O4** to conclude that $0 \leq x^2$. On the other hand, suppose $x \leq 0$. Then, by Part (a), $0 \leq -x$. As above, we conclude that $0 \leq (-x)^2$. Since $(-x)^2 = x^2$ (Exercise 1.3.5), the proof of Part (b) is complete.

Since $1^2 = 1$, Part (b) implies that $0 \leq 1$. By **M3**, $1 \neq 0$ and so $0 < 1$.

Defects of the Rational Field

The rational number system is very satisfying in many ways and is highly useful. However, there are real world mathematic problems that appear to have real world numerical solutions, but these solutions cannot be rational numbers. For example, the Pythagorean Theorem tells us that if the legs of a right triangle have length a and b, then the length c of the hypotenuse satisfies the equation

$$c^2 = a^2 + b^2.$$

However, there are many examples of rational and even integer choices for a and b, such that this equation has no rational solution for c. The simplest example is $a = b = 1$. The Pythagorean Theorem says that a right triangle with legs of length 1 has a hypotenuse of length c satisfying $c^2 = 2$. However, there is no rational number whose square is 2. We will prove this using the following theorem:

Theorem 1.3.9. *If k is an integer and the equation $x^2 = k$ has a rational solution, then that solution is actually an integer.*

Proof. Suppose r is a rational number such that $r^2 = k$. Let $r = \frac{n}{m}$ be r expressed as a fraction in which n and m have no common factors. Then,

$$\left(\frac{n}{m}\right)^2 = k \quad \text{and so} \quad n^2 = m^2 k$$

This equation implies that m divides n^2. However, if $m \neq 1$, then m can be expressed as a product of primes, and each of these primes must also divide n^2. However, if a prime number divides n^2, it must also divide n (Exercise 1.3.14). Thus, each prime factor of m divides n. Since n and m have no common factors, this is impossible. We conclude that $m = 1$ and, hence, that $r = n$ is an integer. □

Now it is easy to see that 2 is not the square of a rational number. If it were, that number would have to be an integer, by the above theorem. The only possibilities are $-1, 0, 1$ since all other integers have squares that are too large. Of course, none of the numbers $-1, 0, 1$ has square equal to 2.

Other geometric objects also lead to the conclusion that the system of rational numbers is not sufficient for the measurement of objects that occur in the natural world. The area π of a circle of radius 1 is not a rational number, for example. In fact, the rational number system is riddled with holes where there ought to be numbers. This problem is fixed by the introduction of the system of real numbers which is the topic of the next section.

Exercise Set 1.3

1. Given that $\mathbb{N}$ has an operation of addition which is commutative and associative, how would you define such an addition operation in $\mathbb{Z}$?

2. Referring to the previous exercise, answer the same question for the operation of multiplication.

3. Prove that if $\mathbb{Z}$ satisfies the axioms for a commutative ring, then $\mathbb{Q}$ satisfies **A1** and **M1**.

4. Prove that if $\mathbb{Z}$ satisfies the axioms for a commutative ring, then $\mathbb{Q}$ satisfies **A2** and **M2**.

   In the next three exercises you are to prove the given statement assuming x, y, z are elements of a field. You may use the results of examples and theorems from this section.

5. $(-x)(-y) = xy$.

6. $xz = yz$ implies $x = y$, provided $z \neq 0$.

7. $xy = 0$ implies $x = 0$ or $y = 0$.

In the next three exercises you are to prove the given statement assuming x, y, z are elements of an ordered field. Again, you may use the results of examples and theorems from this section.

8. $x > 0$ and $y > 0$ imply $xy > 0$.

9. $x > 0$ implies $x^{-1} > 0$.

10. $0 < x < y$ implies $y^{-1} < x^{-1}$.

11. Prove that the equation $x^2 = 5$ has no rational solution.

12. Generalize Theorem 1.3.9 by proving that every rational solution of a polynomial equation

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0,$$

with integer coefficients a_k, is an integer solution.

13. Prove that if m and n are positive integers with no common factors other than 1 (i. e. m and n are relatively prime), then there are integers a and b such that $1 = am + bn$. Hint: let S be the set of all positive integers of the form $am + bn$, where a and b are integers. This set has a smallest element by Exercise 1.2.17. Use the division algorithm (Exercise 1.2.18) to show that this smallest element divides both m and n.

14. Use the result of the preceding exercise to prove that if a prime p divides the product nm of two positive integers, then it divides n or it divides m.

1.4 The Real Numbers

As pointed out in the previous section, the set of rational numbers is riddled with "holes" where there ought to be numbers. Here we will try to make this statement more precise and then indicate how these holes can be "filled" resulting in the system of real numbers. In addition to the ordered field axioms, the real number system satisfies a new axiom **C** – the completeness axiom. Later in the section we will state it and explore its consequences.

The construction of the real numbers that we outline below is motivated by the idea that a "hole" in the rational numbers is a location along the rational number line where there should be a number but there is no rational number. What do we mean by a "location" along the rational number line? Well if this has meaning, then it should make sense to talk about the rational numbers that are to the left of this location and those that are to the right of this location.

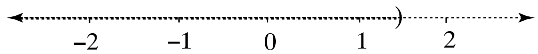

Figure 1.2: A Dedekind Cut in the Rationals.

This should lead to a separation of the rational numbers into two sets – one to the left and one to the right of the given location. In fact, we can *define* a location on the rational line to *be* such a separation. This leads to the notion of a *Dedekind cut*.

Dedekind Cuts

If r is a rational number, consider the infinite interval L_r consisting of all rational numbers to the left of r. That is,

$$L_r = \{x \in \mathbb{Q} : x < r\}. \tag{1.4.1}$$

This set is a non-empty, proper subset of $\mathbb{Q}$. It has no largest element, since, for each $x < r$, there are rational numbers larger than x that are also less than r (for example, $(x + r)/2$ is one such number). It also has the property that if $x \in L_r$, then so is any rational number less than x. A set with these properties is called a *Dedekind cut*. That is,

Definition 1.4.1. A proper subset L of $\mathbb{Q}$ is called a *Dedekind cut*, or simply a *cut* in the rationals, if it satisfies the following conditions:

(a) $L \neq \emptyset$;

(b) L has no largest element;

(c) if $x \in L$ then so is every y with $y < x$.

The reason for calling such a set L a "cut" is that, if R is the complement of L, then each number in L is to the left of each number in R. Thus, the rational line is separated or *cut* into left and right halves. Since each half determines the other, we choose to focus on just the left half in this discussion.

Each rational number r determines a cut – the set L_r of (1.4.1). In this case, r is called the *cut number* for the Dedekind cut. Are there Dedekiind cuts that are not determined in this way? cuts that have no rational cut number?

Example 1.4.2. Describe a Dedekind cut that is not of the form L_r for a rational number r.

Solution: We are guided by the idea that there ought to be a number whose square is 2, but there is no such rational number. If there were a number $\sqrt{2}$ with square 2, then the set of rational numbers less than $\sqrt{2}$ could be described as

$$L = \{r \in \mathbb{Q} : r \geq 0 \text{ and } r^2 < 2\} \cup \{r \in \mathbb{Q} : r < 0\}.$$

We claim this a Dedekind cut not of the form L_r for any $r \in \mathbb{Q}$.

Certainly L is a non-empty, proper subset of $\mathbb{Q}$. It has no largest element because if $\frac{n}{m}$ is any positive element of L, then we can always choose a larger rational number which is still has square less than 2 as follows: $\frac{kn+1}{km} > \frac{n}{m}$ for every $k \in \mathbb{N}$ and

$$\left(\frac{kn+1}{km}\right)^2 = \left(\frac{n}{m}\right)^2 + \frac{1}{km}\left(2\frac{n}{m} + \frac{1}{km}\right).$$

By choosing k large enough, we can make the second term on the right less than $2 - (\frac{n}{m})^2$ and this will imply that $(\frac{kn+1}{km})^2 < 2$. Thus, L has no largest element.

If $x \in L$ and $y < x$, then either y is negative, in which case it is in L, or $0 \le y < x$. In the latter case, $y^2 < x^2 < 2$, and so $y \in L$ in this case as well. Thus L is a Dedekind cut.

We next show that there is no rational number r such that $L = L_r$. If there is such a number r, then r is a positive rational number not in L and so $r^2 \ge 2$. However, there are numbers in L arbitrarily close to r and each of them has square less than 2. It follows that $r^2 \le 2$. This means $r^2 = 2$, which is impossible for a rational number r.

Thus, although it might seem that every Dedekind cut ought to correspond to a cut number, the above example shows that this is not the case. In fact, there are a lot more cuts than there are rational cut numbers. However, we can fix this by enlarging the number system so that there is a cut number for every Dedekind cut. The way this is usually done is to define the new number system to actually *be* the set of all Dedekind cuts of the rationals. Below, we attempt to describe this idea in a way that is somewhat visually intuitive.

We will think of a Dedekind cut L as specifying a certain location (the location between L and its complement R) along the rational number line. We will think of the real number system $\mathbb{R}$ as being the set of all such locations. Then each real number x corresponds to a Dedekind cut L_x, which is to be thought of as the set of all rational numbers to the left of the location x. We next need to define an order relation and operations of addition and multiplication in $\mathbb{R}$.

The order relation on $\mathbb{R}$ is simple: We say $x \le y$ if $L_x \subset L_y$. An element $x \in \mathbb{R}$ is, then, non-negative if $L_0 \subset L_x$. With this definition of order on $\mathbb{R}$ we can assert that

$$L_x = \{r \in \mathbb{Q} : r < x\}$$

for all $x \in \mathbb{R}$ (not just for $x \in \mathbb{Q}$).

Addition of real numbers is defined as follows: If $x, y \in \mathbb{R}$, then we set

$$L_x + L_y = \{r + s : r \in L_x, s \in L_y\}.$$

It is easily verified that this is also a Dedekind cut (Exercise 1.4.10) and, hence, it corresponds to an element of $\mathbb{R}$. We define $x + y$ to be this element.

The product of two non-negative numbers x and y is defined as follows: we set

$$K = \{rs : r \in L_x, r \geq 0, s \in L_y, s \geq 0\} \cup \{t \in \mathbb{Q} : t < 0\}.$$

This is a Dedekind cut (Exercise 1.4.11), and we define xy to be the corresponding element of $\mathbb{R}$. For pairs of numbers where one or both is negative, the definition of product is more complicated due to the fact that multiplication by a negative number reverses order.

Of course $\mathbb{Q} \subset \mathbb{R}$, since each rational number was already the cut number of a Dedekind cut. It is easily checked that the definitions of addition, multiplication and order given above agree with the usual ones in the case that the numbers are rational.

The numbers in $\mathbb{R}$ that are not in $\mathbb{Q}$ are called *irrational* numbers. It turns out that there are many more irrational numbers than there are rational numbers. To make sense of this statement requires a discussion of finite sets and infinite sets, and how some infinite sets are larger than others. We present such a discussion in the appendix.

The Completeness Axiom

This is the property of the real number system that distinguishes it from the rational number system. Without it, most of the theorems of calculus would not be true.

A subset A of an ordered field F is said to be *bounded above* if there is an element $m \in F$ such that $x \leq m$ for every $x \in A$. The element m is called an *upper bound* for A. If, among all upper bounds for A, there is one which is smallest (less than all the others), then we say that A has a *least upper bound.*

Definition 1.4.3. An ordered field F is said to be *complete* if it satisfies:

C. each non-empty subset of F which is bounded above has a least upper bound.

If one defines the real number system $\mathbb{R}$ in terms of Dedekind cuts of the rationals and defines addition, multiplication, and order as above, then one can prove that the resulting system is an ordered field. To carry out all the details of this proof is a long and tedious process and it will not be done here. However, it is quite easy to prove that $\mathbb{R}$, as defined in this way, satisfies the completeness axiom **C**.

Theorem 1.4.4. *If $\mathbb{R}$ is defined using Dedekind cuts of $\mathbb{Q}$, as above, then every bounded subset of $\mathbb{R}$ has a least upper bound.*

Proof. Let A be a bounded subset of $\mathbb{R}$ and let m be any upper bound for A. For each $x \in A$, let L_x be the corresponding cut in $\mathbb{Q}$. Then $x \leq m$ for all $x \in A$ means that $L_x \subset L_m$ for all $x \in A$. We set

$$L = \bigcup_{x \in A} L_x.$$

Then L is a proper subset of $\mathbb{Q}$ because $L \subset L_m$. If $r \in L$ and $s < r$, then $r \in L_x$ for some $x \in A$ and this implies $s \in L_x$ and, hence, $s \in L$. If L had a largest element t, then t would belong to L_x for some x, and it would have to be a largest element for L_x – a contradiction. Thus, L has no largest element. We have now proved that L satisfies (a), (b), and (c) of Definition 1.4.1 and, hence, that L is a Dedekind cut.

If y is the real number corresponding to L, that is if $L = L_y$, then, for all $x \in A$, $L_x \subset L_y$, and this means $x \leq y$. Thus, y is an upper bound for A. Also, $L_y \subset L_m$ means that $y \leq m$. Since m was an arbitrary upper bound for A, this implies that y is the least upper bound for A. This completes the proof. □

This completes our outline of the construction of the real number system beginning with Peano's axioms for the natural numbers. The final result is the following theorem, which we will state without further proof. It will be the starting point for our development of calculus.

Theorem 1.4.5. *The real number system $\mathbb{R}$ is a complete ordered field.*

Example 1.4.6. Find all upper bounds and the least upper bound for the following sets:

$$A = (-1, 2) = \{x \in \mathbb{R} : -1 < x < 2\};$$
$$B = (0, 3] = \{x \in \mathbb{R} : 0 < x \leq 3\}.$$

Solution: The set of all upper bounds for the set A is $\{x \in \mathbb{R} : x \geq 2\}$. The smallest element of this set (the least upper bound of A) is 2. Note that 2 is not actually in the set A.

The set of all upper bounds for B is the set $\{x \in \mathbb{R} : x \geq 3\}$. The smallest element of this set is 3 and so it is the least upper bound of B. Note that, in this case, the least upper bound is an element of the set B.

If the least upper bound of a set A does belong to A, then it is called the *maximum* of A. Note that a set which is bounded above always has a least upper bound, by Axiom **C**. However, the preceding example shows that it need not have a maximum.

The Archimedean Property

An ordered field always contains a copy of the natural numbers and, hence, the integers (Exercise 1.4.5). Thus, the following definition makes sense.

Definition 1.4.7. An ordered field is said to have the Archimedean property if, for every $x \in \mathbb{R}$, there is a natural number n such that $x < n$. An ordered field with the Archimedean property is called an *Archimedean ordered field.*

Theorem 1.4.8. *The field of real numbers has the Archimedean property.*

Proof. We use the completeness property. Suppose there is an x such that $n \leq x$ for all $n \in \mathbb{N}$. Then $\mathbb{N}$ is a bounded subset of $\mathbb{R}$. By the completeness property, there is a least upper bound b for $\mathbb{N}$. Then b is an upper bound for $\mathbb{N}$, but $b-1$ is not. This implies there is an $n \in \mathbb{N}$ such that $b - 1 < n$. Then $b < n + 1$, which contradicts the statement that b is an upper bound for $\mathbb{N}$. Thus, the assumption that $\mathbb{N}$ is bounded above by some $x \in \mathbb{R}$ has led to a contradiction. We conclude that every x in $\mathbb{R}$ is less than some natural number. This completes the proof. □

The Archimedean property can be stated in any one of several equivalent ways. One of these is: for every real number $x > 0$, there is an $n \in N$ such that $1/n < x$ (Example 1.4.9). Another is: given real numbers x and y with $x > 0$, there is an $n \in \mathbb{N}$ such that $nx > y$ (Exercise 1.4.6).

Example 1.4.9. Prove that, in an Archimedean field, for each $x > 0$ there is an $n \in \mathbb{N}$ such that $1/n < x$.

Solution The Archimedean property tells us that there is a natural number $n > 1/x$. Since n and x are positive, this inequaltiy is preserved when we multiply it by x and divide it by n. This yields $1/n < x$, as required.

Another consequence of the Archimedean property is that there is a rational number between each distinct pair of real numbers (Exercise 1.4.7).

Exercise Set 1.4

1. For each of the following sets, describe the set of all upper bounds for the set :
   (a) the set of odd integers;
   (b) $\{1 - 1/n : n \in \mathbb{N}\}$;
   (c) $\{r \in \mathbb{Q} : r^3 < 8\}$;
   (d) $\{\sin x : x \in \mathbb{R}\}$.

2. For each of the sets in $(a), (b), (c)$ of the preceding exercise, find the least upper bound of the set, if it exists.

3. Prove that if a subset A of $\mathbb{R}$ is bounded above, then the set of all upper bounds for A is a set of the form $[x, \infty)$. What is x?

4. Show that the set $A = \{x : x^2 < 1 - x\}$ is bounded above, and then find its least upper bound.

5. If F is an ordered field, prove that there is a sequence of elements $\{n_k\}_{k\in\mathbb{N}}$, all different, such that $n_1 = 1$ (the identity element of F), and $n_{k+1} = n_k + 1$ for each $k \in \mathbb{N}$. Argue that the terms of this sequence form a subset of F which is a copy of the natural numbers, by showing that the correspondence $k \to n_k$ is a one-to-one function from $\mathbb{N}$ onto this subset. By definition it takes the successor $k + 1$ of an element $k \in \mathbb{N}$ to the successor $n_k + 1$ of its image n_k.

6. Let F be an ordered field, We consider $\mathbb{N}$ to be a subset of F as described in the preceding exercise. Prove that F is Archimedean if and only if, for each pair $x, y \in F$ with $x > 0$, there exists a natural number n such that $nx > y$.

7. Prove that if $x < y$ are two real numbers, then there is a rational number r with $x < r < y$. Hint: use the result of Example 1.4.9.

8. Prove that if x is irrational and r is a non-zero rational number, then $x+r$ and rx are also irrational.

9. We know that $\sqrt{2}$ is irrational. Use this fact and the previous exercise to prove that if $r < s$ are rational numbers, then there is an irrational number x with $r < x < s$.

   The following exercises concern Dedekind cuts of the rationals and should be done using only properties of the rational number system and the definition of Dedekind cut.

10. Show that if L_x and L_y are Dedekind cuts defining real numbers x and y, then
$$L_x + L_y = \{r + s : r \in L_x \text{ and } s \in L_y\}$$
is also a Dedekind cut (this is the Dedekind cut determining the sum $x + y$).

11. If L_x and L_y are Dedekind cuts determining positive real numbers x and y, and if we set
$$K = \{rs : 0 \le r \in L_x \text{ and } 0 \le s \in L_y\} \cup \{t \in \mathbb{Q} : t < 0\},$$
then K is also a Dedekind cut (this is the Dedekind cut determining the product xy).

12. If L is the Dedekind cut of Example 1.4.2 and L determines the real number x (so that $L = L_x$), prove that $L_{x^2} = L_2$. Thus, the real number corresponding to L has square 2.

1.5 Sup and Inf

The concept of least upper bound, which appears in the completeness axiom, will be extremely important in this course. It will be examined in detail in this section. We first note that there is a companion concept for sets that are bounded below.

Greatest Lower Bound

We say a set A is bounded below if there is a number m such that $m \le x$ for every $x \in A$. The number m is called a *lower bound* for A. A *greatest lower bound* for A is a lower bound that is larger than any other lower bound.

Theorem 1.5.1. *Every non-empty subset of $\mathbb{R}$ that is bounded below has a greatest lower bound.*

Proof. Suppose A is a non-empty subset of $\mathbb{R}$ which is bounded below . We must show that there is a lower bound for A which is greater than any other lower bound for A. If m is any lower bound for A, then Example 1.3.8 (a) implies that, $-m$ is an upper bound for $-A = \{-a : a \in A\}$. Since $\mathbb{R}$ is a complete ordered field, there is a least upper bound r for $-A$. Then

$$-a \le r \text{ for all } a \in A \quad \text{and} \quad r \le -m.$$

Applying Example 1.3.8 (a) yields that

$$-r \le a \text{ for all } a \in A \quad \text{and} \quad m \le -r.$$

Thus, $-r$ is a lower bound for A and, since m was an arbitrary lower bound, the inequality $m \le -r$ implies that $-r$ is the greatest lower bound. □

The Extended Real Numbers

For many reasons, it is convenient to extend the real number system by adjoining two new points ∞ and $-\infty$. The resulting set is called the *extended real number system*. We declare that ∞ is greater than and $-\infty$ less than every other extended real number. This makes the extended real number system an ordered set. We also define $x + \infty$ to be ∞ if x is any extended real number other than $-\infty$. Similarly, $x - \infty = x + (-\infty)$ is defined to be $-\infty$ if x is any extended real number other than ∞. Of course, there is no reasonable way to make sense of $\infty - \infty$.

The introduction of the extended real number system is just a convenient notational convention. For example, it allows us to make the following definition.

Sup and Inf

Definition 1.5.2. Let A be an arbitrary non-empy subset of $\mathbb{R}$. We define the *supremum* of A, denoted $\sup A$, to be the smallest extended real number M such that $a \le M$ for every $a \in A$.

The infimum of A, denoted $\inf A$, is the largest extended real number m such that $m \le a$ for all $a \in A$.

Note that, if A is bounded above, then $\sup A$ is the least upper bound of A. If A is not bounded above, then the only extended real number M with $a \le M$ for all $a \in A$ is ∞, and so $\sup A = \infty$ in this case. Similarly, $\inf A$ is the greatest lower bound of A if A is bounded below and is $-\infty$ if A is not bounded below. Thus, $\sup A$ and $\inf A$ exist as extended real numbers for any non-empty set A, but they might not be finite. Also note that, even when they are finite real numbers, they may not actually belong to A, as Example 1.4.6 shows.

Example 1.5.3. Find the sup and inf of the following sets:

$$A = (-1, 1] = \{x \in \mathbb{R} : -1 < x \leq 1\};$$
$$B = (-\infty, 5) = \{x \in \mathbb{R} : x < 5\}.$$
$$C = \left\{\frac{n^2}{n+1} : n \in \mathbb{N}\right\} \tag{1.5.1}$$
$$D = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \tag{1.5.2}$$

Solution: Clearly, $\inf A = -1$ and $\sup A = 1$. These are finite, $\sup A$ belongs to A, but $\inf A$ does not.

Also, $\inf B = -\infty$ and $\sup B = 5$. In this case, the inf is not finite. The sup is finite but does not belong to B.

Since $\dfrac{n^2}{n+1} \geq \dfrac{n}{2}$, the set C is unbounded, and so $\sup C = \infty$. Also, we have $n + 1 \leq n^2 + n^2 = 2n^2$, and so

$$\frac{1}{2} \leq \frac{n^2}{n+1}$$

for all $n \in \mathbb{N}$. Thus, $1/2$ is a lower bound for C. It is the greatest lower bound, since it actually belongs to C, due to the fact that $\dfrac{n^2}{n+1} = \dfrac{1}{2}$ when $n = 1$. Thus, $\inf C = 1/2$.

Certainly 0 is a lower bound for the set D. It follows from the Archimedean property (see Example 1.4.9) that there is no $x \in F$ with $x > 0$ which is a lower bound for this set, and so 0 is the greatest lower bound. Thus, $\inf D = 0$. Clearly, $\sup D = 1$.

If A is a set of numbers and $\sup A$ actually belongs to A, then it is called the *maximum* of A and denoted $\max A$. Similarly, if $\inf A$ belongs to A, then it is called the *minimum* of A and is denoted $\min A$.

The following theorem is really just a restatement of the definition of sup, but it may give some helpful insight. It says that $\sup A$ is the dividing point between the numbers which are upper bounds for A (if there are any) and the numbers which are not upper bounds for A. A similar theorem holds for inf. Its formulation and proof are left to the exercises.

Theorem 1.5.4. *Let A be a non-empty subset of $\mathbb{R}$ and x an element of $\mathbb{R}$. Then*

(a) $\sup A \leq x$ *if and only if* $a \leq x$ *for* every $a \in A$;

(b) $x < \sup A$ *if and only if* $x < a$ *for* some $a \in A$.

Proof. (a) By definition $a \leq x$ for every $a \in A$ if and only if x is an upper bound for A.

If x is an upper bound for A, then A is bounded above. This implies its *sup* is its least upper bound, which is necessarily less than or equal to x.

Conversely, if $\sup A \leq x$, then $\sup A$ is finite and is the least upper bound for A. Since $\sup A \leq x$, x is also an upper bound for A. Thus, $\sup A \leq x$ if and only if $a \leq x$ for every $a \in A$.

(b) If $x < \sup A$, then x is not an upper bound for A, which means that $x < a$ for some $a \in A$. Conversely, if $x < a$ for some $a \in A$, then $x < \sup A$, since $a \leq \sup A$. Thus, $x < \sup A$ if and only if $x < a$ for some $a \in A$. □

Example 1.5.5. If $A = \left\{ \frac{4n-1}{6n+3} : n \in \mathbb{N} \right\}$, find the set of all upper bounds for A.

Solution: Long division yields

$$\frac{4n-1}{6n+3} = \frac{2}{3} - \frac{1}{2n+1} \leq \frac{2}{3}.$$

Thus, $2/3$ is an upper bound for A. If $x < 2/3$, then $\epsilon = 2/3 - x$ is positive, and the Archimedean Property implies we can choose n large enough that

$$\frac{1}{2n+1} < \frac{1}{n} < \epsilon.$$

Then

$$x < \frac{2}{3} - \frac{1}{2n+1} = \frac{4n-1}{6n+3}$$

for such an n, which means that x is not an upper bound for A.

We conclude that $2/3$ is the least upper bound for A – that is $\sup A = 2/3$. By the previous theorem, the set of all upper bounds for A is the interval $[2/3, \infty)$.

Example 1.5.6. If $A = \left\{ \frac{n^2}{n+1} : n \in \mathbb{N} \right\}$, find $\sup A$ and the set of all upper bounds for A.

Solution: Long division yields

$$\frac{n^2}{n+1} = n - 1 + \frac{1}{n+1} \geq n - 1.$$

Then the Archmedean Property implies that there are no upper bounds for A, since, for every $x \in \mathbb{R}$, there is an $n \in \mathbb{N}$ for which $n - 1$ is larger than x. Thus, the set of upper bounds for A is the empty set and $\sup A = \infty$.

Properties of Sup and Inf

The next theorem uses the following notation concerning subsets A and B of $\mathbb{R}$:

$$\begin{aligned} -A &= \{-a : a \in A\}; \\ A + B &= \{a + b : a \in A, b \in B\} \\ A - B &= \{a - b : a \in A, b \in B\}. \end{aligned}$$

Theorem 1.5.7. *Let A and B be non-empty subsets of $\mathbb{R}$. Then*

(a) $\inf A \leq \sup A$*;*

(b) $\sup(-A) = -\inf A$ *and* $\inf(-A) = -\sup A$*;*

(c) $\sup(A + B) = \sup A + \sup B$ *and* $\inf(A + B) = \inf A + \inf B$*;*

(d) $\sup(A - B) = \sup A - \inf B$*;*

(e) if $A \subset B$, then $\sup A \leq \sup B$ *and* $\inf B \leq \inf A$.

Proof. We will prove (a), (b), and (c) and leave (d) and (e) to the exercises.

(a) If A is non-empty, then there is an element $a \in A$. Since $\inf A$ is a lower bound and $\sup A$ an upper bound for A, we have $\inf A \leq a \leq \sup A$.

(b) A number x is a lower bound for the set A ($x \leq a$ for all $a \in A$) if and only if $-x$ is an upper bound for the set $-A$ ($-a \leq -x$ for all $a \in A$). Thus, if L is the set of all lower bounds for A, then $-L$ is the set of all upper bounds for $-A$. Furthermore, the largest member of L and the smallest member of $-L$ are negatives of each other. That is, $-\inf A = \sup(-A)$. This is the first equality in (b) If we apply this result with $-A$ replacing A, we have $-\inf(-A) = \sup A$. If we multiply this by -1, we get the second equality in (b).

(c) Since $a \leq \sup A$ and $b \leq \sup B$ for all $a \in A$, $b \in B$, we have

$$a + b \leq \sup A + \sup B \quad \text{for all} \quad a \in A,\ b \in B.$$

It follows that

$$\sup(A + B) \leq \sup A + \sup B.$$

Let x be any number less than $\sup A + \sup B$. We claim that there are elements $a \in A$ and $b \in B$ such that

$$x < a + b. \tag{1.5.3}$$

Once proved, this will imply that no number less than $\sup A + \sup B$ is an upper bound for $A + B$. Thus, proving this claim will establish that $\sup(A + B) = \sup A + \sup B$.

There are two cases to consider: $\sup B$ finite and $\sup B = \infty$. If $\sup B$ is finite, then $x - \sup B < \sup A$, and Theorem 1.5.4 implies there is an $a \in A$ with $x - \sup B < a$. Then $x - a < \sup B$. Applying Theorem 1.5.4 again, we conclude there is an $b \in B$ with $x - a < b$. This implies (1.5.3), and proves our claim in the case where $\sup B$ is finite.

Now suppose $\sup B = \infty$. Let a be any element of A. Then $x - a < \sup B = \infty$ and so, as above, we conclude from Theorem 1.5.4 that there is a $b \in B$ satisfying $x - a < b$. This implies (1.5.3), which establishes our claim in this case and completes the proof. □

Sup and Inf for Functions

If f is a real valued function defined on some set X and if A is a subset of X, then

$$f(A) = \{f(x) : x \in A\}$$

is a set of real numbers, and so we can take its sup and inf.

Definition 1.5.8. If $f : X \to \mathbb{R}$ is a function and $A \subset X$, then we set

$$\sup_A f = \sup\{f(x) : x \in A\} \quad \text{and} \quad \inf_A f = \inf\{f(x) : x \in A\}.$$

Thus, $\sup_A f$ is the supremum of the set of values that f assumes on A and $\inf_A f$ is the infimum of this set. They themselves may or may not be values that f assumes on A. If $\sup_A f$ is a value that f assumes on A, then it is called the *maximum* of f on A. Similarly, if $\inf_A f$ is a value assumed by f somewhere on A, then it is called the *minimum* of f on A.

Example 1.5.9. Find $\sup_I f$ and $\inf_I f$ if

(a) $f(x) = \sin x$ and $I = [-\pi/2, \pi/2)$;

(b) $f(x) = 1/x$ and $I = (0, \infty)$.

Solution: (a) The function $\sin x$ takes on all values in the interval $[-1, 1)$ on I, but does not take on the value 1. Thus, $\inf_I f = -1$ and $\sup_I f = 1$. In this case, $\inf_I f$ is a value assumed by f on I, but $\sup_I f$ is not.

(b) the function $1/x$ takes on all values in the open interval $(0, \infty)$. Thus, $\inf_I f = 0$ and $\sup_i f = \infty$ in this case. Neither one of these extended real numbers is a value taken on by f on I.

The following theorem concerning sup and inf for functions follows easily from Theorem 1.5.7. We leave the details to the exercises.

Theorem 1.5.10. *Let f and g be functions defined on a set containing A as a subset, and let $c \in \mathbb{R}$ be a positive constant. Then*

(a) $\sup_A cf = c \sup_A f$ *and* $\inf_A cf = c \inf_A f$;

(b) $\sup_A(-f) = -\inf_A f$;

(c) $\sup_A(f + g) \le \sup_A f + \sup_A g$ *and* $\inf_A f + \inf_A g \le \inf_A(f + g)$;

(d) $\sup\{f(x) - f(y) : x, y \in A\} \le \sup_A f - \inf_A f$.

Exercise Set 1.5

1. For each of the following sets, find the set of all extended real numbers x that are greater than or equal to every element of the set. Then find the sup of the set. Does the set have a maximum?

   (a) $(-10, 10)$;

   (b) $\{n^2 : n \in \mathbb{N}\}$;

   (c) $\left\{\dfrac{2n+1}{n+1}\right\}$.

2. Find the sup and inf of the following sets. Tell whether each set has a maximum or a minimum.

   (a) $(1, 8]$;

   (b) $\left\{\dfrac{n+2}{n^2+1}\right\}$;

   (c) $\{n/m : n, m \in \mathbb{Z},\ n^2 < 5m^2\}$;

3. Prove that if $\sup A < \infty$, then for each $n \in \mathbb{N}$ there is an element $a_n \in A$ such that $\sup A - 1/n < a_n \le \sup A$.

4. Prove that if $\sup A = \infty$, then for each $n \in \mathbb{N}$ there is an element $a_n \in A$ such that $a_n > n$.

5. Formulate and prove the analog of Theorem 1.5.4 for inf.

6. Prove part (d) of Theorem 1.5.7.

7. Prove (e) of Theorem 1.5.7.

8. if A and B are two non-empty sets of real numbers, then prove that

$$\sup(A \cup B) = \max\{\sup A, \sup B\} \quad \text{and} \quad \inf(A \cup B) = \min\{\inf A, \inf B\}.$$

9. Find $\sup_I f$ and $\inf_I f$ for the following functions f and sets I. Which of these is actually the maximum or the minimum of the function f on I.

   (a) $f(x) = x^2$, $I = [-1, 1]$;

   (b) $f(x) = \dfrac{x+1}{x-1}$, $I = (1, 2)$;

   (c) $f(x) = 2x - x^2$, $I = [0, 1)$.

10. Prove (a) of Theorem 1.5.10

11. Prove (b) of Theorem 1.5.10

12. Prove (c) of Theorem 1.5.10

13. Prove (d) of Theorem 1.5.10